Asia Pac. J. Math. 2022 **9**:11

MULTIPLE SOLUTIONS FOR ANISOTROPIC NONLINEAR DISCRETE DIRICHLET BOUNDARY VALUE PROBLEMS IN A TWO-DIMENSIONAL HILBERT SPACE

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Received Feb. 15, 2022

Abstract. In this paper, using minimization method, Mountain Pass geometry lemma and Ekelands variational principle we study the existence of a weak nontrivial solution for a family of nonlinear discrete Dirichlet boundary value problems where the solution lies in a discrete two-dimensional Hilbert space.

2010 Mathematics Subject Classification. 47A75, 35B38, 35P30, 34L05, 34L30.

Key words and phrases. discrete boundary value problem; critical point; weak solution; two-dimensional Hilbert space; Mountain Pass geometry lemma; Palais-Smale condition.

1. Introduction

In the last few years, the nonlinear equations have been of great interest because of their important applications appearing in various fields of research, such as numerical analysis, computer science, mechanical engineering, control systems, artificial or biological neural networks and social sciences, suh as economics. For background and recent results, we refer the reader to [1,2,4,6,14–16] and the references therein. Here, we are interested in investigating nonlinear discrete boundary value problems in two-dimensional Hilbert space. For example, based on the method of minimization, in [8], Ouaro and al. proved the existence of at least

DOI: 10.28924/APJM/9-11

one weak solution to the two-dimensional following Dirichlet problem,

$$\begin{cases} -\Delta(a(k-1,h-1,\Delta u(k-1,h-1))) = g(k,h,u(k,h)),\ (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \\ \\ u(k,h)=0,\ (k,h)\in\Gamma. \end{cases}$$

In this paper, motivated by the above facts, we propose the following anisotropic nonlinear discrete Dirichlet problem :

$$(1.1)\qquad \begin{cases} -\Delta(a(k-1,h-1,\Delta u(k-1,h-1))) = \lambda g(k,h,u(k,h)), \\ \\ (k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2], \\ \\ u(k,h)=0,\ (k,h)\in\Gamma \end{cases}$$

where

$$\Gamma = (\{0,T_1+1\}\times\mathbb{N}[0,T_2+1])\cup(\mathbb{N}[0,T_1+1]\times\{0,T_2+1\})$$

is the boundary of the domain $\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]$; $\Delta u(k,h) = u(k+1,h+1)-u(k,h)$ is the forward difference operateur, λ is a positive real parameter and

$$a:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}\longrightarrow\mathbb{R},\qquad g:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}\longrightarrow\mathbb{R}$$

are functions to be defined later.

The current article is organized in the following way. In Section 2, we give some preliminary results. In section 3, we show that problem (1.1) admits at least one weak nontrivial solution under suitable hypothesis on the data such as [12].

In the last section of this paper we study an extension of problem (1.1).

2. Some Preliminary results

We define the $(T_1\times T_2)$-dimensional Hilbert space,

$$W =: \{u:\mathbb{N}[0,T_1+1]\times\mathbb{N}[0,T_2+1]\to\mathbb{R} \text{ such that } u(k,h)=0,\ \forall(k,h)\in\Gamma\}$$

with the inner product

$$(u,v) = \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} u(k,h)v(k,h)$$

and the associated norm defined by

$$\|u\| = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^2 \right)^{\frac{1}{2}} .$$

However, we introduce another norm on the space W, namely

$$|u|_m = \left(\sum_{k=1}^{T_1} \sum_{h=1}^{T_2} |u(k,h)|^m \right)^{\frac{1}{m}} , \ \forall m \geq 2.$$

For the data g and a we impose the following conditions :

$$a(k,h,.) : \mathbb{R} \longrightarrow \mathbb{R} \text{ is continuous } \forall (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$$

and there exists a mapping $A : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \times \mathbb{R} \longrightarrow \mathbb{R}$ which satisfies

$$a(k,h,x) = \frac{\partial}{\partial x} A(k,h,x), \quad A(k,h,0) = 0, \quad \forall (k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]. \tag{2.1}$$

We also suppose that there exists a positive constant C_3 such that

$$|a(k,h,x)| \leq C_3 \left(1 + |x|^{p(k,h)-1}\right) \tag{2.2}$$

and

$$|x|^{p(k,h)} \leq a(k,h,x)x \leq p(k,h)A(k,h,x), \quad \forall x \in \mathbb{R}. \tag{2.3}$$

For each couple $(k,h) \in \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2]$, the function $g(k,h,.) : \mathbb{R} \longrightarrow \mathbb{R}$ is jointly continuous and there exist the functions $\sigma_1, \sigma_2 : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow (-\infty, 0)$ and $\phi_1, \phi_2 : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow (0,+\infty)$ and a function
$\gamma : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow [2,+\infty)$ such that

$$\sigma_1(k,h) + \phi_1(k,h)|x|^{\gamma(k,h)-1} \leq g(k,h,x) \leq \sigma_2(k,h) + \phi_2(k,h)|x|^{\gamma(k,h)-1}. \tag{2.4}$$

Where

$$-\infty < \underline{\sigma_1} = \inf_{k\in\mathbb{N}[1,T_1]} \left(\inf_{h\in\mathbb{N}[1,T_2]} \sigma_1(k,h) \right); \quad \overline{\sigma_1} = \sup_{k\in\mathbb{N}[1,T_1]} \left(\sup_{h\in\mathbb{N}[1,T_2]} \sigma_1(k,h) \right) < 0\,,$$

$$-\infty < \underline{\sigma_2} = \inf_{k\in\mathbb{N}[1,T_1]} \left(\inf_{h\in\mathbb{N}[1,T_2]} \sigma_2(k,h) \right); \quad \overline{\sigma_2} = \sup_{k\in\mathbb{N}[1,T_1]} \left(\sup_{h\in\mathbb{N}[1,T_2]} \sigma_2(k,h) \right) < 0,$$

$$0 < \underline{\phi_1} = \inf_{k\in\mathbb{N}[1,T_1]} \left(\inf_{h\in\mathbb{N}[1,T_2]} \phi_1(k,h) \right); \quad \overline{\phi_1} = \sup_{k\in\mathbb{N}[1,T_1]} \left(\sup_{h\in\mathbb{N}[1,T_2]} \phi_1(k,h) \right) < +\infty,$$

$$0 < \underline{\phi_2} = \inf_{k\in\mathbb{N}[1,T_1]} \left(\inf_{h\in\mathbb{N}[1,T_2]} \phi_2(k,h) \right); \quad \overline{\phi_2} = \sup_{k\in\mathbb{N}[1,T_1]} \left(\sup_{h\in\mathbb{N}[1,T_2]} \phi_2(k,h) \right) < +\infty$$

We denote

$$G(k,h,x)=\int_0^x g(k,h,s)\mathrm{d}s \quad \text{for} \quad (k,h,x)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\times\mathbb{R}. \tag{2.5}$$

Example 2.1. *We can give the following function:*

$$A(k,h,x)=\frac{1}{p(k,h)}\left(\left(1+|x|^2\right)^{\frac{p(k,h)}{2}}-1\right)$$

where

$$a(k,h,x)=\left(1+|x|^2\right)^{\frac{p(k,h)-2}{2}}x, \qquad (k,h,)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2],\ x\in\mathbb{R}$$

and

$$g(k,h,x)=-1+|x|^{\gamma(k,h)-1}.$$

In this paper, we assume that the function

$$p:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow(1,+\infty).$$

We will use the following notations:

$$p^-=\min_{k\in\mathbb{N}[1,T_1]}\left(\min_{h\in\mathbb{N}[1,T_2]}p(k,h)\right) \quad ; \quad p^+=\max_{k\in\mathbb{N}[1,T_1]}\left(\max_{h\in\mathbb{N}[1,T_2]}p(k,h)\right)$$

and

$$\gamma:\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\longrightarrow[2,+\infty)$$

with

$$\gamma^-=\min_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}}\gamma(k,h) \quad ; \quad \gamma^+=\max_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}}\gamma(k,h).$$

Lemma 2.1.

(1) *For any function $u\in W$ with $\|u\|>1$, there exist constants C_6, $C_7>0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\geq C_6\|u\|^{p^-}-C_7. \tag{2.6}$$

(2) *For any function $u\in W$ with $\|u\|\leq 1$, there exists constant $C_8>0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\geq C_8\|u\|^{p^+}. \tag{2.7}$$

(3) *For any function $u\in W$ there exist constants $c_1,\ c_2>0$ such that*

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1)|^{p(k-1,h-1)}\leq c_1\|u\|^{p^+}+c_2. \tag{2.8}$$

Proof. To have *1.* see [8].

For *2.* Let

$$v : \mathbb{N}[0, T_1 + 1] \longrightarrow \mathbb{R},\ k \mapsto v(k) = u(k, h)$$

and

$$q : \mathbb{N}[0, T_1] \longrightarrow (1, +\infty),\ k \mapsto q(k) = p(k, h),\ \text{with } h \text{ fixed in } \mathbb{N}[0, T_2 + 1].$$

According to [7] we have

$$\sum_{k=1}^{T_1+1} |\Delta v(k-1)|^{q(k-1)} \geq T_1^{-\frac{2-q^+}{2}} \|v\|^{q^+}.$$

Then there exists constant $C_8 > 0$ such that

$$\sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} |\Delta u(k-1, h-1)|^{p(k-1,h-1)} \geq C_8 \|u\|^{p^+}.$$

To see *3.* Taking $u \in W$, let

$$v : \mathbb{N}[0, T_1 + 1] \longrightarrow \mathbb{R},\ k \mapsto v(k) = u(k, h)$$

and

$$q : \mathbb{N}[0, T_1] \longrightarrow (1, +\infty),\ k \mapsto q(k) = p(k, h),\ \text{with } h \text{ fixed in } \mathbb{N}[0, T_2 + 1].$$

According to lemma 7 in [7] we have

$$\sum_{k=1}^{T_1+1} |\Delta v(k-1)|^{q(k-1)} \leq (T_1 + 1)\|v\|^{q^+} + T_1 + 1.$$

Then there exist constants c_1, $c_2 > 0$ such that

$$\sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} |\Delta u(k-1, h-1)|^{p(k-1,h-1)} \leq c_1 \|u\|^{p^+} + c_2.$$

□

Theorem 2.1. [9] *Let X be reflexive Banach space. If a functional $I \in C^1(X, \mathbb{R})$ is weakly lower semi-continous and coercive, i.e $\lim_{\|u\| \to +\infty} I(u) = +\infty$, then there exists u_0 such that*

$$I(u_0) = \inf_{u \in X} I(u)$$

and u_0 is also a critical point of I, i.e. $I'(u_0) = 0$. Moreover, if I is strictly convex, then a critical point is unique.

Theorem 2.2. [3] (*Ekeland's principle*) *Let* X *be a complete metric space and* $\Phi : X \longrightarrow \mathbb{R}$ *a lower semi-continuous function that is bounded below. Let* $\epsilon > 0$ *and* $\bar{u} \in X$ *be given such that*

$$\Phi(\bar{u}) \leq \inf_{u \in X} \Phi(u) + \frac{\epsilon}{2}.$$

Then given $\lambda > 0$ *there exists* $u_\lambda \in X$ *such that*

(1) $\Phi(u_\lambda) \leq \Phi(\bar{u})$,

(2) $d(u_\lambda, \bar{u}) < \lambda$,

(3) $\Phi(u_\lambda) < \Phi(u) + \frac{\epsilon}{\lambda} d(u, u_\lambda)$ *for all* $u \neq u_\lambda$.

Definition 2.1. *Let* X *be a real Banach space. We say that a functional* $I : X \longrightarrow \mathbb{R}$ *satisfies the Palais-Smale condition if every sequence* $\{u_n\}$ *such that* $\{I(u_n)\}$ *is bounded and* $I'(u_n) \longrightarrow 0$ *has a convergent subsequence.*

Lemma 2.2. [5] *Let* X *be a Banach space and* $I \in C^1(X, \mathbb{R})$ *satisfies the Palais-Smale condition. Assume that there exist* u_0, $u_1 \in X$ *and a Bounded open neighborhood* Ω *of* u_0 *such that* $u_1 \notin \bar{\Omega}$ *and*

$$\max\{I(u_0), I(u_1)\} < \inf_{u \in \partial\Omega} I(u).$$

Let

$$\Gamma_1 = \{h \in C([0,1], X) : h(0) = u_0, \quad h(1) = u_1\}$$

and

$$c = \inf_{h \in \Gamma_1} \max_{x \in [0,1]} I(h(x)).$$

Then c *is a critical value of* I*; that is, there exists* $u^* \in X$ *such that* $I'(u^*) = 0$ *and* $I(u^*) = c$*, where* $c > \max\{I(u_0), I(u_1)\}$.

3. Existence of solutions

In this section, first we use the minimization method, in the second we apply Mountain Pass geometry lemma, then in the third we apply the Ekelands variational principle to show that the problem (1.1) has at least one weak nontrivial solution.

We define the energy functional $I_\lambda : W \longrightarrow \mathbb{R}$ by

$$I_\lambda(u) = \sum_{k=1}^{T_1+1} \sum_{h=1}^{T_2+1} A(k-1, h-1, \Delta u(k-1, h-1)) - \lambda \sum_{k=1}^{T_1} \sum_{h=1}^{T_2} G(k, h, u(k, h)).$$

In ([8]), the functional I_λ is continuous Gateaux differentiable and its Gateaux derivate I'_λ at u reds

$$\begin{aligned}(I'_\lambda(u),v) &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)\\ &\quad -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} g(k,h,u(k,h))v(k,h),\end{aligned}$$

for all $v\in W$. If $u\in W$ is a critical point to I_λ, namely $(I'_\lambda(u),v)=0$ for all $v\in W$, we observe that

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} g(k,h,u(k,h))v(k,h)=0.$$

Since v is any in W, we see that the critical point u to I_λ satisfies the problem (1.1).

Lemma 3.1. *Suppose that (2.1) − (2.4), (2.5), (2.8) are satisfied and $\gamma^- > p^+$. Then for any $\lambda > 0$ the functional I_λ satisfies the Palais-Smale condition.*

Proof. We have W finitely dimensional Banach space, we only need to show that $I_\lambda(u_n)\longrightarrow -\infty$ when $\|u_n\|\longrightarrow +\infty$. From (2.1) − (2.4), (2.5), (2.8) we have

$$I_\lambda(u_n)=\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u_n(k-1,h-1))-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k,h,u_n(k,h))$$

then

$$\begin{aligned}I_\lambda(u_n) &\le \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u_n(k-1,h-1)|+\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_3|\Delta u_n(k-1,h-1)|^{p(k-1,h-1)}}{p(k-1,h-1)}\\ &\quad -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k,h,u_n(k,h))\\ &\le \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u_n(k-1,h-1)|+\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_3}{p^-}|\Delta u_n(k-1,h-1)|^{p(k-1,h-1)}\\ &\quad -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k,h,u_n(k,h)).\end{aligned}$$

We show that $\displaystyle\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u_n(k-1,h-1)|\le\alpha\|u_n\|.$

We have

$$\begin{aligned}
|\Delta u_n(k-1,h-1)| &= |u_n(k,h)-u_n(k-1,h-1)| \\
|u_n(k,h)-u_n(k-1,h-1)| &\le |u_n(k,h)|+|u_n(k-1,h-1)| \\
\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u_n(k-1,h-1)| &\le \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|u_n(k,h)|+\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|u_n(k-1,h-1)| \\
&\le \sqrt{(T_1+1)(T_2+1)}|u_n|_2+\sum_{k=0}^{T_1}\sum_{h=0}^{T_2}|u_n(k,h)| \\
&\le 2\sqrt{(T_1+1)(T_2+1)}|u_n|_2 \\
&\le 2\sqrt{(T_1+1)(T_2+1)}\|u_n\| \\
\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}C_3|\Delta u_n(k-1,h-1)| &\le \alpha\|u_n\|.
\end{aligned}$$

Then we show that

$$-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}G(k,h,u(k,h)) \le -\lambda\left[-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\|+\frac{\underline{\underline{\phi_1}}}{\gamma^+}\|u_n\|^{\gamma^-}\right].$$

From (2.4) and (2.5) we have

$$\begin{aligned}
-\lambda G(k,h,u_n) &\le -\lambda\int_0^{u_n}\left(\sigma_1(k,h)+\phi_1(k,h)|s|^{\gamma(k,h)-1}\mathrm{d}s\right) \\
&\le -\lambda\left[\sigma_1(k,h)s+\phi_1(k,h)\frac{|s|^{\gamma(k,h)}}{\gamma(k,h)}\right]_0^{u_n} \\
&\le -\lambda\left(\sigma_1(k,h)|u_n(k,h)|+\phi_1(k,h)\frac{|u_n(k,h)|^{\gamma(k,h)}}{\gamma(k,h)}\right).
\end{aligned}$$

Now,

$$\begin{aligned}
-\lambda\sigma_1(k,h) &\le -\lambda\underline{\sigma_1}=-\lambda(-|\underline{\sigma_1}|) \\
-\lambda\sigma_1(k,h)|u_n(k,h)| &\le -\lambda(-|\underline{\sigma_1}||u_n(k,h)|).
\end{aligned}$$

We have $\gamma^-\le\gamma(k,h)\le\gamma^+\Longrightarrow\frac{1}{\gamma^+}\le\frac{1}{\gamma(k,h)}\le\frac{1}{\gamma^-}$; we get

$$\begin{aligned}
\frac{\phi_1(k,h)}{\gamma(k,h)} &\ge \frac{\phi_1(k,h)}{\gamma^+}\ge\frac{\underline{\underline{\phi_1}}}{\gamma^+} \\
-\lambda\frac{\phi_1(k,h)}{\gamma(k,h)}|u_n(k,h)|^{\gamma(k,h)} &\le -\lambda\frac{\underline{\underline{\phi_1}}}{\gamma^+}|u_n(k,h)|^{\gamma^-}.
\end{aligned}$$

$$-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}G(k,h,u_n) \leq -\lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}-|\underline{\sigma_1}||u_n(k,h)|+\frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u_n(k,h)|^{\gamma^-}\right)$$

$$-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}G(k,h,u_n) \leq -\lambda\left[-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\|+\frac{\underline{\phi_1}}{\gamma^+}\|u_n\|^{\gamma^-}\right].$$

Finally we have

$$I_\lambda(u_n) \leq \alpha\|u_n\|+\frac{c_1C_3}{p^-}\|u_n\|^{p^+}+\frac{c_1C_3}{p^-}-\lambda\left[-|\underline{\sigma_1}|\sqrt{T_1\times T_2}\|u_n\|+\frac{\underline{\phi_1}}{\gamma^+}\|u_n\|^{\gamma^-}\right].$$

Since $\gamma^->p^+$ we have $\lim\limits_{\|u_n\|\to+\infty}I_\lambda(u_n)=-\infty$. □

Proposition 3.1. *Suppose that* (2.3), (2.6) *are satisfied and* $p^->\gamma^+$. *Then* I_λ *is coercive for all* $\lambda\in(0,+\infty)$.

Proof. From (2.3) and (2.6), We have

$$I_\lambda(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}A(k-1,h-1,\Delta u(k-1,h-1))-\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}G(k,h,u(k,h))$$

$$\geq \frac{1}{p^+}\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1))|^{p(k-1,h-1)}$$

$$-\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma(k,h)}+|\underline{\sigma_2}|\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|\right)$$

$$I_\lambda(u) \geq \frac{1}{p^+}\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1))|^{p(k-1,h-1)}$$

$$-\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^+}+\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^-}+|\underline{\sigma_2}|\sqrt{T_1\times T_2}|u|_2\right)$$

$$I_\lambda(u) \geq \frac{C_6}{p^+}\|u\|^{p^-}-\frac{C_7}{p^+}-\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^+}+\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^-}+|\underline{\sigma_2}|\sqrt{T_1\times T_2}\|u\|\right).$$

Since $p^->\gamma^+$ and $\|u\|>1$, $\lim\limits_{\|u\|\to+\infty}I_\lambda(u)=+\infty$. Thus the functional I_λ is coercive. □

Theorem 3.1. *Suppose that* $p^->\gamma^+$ *and* $\dfrac{\underline{\phi_1}}{|\underline{\sigma_1}|}>\gamma^+$. *Then there exists* $\lambda_0>0$ *such that for any* $\lambda>\lambda_0$ *the problem* (1.1) *has at least one weak nontrivial solution.*

Proof. $I_\lambda\in C^1(W,\mathbb{R})$ and weakly lower semi-continuous (see [8]).

Consider $u_\epsilon(k,h)\in W$ solution of problem (1.1). We show that u_ϵ is not trivial for $p^->\gamma^+$

and $\lambda > \lambda_0$. Let t_0 be a fixed real in $(1, +\infty)$ and
$(k_0, h_0) \in \mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1]$. We define $u_0 \in W$ in the following way

$$\begin{cases} u_0(k_0, h_0) = t_0 \\ \\ u_0(k, h) = 0, \quad (k, h) \in (\mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1]) \setminus \{(k_0, h_0)\}. \end{cases}$$

From (2.3), (2.4) and (2.5) we have

$$\begin{aligned}
I_\lambda(u_0) &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u_0(k-1, h-1)| + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u_0(k-1, h-1)|^{p(k-1,h-1)} \\
&\quad -\lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u_0(k, h)) \\
&\leq C_3|\Delta u_0(k_0 - 1, h_0 - 1)| + \sum_{k\neq k_0}^{T_1}\sum_{h\neq h_0}^{T_2} C_3|\Delta u_0(k-1, h-1)| \\
&\quad + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u_0(k-1, h-1)|^{p(k-1,h-1)} - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u_0(k, h)) \\
&\leq 2C_3 t_0 + \frac{C_3|\Delta u_0(k_0 - 1, h_0 - 1)|^{p(k_0-1,h_0-1)}}{p^-} \\
&\quad + \sum_{k\neq k_0}^{T_1}\sum_{h\neq h_0}^{T_2} \frac{C_3}{p^-}|\Delta u_0(k-1, h-1)|^{p(k-1,h-1)} - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u_0(k, h)) \\
&\leq C_3\left(2t_0 + \frac{t_0^{p(k_0,h_0)} + t_0^{p(k_0-1,h_0-1)}}{p^-}\right) - \lambda\left[-|\underline{\sigma_1}|t_0 + \frac{\phi_1}{\gamma^+}t_0^{\gamma^-}\right] \\
&\leq 4C_3 t_0^{p^+} - \lambda\left[-|\underline{\sigma_1}|t_0 + \frac{\phi_1}{\gamma^+}t_0^{\gamma^-}\right] \\
I_\lambda(u_0) &\leq 4C_3 t_0^{p^+} - \lambda t_0\left[-|\underline{\sigma_1}| + \frac{\phi_1}{\gamma^+}\right]
\end{aligned}$$

Since $\lambda > \lambda_0$ with $\lambda_0 = \dfrac{4C_3 t_0^{p^+-1}}{-|\underline{\sigma_1}| + \dfrac{\phi_1}{\gamma^+}}$, we get $I_\lambda(u_0) < 0$. Thus $I_\lambda(u_\epsilon) < 0$ for any $\lambda > \lambda_0$.
By [10] we have u_ϵ is a weak nontrivial solution of problem (1.1). □

Note that when $\gamma^- > p^+$, the functional I_λ satisfies the Palais-Smale (PS) condition.
Now, we use the Mountain Pass geometry lemma to prove that the problem (1.1) admits at least one weak nontrivial solution.

Theorem 3.2. *Suppose that (2.3), (2.4), (2.5), (2.7) are satisfied and $\gamma^- > p^+$. Then there exists $\lambda_1 \in (0, +\infty)$ such that for $\lambda \in (0, \lambda_1)$ the problem (1.1) has at least one weak nontrivial solution.*

Proof. Let

$$\Omega := \{u \in W : \|u\| \leq \alpha\}$$

with $\alpha \in (0, 1)$. For $u \in \Omega$, from (2.3), (2.4), (2.5) and (2.7), we have

$$I_\lambda(u) = \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1, h-1, \Delta u(k-1, h-1)) - \lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u(k, h))$$

$$\begin{aligned} I_\lambda(u) &\geq \frac{1}{p^+}\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} |\Delta u(k-1, h-1))|^{p(k-1,h-1)} \\ &\quad -\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k, h)|^{\gamma(k,h)} + |\underline{\sigma_2}|\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k, h)|\right) \end{aligned}$$

$$I_\lambda(u) \geq \frac{C_8}{p^+}\|u\|^{p^+} - \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\|u\|^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1 \times T_2}\|u\|\right).$$

Taking $u \in \partial\Omega$, we get

$$\begin{aligned} I_\lambda(u) &\geq \frac{C_8}{p^+}\alpha^{p^+} - \lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\alpha^{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1 \times T_2}\alpha\right) \\ I_\lambda(u) &\geq \frac{C_8}{p^+}\alpha^{p^+} - \lambda\alpha\left(\frac{\overline{\phi_2}}{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1 \times T_2}\right). \end{aligned}$$

So for every $\lambda < \lambda_1$, with $\lambda_1 = \dfrac{\frac{C_8\alpha^{p^+-1}}{p^+}}{\frac{\overline{\phi_2}}{\gamma^-} + |\underline{\sigma_2}|\sqrt{T_1 \times T_2}}$; $I_\lambda(u) > 0$ for all $u \in \partial\Omega$.

For $u \in W$ such that $u(k, h) > 1$, and $(k, h) \in \mathbb{N}[1, T_1 + 1] \times \mathbb{N}[1, T_2 + 1]$; from (2.1) $-$ (2.4), we have

$$\begin{aligned} I_\lambda(u) &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u(k-1, h-1)| + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u(k-1, h-1)|^{p(k-1,h-1)} \\ &\quad -\lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k, h)| + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k, h)|^{\gamma^-}\right). \end{aligned}$$

Consider $u_t \in W$ defined in the following way

$$\begin{cases} u_t(k, h) = t \quad for \quad (k, h) \in (\mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1]) \setminus \{\Gamma\} \\ \\ u_t(k, h) = 0 \quad in \quad \Gamma. \end{cases}$$

Then there exist integers N_1, N_2, N_3 and N such that

$$\begin{aligned} I_\lambda(u) &\leq C_3(N_1+N_2)t + \frac{C_3}{p^-}(N_1 t^{p^+} + N_2 t^{p^+}) - \lambda N_3 \left[-|\underline{\sigma_1}|t + \frac{\phi_1}{\gamma^+} t^{\gamma^-} \right] \\ I_\lambda(u) &\leq [2C_3(N_1+N_2)]\, t^{p^+} - \lambda N_3 \left[-|\underline{\sigma_1}|t + \frac{\phi_1}{\gamma^+} t^{\gamma^-} \right] \\ I_\lambda(u) &\leq 4C_3 N t^{p^+} - \lambda N_3 \left[-|\underline{\sigma_1}|t + \frac{\phi_1}{\gamma^+} t^{\gamma^-} \right] \end{aligned}$$

with $N = \max\{N_1, N_2\}$.
Since $\gamma^- > p^+$, then $\lim_{t\to+\infty} I_\lambda(u) = -\infty$. Thus there exists t_0 such that for $u_{t_0} \in W \setminus \Omega,\ I_\lambda(u_{t_0}) < \min_{u\in\partial\Omega} I_\lambda(u)$. According to lemma (2.2) the problem (1.1) has at least one weak nontrivial solution. □

Theorem 3.3. *Suppose that* $p^- > \gamma^-$ *and* $\dfrac{\phi_1}{|\underline{\sigma_1}|} > \gamma^+ a_0$. *For any* $\lambda \in (0, \lambda_1)$ *the problem* (1.1) *has at least one weak nontrivial solution.*

Proof. In the proof of theorem 3.2, for $u \in \partial\Omega$ and $\lambda < \lambda_1$, we have $I_\lambda(u) > 0$. From Weierstrass theorem we deduce that

$$\min_{u\in\partial\Omega} I_\lambda(u) > 0.$$

Taking $u(k,h) \in (0,\alpha)$. From (2.2) and (2.4) – (2.5) we have

$$\begin{aligned} I_\lambda(u) \leq & \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u(k-1,h-1)| + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} \\ & -\lambda \left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k,h)| + \frac{\phi_1}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{\gamma^-} \right). \end{aligned}$$

For $s \in (0,\alpha)$, we suppose that $s < \sqrt[(p^- - \gamma^-)]{\dfrac{\lambda\left(-|\underline{\sigma_1}|a_0 + \dfrac{\phi_1}{\gamma^+}\right)}{2C_3\left(a_2 + \dfrac{1}{p^-}\right)}}$.
We choose $(k_0,h_0) \in (\mathbb{N}[0,T_1+1] \times \mathbb{N}[0,T_2+1]) \setminus \{\Gamma\}$ such that $\gamma(k_0,h_0) = \gamma^-$.
Let $u_0 \in W$ be a function such that $u_0(k_0,h_0) = s$ and $u_0(k,h) = 0$ for any

$(k,h) \in (\mathbb{N}[1,T_1+1] \times \mathbb{N}[1,T_2+1]) \setminus \{(k_0,h_0)\}$. We obtain

$$\begin{aligned}
I_\lambda(u_0) &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u_0(k-1,h-1)| + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u_0(k-1,h-1)|^{p(k-1,h-1)} \\
&\quad -\lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u_0(k,h)| + \frac{\underline{\underline{\phi_1}}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u_0(k,h)|^{\gamma^-}\right) \\
I_\lambda(u_0) &\leq C_3\left(2s + \frac{s^{p(k_0,h_0)} + s^{p(k_0-1,h_0-1)}}{p^-}\right) - \lambda\left[-|\underline{\sigma_1}|s + \frac{\underline{\underline{\phi_1}}}{\gamma^+}s^{\gamma^-}\right] \\
&\leq C_3\left(2s + \frac{2s^{p^-}}{p^-}\right) - \lambda\left[-|\underline{\sigma_1}|s + \frac{\underline{\underline{\phi_1}}}{\gamma^+}s^{\gamma^-}\right] \\
I_\lambda(u_0) &\leq 2C_3\left(s + \frac{s^{p^-}}{p^-}\right) - \lambda\left[-|\underline{\sigma_1}|s + \frac{\underline{\underline{\phi_1}}}{\gamma^+}s^{\gamma^-}\right].
\end{aligned}$$

There exist a_2, $a_0 > 1$ such that $a_2 s^{p^-} \geq s$ and $a_0 s^{\gamma^-} \geq s$. Then, we deduce that

$$I_\lambda(u_0) \leq 2C_3 s^{p^-}\left(a_2 + \frac{1}{p^-}\right) - \lambda\left[-|\underline{\sigma_1}|a_0 + \frac{\underline{\underline{\phi_1}}}{\gamma^+}\right]s^{\gamma^-}$$

Thus, $I_\lambda(u_0) < 0$ for $u_0 \in int(\Omega)$.

Therefore,

$$-\infty < \inf_{u\in int(\Omega)} I_\lambda(u) < 0.$$

So, we have

$$\inf_{u\in int(\Omega)} I_\lambda(u) < \inf_{u\in\partial\Omega} I_\lambda(u).$$

Using [11] we have

$$\inf_{u\in\partial\Omega} I_\lambda(u) - \inf_{u\in int(\Omega)} I_\lambda(u) > \epsilon > 0.$$

Applying Ekeland's variational principle to the functional $I_\lambda : \Omega \longrightarrow \mathbb{R}$,

we find $u_\epsilon \in \Omega$ such that

$$I_\lambda(u_\epsilon) < \inf_{u\in\Omega} I_\lambda(u) + \epsilon$$

and

$$I_\lambda(u_\epsilon) < I_\lambda(u) + \epsilon\|u - u_\epsilon\| \qquad \text{for} \qquad u \neq u_\epsilon.$$

Since

$$I_\lambda(u_\epsilon) < \inf_{u\in\Omega} I_\lambda(u) + \epsilon \leq \inf_{u\in int(\Omega)} I_\lambda(u) + \epsilon < \inf_{u\in\partial\Omega} I_\lambda(u),$$

we deduce that $u_\epsilon \in int(\Omega)$. Now, we define $H_\lambda : \Omega \longrightarrow \mathbb{R}$ by

$$H_\lambda(u) = I_\lambda(u) + \epsilon\|u - u_\epsilon\| \qquad \text{for} \qquad u \neq u_\epsilon.$$

We have u_ϵ as a minimum of the functional H_λ and therefore $\forall u \in \Omega$

$$H_\lambda(u) \geq H_\lambda(u_\epsilon).$$

Taking $u = u_\epsilon + sv$, $v \in \Omega$ and $s > 0$ we have

$$\frac{H_\lambda(u_\epsilon + sv) - H_\lambda(u_\epsilon)}{s} \geq 0$$

we deduce that

$$\frac{I_\lambda(u_\epsilon + sv) - I_\lambda(u_\epsilon)}{s} + \epsilon\|v\| \geq 0.$$

Let $s \longrightarrow 0$, it follows that

$$(I'_\lambda(u_\epsilon), v) + \epsilon\|v\| \geq 0.$$

We obtain

$$\|I'_\lambda(u_\epsilon)\| \leq \epsilon.$$

Taking $\epsilon = \dfrac{1}{n^2}$ there exists a sequence $\{x_n\} \subset int(\Omega)$, (see [13]) such that

$$I_\lambda(x_n) \longrightarrow \inf_{u\in\Omega} J_\lambda(u) \quad \text{and} \quad I'_\lambda(x_n) \longrightarrow 0.$$

Since $\{x_n\}$ is bounded in W there exists $x_0 \in W$ such that, up to a subsequence $\{x_n\}$ converge to $x_0 \in W$. Thus

$$I_\lambda(x_0) = \inf_{u\in\Omega} I_\lambda(u) \quad \text{and} \quad I'_\lambda(x_0) = 0.$$

x_0 is one weak nontrivial solution for the problem (1.1). □

4. An extension

This section is devoted to study an extension of problem (1.1). For that, we consider the general boundary value problems,

$$(4.1) \qquad \begin{cases} -\Delta(a(k-1, h-1, \Delta u(k-1, h-1))) + \varphi_r(u) = \lambda g(k, h, u(k, h)), \\ \\ (k, h) \in \mathbb{N}[1, T_1] \times \mathbb{N}[1, T_2], \\ \\ u(k, h) = 0, \ (k, h) \in \Gamma \end{cases}$$

with $\varphi_r(u) = |u(k,h)|^{r(k,h)-2}u(k,h)$ and $r : \mathbb{N}[1,T_1] \times \mathbb{N}[1,T_2] \longrightarrow (2,+\infty)$.

We denote by

$$r^- = \min_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}} r(k,h) \quad \text{and} \quad r^+ = \max_{\{(k,h)\in\mathbb{N}[1,T_1]\times\mathbb{N}[1,T_2]\}} r(k,h).$$

A function $u \in W$ is a solution of problem (4.1) if for any $v \in W$,

$$\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1)$$
$$+\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{r(k,h)-2}u(k,h)v(k,h) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} g(k,h,u(k,h))v(k,h) = 0.$$

Theorem 4.1. *We suppose that (2.3), (2.4), (2.5) are satisfied and* $p^- > \gamma^+$, $\dfrac{\phi_1}{|\sigma_1|} > \gamma^+$. *Then there exists* $\lambda_2 > 0$ *such that for* $\lambda \in (\lambda_2, +\infty)$ *the problem (4.1) has at least one weak nontrivial solution.*

Proof. We associate to the problem (4.1) the energy functional I_λ defined by

$$\begin{aligned} I_\lambda(u) &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1)) + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r(k,h)}|u(k,h)|^{r(k,h)} \\ &\quad -\lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k,h,u(k,h)). \end{aligned}$$

From [8] the functional I_λ is of class $C^1(W,\mathbb{R})$ and weakly lower semi-continuous. It's derivate given by

$$\begin{aligned} (I'_\lambda(u),v) &= \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} a(k-1,h-1,\Delta u(k-1,h-1))\Delta v(k-1,h-1) \\ &\quad +\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} |u(k,h)|^{r(k,h)-2}u(k,h)v(k,h) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} g(k,h,u(k,h))v(k,h). \end{aligned}$$

Since

$$\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r(k,h)}|u(k,h)|^{r(k,h)} \geq 0$$

we obtain

$$I_\lambda(u) \geq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} A(k-1,h-1,\Delta u(k-1,h-1)) - \lambda\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k,h,u(k,h)).$$

We deduce by proposition (3.1) that the functional I_λ is coercive. Let u_λ be a global minimizer of I_λ. For t_0 be a fixed real in $(1,+\infty)$ and

$(k_0, h_0) \in \mathbb{N}[0, T_1+1] \times \mathbb{N}[0, T_2+1]$. We define $u_0 \in W$ such that

$$\begin{cases} u_0(k_0, h_0) = t_0 \\ \\ u_0(k, h) = 0, \quad (k, h) \in (\mathbb{N}[0, T_1+1] \times \mathbb{N}[0, T_2+1]) \setminus \{(k_0, h_0)\}. \end{cases}$$

From (2.3), (2.4) and (2.5), we obtain

$$I_\lambda(u_0) \leq 4C_3 t_0^{p^+} + \frac{t_0^{r^+}}{r^-} - \lambda t_0 \left[-|\underline{\sigma_1}| + \frac{\phi_1}{\gamma^+} \right]$$

where

$$\lambda_2 = \frac{4C_3 t_0^{p^+-1} + \dfrac{t_0^{r^+}}{r^-}}{-|\underline{\sigma_1}| + \dfrac{\phi_1}{\gamma^+}}.$$

Therefore $I_\lambda(u_\lambda) < 0$ for any $\lambda \in (\lambda_2, +\infty)$. Then u_λ is a weak nontrivial solution of problem (4.1). □

Lemma 4.1. *Suppose that* (2.1) − (2.4), (2.5) *are satisfied and* $\gamma^- > \max\{p^+, r^+\}$. *Then for any* $\lambda > 0$ *the functional* I_λ *satisfies the Palais-Smale condition.*

Proof. Using the proof of lemma (3.1) we obtain

$$\begin{aligned} I_\lambda(u_n) &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u_n(k-1, h-1)|^{p(k-1,h-1)} + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} \frac{1}{r(k,h)}|u_n(k,h)|^{r(k,h)} \\ &\quad -\lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u_n(k,h)) \\ &\leq \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} \frac{C_3}{p^-}|\Delta u_n(k-1, h-1)|^{p(k-1,h-1)} + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} \frac{1}{r^-}|u_n(k,h)|^{r^+} \\ &\quad -\lambda \sum_{k=1}^{T_1}\sum_{h=1}^{T_2} G(k, h, u_n(k,h)) \\ I_\lambda(u_n) &\leq \alpha\|u_n\| + \frac{c_1 C_3}{p^-}\|u_n\|^{p^+} + \frac{c_1 C_3}{p^-} + \frac{1}{r^-}\|u_n\|^{r^+} - \lambda \left[-|\underline{\sigma_1}|\sqrt{T_1 \times T_2}\|u_n\| + \frac{\phi_1}{\gamma^+}\|u_n\|^{\gamma^-} \right]. \end{aligned}$$

Since $\gamma^- > \max\{p^+, r^+\}$, we have $\lim\limits_{\|u_n\| \to +\infty} I_\lambda(u_n) = -\infty$. □

Theorem 4.2. *Suppose that* (2.3), (2.4), (2.5), (2.7) *are satisfied and* $\gamma^- > \max\{p^+, r^+\}$. *Then for* $\lambda \in (0, \lambda_1)$, *the problem* (4.1) *has at least one weak nontrivial solution.*

Proof. From the proof of theorem (3.2), consider

$$\Omega := \{u \in W : \|u\| \leq \alpha\}$$

with $\alpha \in (0,1)$.

For $u \in \Omega$

$$\begin{aligned} I_\lambda(u) \geq & \frac{1}{p^+}\sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}|\Delta u(k-1,h-1))|^{p(k-1,h-1)} \\ & -\lambda\left(\frac{\overline{\phi_2}}{\gamma^-}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma(k,h)} + |\underline{\sigma_2}|\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|\right). \end{aligned}$$

For all $\lambda < \lambda_1$ we obtain $\inf_{u\in\partial\Omega} I_\lambda(u) > 0$. Let $u \in W$ such that $u(k,h) > 1$, for $(k,h) \in \mathbb{N}[1, T_1+1] \times \mathbb{N}[1, T_2+1]$. From (2.3), (2.4), (2.5), (2.7) we have

$$\begin{aligned} I_\lambda(u) \leq & \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1} C_3|\Delta u(k-1,h-1)| + \sum_{k=1}^{T_1+1}\sum_{h=1}^{T_2+1}\frac{C_3}{p^-}|\Delta u(k-1,h-1)|^{p(k-1,h-1)} \\ & + \sum_{k=1}^{T_1}\sum_{h=1}^{T_2}\frac{1}{r^-}|u(k,h)|^{r^+} - \lambda\left(\sum_{k=1}^{T_1}\sum_{h=1}^{T_2} -|\underline{\sigma_1}||u(k,h)| + \frac{\underline{\phi_1}}{\gamma^+}\sum_{k=1}^{T_1}\sum_{h=1}^{T_2}|u(k,h)|^{\gamma^-}\right). \end{aligned}$$

Let $u_t \in W$ defined in the following way

$$\begin{cases} u_t(k,h) = t \quad for \quad (k,h) \in (\mathbb{N}[0,T_1+1] \times \mathbb{N}[0,T_2+1]) \setminus \{\Gamma\} \\ \\ u_t(k;h) = 0 \quad in \quad \Gamma. \end{cases}$$

We obtain

$$I_\lambda(u) \leq 4C_3N\left(t^{p^+} + t^{r^+}\right) - \lambda N_3\left[-|\underline{\sigma_1}|t + \frac{\underline{\phi_1}}{\gamma^+}t^{\gamma^-}\right].$$

Since $\gamma^- > \max\{p^+, r^+\}$, then $\lim_{t\to+\infty} I_\lambda(u) = -\infty$. Thus there exists t' such that for $u_{t'} \in W \setminus \Omega$, $J_\lambda(u_{t'}) < \min_{u\in\partial\Omega} I_\lambda(u)$. According to lemma (2.2); the problem (4.1) has at least one weak nontrivial solution. □

Now, we apply Ekeland's variational principle. We use the result when $p^- > \gamma^-$ and $\frac{\underline{\phi_1}}{|\underline{\sigma_1}|} > \gamma^+ a_0$. For $\lambda < \lambda_1$, one has $\min_{u\in\partial\Omega} I_\lambda(u) > 0$. Taking $s \in (0,\alpha)$ and suppose that

$$s < \sqrt[\left(\min\{p^-,r^-\}-\gamma^-\right)]{\frac{\lambda\left(-|\underline{\sigma_1}|a_0 + \frac{\underline{\phi_1}}{\gamma^+}\right)}{2C_3\left(a_2 + \frac{1}{p^-} + \frac{1}{2C_3 r^-}\right)}}.$$

For $(k_0, h_0) \in (\mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1]) \setminus \{\Gamma\}$ such that $\gamma(k_0, h_0) = \gamma^-$.

Let $u_0 \in W$ be a function such that $u_0(k_0, h_0) = s$ and $u_0(k, h) = 0$ for any $(k, h) \in (\mathbb{N}[0, T_1 + 1] \times \mathbb{N}[0, T_2 + 1]) \setminus \{(k_0, h_0)\}$.

We obtain

$$I_\lambda(u_0) \leq 2C_3 \left(s + \frac{s^{p^-}}{p^-} \right) + \frac{s^{q^-}}{q^-} - \lambda \left[-|\underline{\sigma_1}| s + \frac{\underline{\phi_1}}{\gamma^+} s^{\gamma^-} \right].$$

There exist a_2, $a_0 > 1$ such that $a_2 s^{\min\{p^-, r^-\}} \geq s$ and $a_0 s^{\gamma^-} \geq s$. We have

$$\begin{aligned} I_\lambda(u_0) &\leq 2C_3 s^{p^-} \left(a_2 + \frac{1}{p^-} \right) + \frac{s^{r^-}}{r^-} - \lambda \left[-|\underline{\sigma_1}| a_0 + \frac{\underline{\phi_1}}{\gamma^+} \right] s^{\gamma^-} \\ I_\lambda(u_0) &\leq 2C_3 s^{\min\{p^-, r^-\}} \left(a_2 + \frac{1}{p^-} + \frac{1}{2C_3 r^-} \right) - \lambda \left[-|\underline{\sigma_1}| a_0 + \frac{\underline{\phi_1}}{\gamma^+} \right] s^{\gamma^-} < 0. \end{aligned}$$

Since $\min\{p^-, r^-\} > \gamma^-$, $I_\lambda(u_0) < 0$ for $u_0 \in int(\Omega)$.

By the same reasoning as in the proof of theorem (3.3) we show that the problem (4.1) has at least one weak nontrivial solution.

Acknowledgement

The authors want to thank the anonymous referees for their valuables comments on the paper.

References

[1] R.P. Agarwal, Difference equations and inequalities: Theory, methods, and applications, CRC Press, 2000. https://doi.org/10.1201/9781420027020.

[2] X. Cai, J. Yu, Existence theorems for second-order discrete boundary value problems, J. Math. Anal. Appl. 320 (2006), 649–661. https://doi.org/10.1016/j.jmaa.2005.07.029.

[3] D.G. de Figueiredo, Lectures on the Ekeland variational principle with applications and detours, Springer, Heidelberg, 1989.

[4] S. Du, Z. Zhou, Multiple solutions for partial discrete Dirichlet problems involving the p-Laplacian, Mathematics. 8 (2020), 2030. https://doi.org/10.3390/math8112030.

[5] G. Dajun, Nonlinear functional analysis, Shandong Science and Technology Press, Jinan, 1985 (in Chinese).

[6] M. Galewski, S. Glab, R. Wieteska, Positive solutions for anisotropic discrete boundary-value problems, Electron. J. Differ. Equ. 2013 (2013), 32.

[7] M. Galewski, R. Wieteska, Existence and multiplicity of positive solutions for discrete anisotropic equations, Turk. J. Math. 38 (2014), 297-310. https://doi.org/10.3906/mat-1303-6.

[8] I. Ibrango, B. Koné, A. Guiro, S. Ouaro, Weak solutions for anisotropic nonlinear discrete dirichlet boundary value problems in a two-dimensional Hilbert space, Nonlinear Dyn. Syst. Theory. 21 (2021), 90-99.

[9] J. Mawhin, Problèmes de dirichlet variationnels non linéaires, Les Presses de l'Université de Montréal, 1987.

[10] C.Y. Lei, J.F. Liao, C.L. Tang, Multiple positive solutions for Kirchhoff type of problems with singularity and critical exponents, J. Math. Anal. Appl. 421 (2015), 521–538. https://doi.org/10.1016/j.jmaa.2014.07.031.

[11] M. Mihăilescu, V. Rădulescu, S. Tersian, Eigenvalue problems for anisotropic discrete boundary value problems, J. Differ. Equ. Appl. 15 (2009), 557–567. https://doi.org/10.1080/10236190802214977.

[12] R. Sanou, B. Kone, Weak nontrivial solutions of dirichlet discrete nonlinear problems, Asia Pac. J. Math, 7 (2020), 33. https://doi.org/10.28924/APJM/7-33.

[13] A. Szulkin, Minimax principles for lower semicontinuous functions and applications to nonlinear boundary value problems, Ann. Inst. H. Poincaré Anal. Non Linéaire. 3 (1986), 77–109. https://doi.org/10.1016/s0294-1449(16)30389-4.

[14] Y. Tian, W. Ge, Existence of multiple positive solutions for discrete problems with p-laplacian via variational methods, Electron. J. Differ. Equ. 2011 (2011), 45.

[15] G. Zhang, S. Liu, On a class of semipositone discrete boundary value problems, J. Math. Anal. Appl. 325 (2007), 175–182. https://doi.org/10.1016/j.jmaa.2005.12.047.

[16] V.V. Zhikov, Averaging of functionals of the calculus of variations and elasticity theory, Math. USSR Izv. 29 (1987), 33–66. https://doi.org/10.1070/IM1987v029n01ABEH000958.