Asia Pac. J. Math. 2022 **9**:14

AN ECO-EPIDEMIOLOGICAL MODEL ON THE INTERACTION OF WILDEBEEST, ZEBRA AND LION PREY-PREDATOR SYSTEMS IN SERENGETI ECOSYSTEM

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Received Mar. 16, 2022

Abstract. In this paper we have developed an eco-epidemic model of wildebeest, zebra and lion prey-predator system of Serengeti ecosystem where both prey species and predator are infected by infectious disease. The interaction between lion(a predator) and prey (wildebeest and zebra) is assumed to be governed by Holling type II functional response and disease transmission is assumed to follow a simple kinetic mass action function. The boundness, positivity and existence of the solutions of the model has been checked and established. Steady states of the model are identified and the local stability analysis of the axial equilibrium, trivial equilibrium and disease free equilibrium point are established with the method of Routh-Hurwitz criterion and the Jacobian matrix. The axial equilibrium point is stable if and only if $\beta_1 k - t_1 - d_2 < 0$, $s < 0$, $-t_2 - d_3 < 0$, $\frac{qp_1 k}{1+ak} - d_5 < 0$, $\frac{qp_2 k}{1+ak} - d_4 - t_3 < 0$ and $-(d_1 + r_1 + r_2 + r_3) < 0$. Treatment is a helpful tool to minimize or eradicate infections in prey-predator system. Therefore, providing treatment in an infected prey-predator system creates an opportunity to recover from illness and the prey-predator population can be saved and exist in a stable situation.

2010 Mathematics Subject Classification. 92D30.

Key words and phrases. prey; predator; Serengeti ecosystem; infections; treatment.

1. Introduction

The occurrence of disease induced mortality has been not recorded in Serengeti ecosystem since the removal of rinderpest [5]. It is therefore assumed that the population dynamics in Serengeti has not been regulated or influenced by disease. However this may not be a correct

DOI: 10.28924/APJM/9-14

supposition as [5] states that zebra are sensitive to regular epizootics as the consequences of their social structure. He found that the female remained in close contact with each other within groups, this maximizes the probability of epizootic transmissions. Sometimes infections in the ecosystem are not straight forward as the disease related mortality are generally dependent on an animal's nutritional status [6]. For instance, for ecto-endoparasites to reach fatal number, an animal must have inadequate disease resistance which tends to occur when it is undernaurished. Studying the influence of disease in the population dynamics of species in Serengeti ecosystem is potentially significant because they play a great role in stabilizing or distabilizing the ecosystem. For instance, it took more than ten years for Serengeti ecosystem to stabilize following the eruption of rinderpest disease in wildebeest as well as canine dispenser virus of 1994 in lion [6]. So far in many ecological studies of prey-predator systems with disease, it has been reported that predator take high parasites of infected prey [3]. Hence there has been a growing interest in the study of disease in prey-predator systems and mathematical modelling in both ecology and epidemiology are the important tool in understanding and analysing the dynamics of prey-predator systems. Anderson and May were the first to combine these two modelling systems [9], [10], [8], while Chattopadhyay and Arino were the first who used the term eco-epidemiology [7]. Several mathematical models have been proposed and studied on prey-predator systems [1]. Many studies focused on the study of disease in prey only [4]. Other researchers were interested in the study of disease within predator only [10], [2] and there are also some studies on disease on both prey and predator. In this paper we proposed and studied disease in both prey species and predator with treatment given to infected prey and infected predator. Particularly focus on the disease outbreak among wildebeest, zebra and lion prey-predator systems in Serengeti ecosystem.

2. Model Formulation

In this paper the model population is divided into seven compartments. Let us denote $x_1(t)$ as the population of susceptible wildebeest, $x_2(t)$ as the population of infected wildebeest, $y_1(t)$ for susceptible zebra, $y_2(t)$ for the population of infected zebra, $z_1(t)$ for susceptible lion and $z_2(t)$ for infected lion. This dynamics is assumed to follow Michaelis-Menten kinetics Holling type II function response. In formulating the model, the following assumptions are taken into considerations.

i In the absence of infectious disease the susceptible wildebeest and zebra grows logistically with intrinsic growth rate r and s respectively and environmental carrying capacities k and l respectively.

ii In the presence of infectious disease, susceptible lion become infected lion during predation not genetically inherited.

iii Only the susceptible prey can reproduce. Logistic law is used to model the birth process with assumption that births should always be positive. The infected prey is removed with the positive death rate or by predation before the possibility of reproducing. However the infected wildebeest x_2 and infected zebra y_2 contribute with susceptible ones, x_1 and y_1 to growth towards the carrying capacity k and l respectively.

iv Susceptible prey become infected when they comes in contact with the infected when they comes in contact with the infected prey and this contact process is assumed to follow the simple mass action kinetics with β_1, β_2 and α as the rate of conversion.

v The infected lion can recover by treatment at the rates t_3 and posesses a death rate of $(d_4 + e)$ where d_4 and e are the deaths due to infections an nature respectively.

vi The predation functional response of lion towards both susceptible and infected prey are assume to follow Michaelis-Menten kinetics and is modelled using Hollings type II functional form with predation coefficient $p_i (p_i > 0)$ and half saturation constants a and d $(a > 0, d > 0)$. According to the above assumptions we have the following set of differential equations;

$$
(1) \qquad \begin{cases}
\frac{dx_1}{dt} = rx_1(1 - \frac{x_1+x_2}{k}) - \beta_1 x_1 x_2 + r_1 T - \frac{p_1 x_1 z_1}{a+x_1} - \frac{p_2 x_1 z_2}{a+x_1} \\
\frac{dx_2}{dt} = \beta_1 x_1 x_2 - t_1 x_2 - d_2 x_2 - \frac{p_3 x_2 z_1}{a+x_2} - \frac{p_4 x_2 z_2}{a+x_2} \\
\frac{dy_1}{dt} = sy_1(1 - \frac{y_1+y_2}{l}) - \beta_2 y_1 y_2 + r_2 T - \frac{p_5 y_1 z_1}{d+y_1} - \frac{p_6 y_1 z_2}{d+y_1} \\
\frac{dy_2}{dt} = \beta_2 y_1 y_2 - t_2 y_2 - d_3 y_2 - \frac{p_7 y_2 z_1}{d+y_2} - \frac{p_8 y_2 z_2}{d+y_2} \\
\frac{dz_1}{dt} = \frac{q p_1 x_1 z_1}{a+x_1} + \frac{q p_5 y_1 z_1}{d+y_1} + r_3 T - \alpha z_1 z_2 - d_5 z_1 \\
\frac{dz_2}{dt} = \alpha z_1 z_2 + \frac{q p_2 x_1 z_2}{a+x_1} + \frac{q p_4 x_2 z_2}{a+x_2} + \frac{q p_6 y_1 z_2}{d+y_1} + \frac{q p_8 y_2 z_2}{d+y_2} - t_3 z_2 - d_4 z_2 \\
\frac{dT}{dt} = t_1 x_2 + t_2 y_2 + t_3 z_2 - d_1 T - r_1 T - r_2 T - r_3 T.
\end{cases}
$$

With initial conditions $x_1(0) \geq 0$; $x_2(0) \geq 0$; $y_1(0) \geq 0$; $y_2(0) \geq 0$; $z_1(0) \geq 0$; $z_2(0) \geq 0$; $T(0) \geq 0$. Expressing the model in a more compact form we get;

$$(2)\quad \begin{cases} \frac{dx_1}{dt} = x_1 \left\{ r\left(1 - \frac{x_1+x_2}{k}\right) - \left(\beta_1 x_2 + \frac{p_1 z_1 + p_2 z_2}{a+x_1}\right) \right\} + r_1 T \\ \frac{dx_2}{dt} = x_2 \left\{ \beta_1 x_1 - \left(t_1 + d_2 + \frac{p_3 z_1 + p_4 z_2}{a+x_2}\right) \right\} \\ \frac{dy_1}{dt} = y_1 \left\{ s\left(1 - \frac{y_1+y_2}{l}\right) - \left(\beta_2 y_2 + \frac{p_7 z_1 + p_6 z_2}{d+y_2}\right) \right\} + r_2 T \\ \frac{dy_2}{dt} = y_2 \left\{ \beta_2 y_1 - \left(t_2 + d_3 + \frac{p_7 z_1 + p_8 z_2}{d+y_2}\right) \right\} \\ \frac{dz_1}{dt} = z_1 \left\{ q\left(\frac{p_1 x_1}{a+x_1} + \frac{p_5 y_1}{d+y_1}\right) - (\alpha z_2 + d_5) \right\} + r_3 T \\ \frac{dz_2}{dt} = z_2 \left\{ q\left\{\frac{p_2 x_1}{a+x_1} + \frac{p_4 x_2}{a+x_2} + \frac{p_6 y_1}{d+y_1} + \frac{p_8 y_1}{d+y_2}\right) + (\alpha z_1 - d_4 - t_3) \right\} \\ \frac{dT}{dt} = t_1 x_1 + t_2 y_2 + t_3 z_2 - (d_1 + r_1 + r_2 + r_3)\, T \end{cases}$$

Parameters r_1, r_2 and r_3 are recovery rates of wildebeest, zebra and lion respectively, t_1, t_2 and t_3 are treatment rate, d_1, d_2 and d_3 are the death rates while q is the conversion rate of prey by predator.

Theorem 1. (*Boundness*): *All solutions of the system* (1) *are uniformly bounded within* R^7_+.

Proof: Assume $\{x_1(t), x_2(t), y_1(t), y_2(t), z_1(t), z_2(t), T(t)\}$ to be solutions of the system (1). Let $U = x_1 + x_2 + y_1 + y_2 + z_1 + z_2$ Then, the derivative of U with respect to time (t) along the trajectory of the system (1) can be indicated as;

$\frac{dU}{dt} = \frac{dx_1}{dt} + \frac{dx_1}{dt} + \frac{dy_1}{dt} + \frac{dz_1}{dt} + \frac{dz_3}{dt} + \frac{dT}{dt}$, substuting the model equations one gets;

$\frac{dU}{dt} \leq rx_1\left(1 - \frac{x_1+x_2}{k}\right) + r_1 T - (t_1 + d_2)x_2 + r_2 T - (t_2 + d_3)y_2$

$-d_5 z_1 + r_3 T - d_4 z_2 - t_3 z_2 + t_1 x_1 + t_2 x_2 + t_3 z_2 - (d_1 + r_1 + r_2 + r_3)T.$

$\frac{dU}{dt} \leq rx_1 - d_2 x_2 - d_3 y_2 - d_4 z_2 - d_5 z_1 - d_1 T$

$\leq (r+1)x_1 - (x_1 + d_2 x_2 + d_3 y_2 + d_4 z_2 - d_5 z_1 + d_1 T) \leq k(r+1) - hU$

Where $k = max\{x(0), k\}$, $h = min\{1 + d_1 + d_2 + d_3 + d_4 + d_5\}$

The equation $\frac{dU}{dt} + hU \leq k(r+1)$ has a solution $U \leq \frac{k}{h}(r+1)(1 - e^{-ht})$

As $t \to \infty$, we have $U \leq \frac{k}{h}(r+1)$, implying that the solution is bounded for $0 \leq U \leq \frac{k}{h}(r+1)$.

Therefore, all solutions of the model (1) in R^7_+ are confined in the region:

$\tau = \left\{(x_1, x_2, y_1, y_2, z_1, z_2, T) \in R^7_+ : U \leq \frac{k}{h}(r+1) + \epsilon\right\}$ for all $\epsilon \geq 0$ and $t \to \infty$

Theorem 2. (*Positivity*): *All solutions of the model*(1) *are positive.*

Proof: To prove the theorem, we have to show that variables $x_1(t), x_2(t), y_1(t)$ $, y_2(t), z_1(t), z_2(t), T(t)$ of the model (1) are non-negative $\forall t > 0$.From the susceptible wildebeest of the model;

$\frac{dx_1}{dt} = rx_1(1 - \frac{x_1+x_2}{k}) - \beta_1 x_1 x_2 + r_1 T - \frac{p_1 x_1 z_1}{a+x_1} - \frac{p_2 x_2 z_2}{a+x_1}$

Without loss of generality, after removing all the positive terms from the right hand side of the differential equation, we have the following differential enequality;

$\frac{dx_1}{dt} \geq -\left(\frac{rx_1^2+rx_1x_2}{k} + \beta_1 x_1 x_2 + \frac{p_1 x_1 z_1 + p_2 x_1 z_2}{a+x_1}\right)$, divide both sides by negative yields;

$-\frac{dx_1}{dt} \leq \frac{rx_1^2+rx_1x_2}{k} + \beta_1 x_1 x_2 + \frac{p_1 x_1 z_1 + p_2 x_1 z_2}{a+x_1}$

$\frac{rx_1^2+rx_1x_2}{k} + \beta_1 x_1 x_2 + \frac{p_1 x_1 z_1 + p_2 x_1 z_2}{a+x_1}$

$\leq rx_1^2 + rx_1x_2 + \beta_1 x_1 x_2 + p_1 x_1 z_1 + p_2 x_1 z_2 = x_1(rx_1 + rx_2 + \beta_1 x_2 + p_1 z_1 + p_2 z_2)$

Assume $rx_2 + \beta_1 x_2 + p_1 z_1 + p_2 z_2 = C$, the the differential inequality is reduced to $\frac{dx_1}{dt} \leq x_1(rx_1 + C)$. This inequality can be arranged for integration by partial fractions as;

$\int \left(\frac{1}{x_1}(rx_1 + C)\right) dx_1 \geq \int -dt$

$\int \left(r + \frac{C}{x_1}\right) dx_1 \geq \int -dt$

$\frac{1}{C}\ln|x_1| - \frac{1}{C}\ln|rx_1 + C| \geq -t + Q$, where Q is the integration constant.

$\ln|\frac{x_1}{rx_1+C}| \geq -Ct + CQ$. Solving for x_1, will give us;

$x_1(t) \geq \frac{ACe^{-ct}}{1-rAe^{-ct}}$, for $A = e^{CQ}$.

Therefore; $x_1(t) > 0$ for $1 - rAe^{-Ct} > 0$. That is $x_1(t)$ is non negative for $t > \frac{1}{C}\ln(rA)$. By similar proofs, the solutions of the other model equations can be shown to be positive.

3. Equilibrium analysis

In the absence of predator: In the absence of predator, that is, when $z_1(t)$ and $z_2(t)$ are zero, then the model (1) become;

$$(3) \qquad \begin{cases} \frac{dx_1}{dt} = rx_1\left(1 - \frac{x_1+x_2}{k}\right) - \beta_1 x_1 x_2 - r_1 T \\ \frac{x_2}{dt} = \beta_1 x_1 x_2 - t_1 x_2 - d_2 x_2 \\ \frac{dy_1}{dt} = sy_1\left(1 + \frac{y_1+y_2}{l}\right) - \beta_2 y_1 y_2 + r_2 T \\ \frac{y_2}{dt} = \beta_2 y_1 y_2 - t_2 y_2 - d_3 y_2 \\ \frac{dT}{dt} = t_1 x_1 + t_2 y_2 - (d_3 + r_1 + r_2)T. \end{cases}$$

The system has trivial $E_0(0,0,0,0,0)$, axial $E_A(k,0,0,0,0)$ and positive $E_1(x_1, x_2, y_1, y_2, T)$ where;

$x_1 = k - \frac{k\beta_1}{r} - \frac{\beta_1}{d_2+t_1} + \frac{k\beta_1 r_1 t_1}{r(d_1+r_1)(d_2+t_1)}, \quad x_2 = \frac{\beta_1}{d_2+t_1}$

$y_1 = k - \frac{l\beta_2}{s} - \frac{\beta_2}{d_3+t_1} + \frac{l\beta_2 s_1 t_2}{s(d_3+r_2)(d_3+t_2)}, \quad y_2 = \frac{\beta_2}{d_3+t_2}$

$T = \frac{t_1 x_1^* + t_2 y_2^*}{d_1+r_1+r_2}$

In the absence of Infectious disease: In the absence of infectious disease, that is when $x_2(t)$, $y_1(t)$,

$y_2(t)$, $z_2(t)$ and $T(t)$ are zero, then the model(1) become;

$$(4)\qquad \begin{cases} \frac{dx_1}{dt} = rx_1(1-\frac{x_1}{k}) - \frac{p_1x_1z_1}{1+ax_1} \\ \frac{dy_1}{dt} = sy_1(1-\frac{y_1}{l}) - \frac{p_5y_1z_1}{1+dy_1} \\ \frac{dz_1}{dt} = \frac{qp_1x_1z_1}{1+az_1} + \frac{qp_5y_1z_1}{1+dy_1} - d_5z_1 \end{cases}$$

The system consists of trivial $E_0(0,0,0)$, axial $E_A(k,0,0)$ and positive $E_1(x_1,y_1,z_1(t))$ equilibrium points, where;

$$x_1 = \frac{r(k-a)+\sqrt{r}\sqrt{k^2r+2kra+rx_1a^2+4kp_1}}{2r}$$
$$y_1 = \frac{s(l-d)+\sqrt{s}\sqrt{l^2s+2lsd+sy_1d^2+4lp_5}}{2s}$$
$$z_1 = \frac{2rqp_1}{d_5\left(a+r(k-a)+\sqrt{r}\sqrt{k^2r+2kra+rx_1a^2+4kp_1}\right)} + \frac{2sqp_5}{d_5\left(d+s(l-d)+\sqrt{s}\sqrt{l^2s+2lsd+sy_1d^2+4lp_5}\right)}$$

4. Stability analysis

Stability analysis in the absence of predator: In the absence of predator the system become;

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1(1-\frac{x_1+x_2}{k}) - \beta_1x_1x_2 + r_1T \\ \frac{dx_2}{dt} &= \beta_1x_1x_2 - t_1x_2 - d_2x_2 \\ \frac{dy_1}{dt} &= sy_1(1-\frac{y_1+y_2}{l}) - \beta_2y_1y_2 + r_2T \\ \frac{dy_2}{dt} &= \beta_2y_1y_2 - t_2y_2 - d_3y_2 \\ \frac{dT}{dt} &= t_1x_2 + t_2y_2 - d_1T - r_1T - r_2T \end{aligned}$$

The Jacobian matrix of the system is given as;

$$(5)\qquad J = \begin{pmatrix} A & \frac{rx_1}{k} - \beta_1x_1 & 0 & 0 & r_1 \\ \beta_1x_2 & B & 0 & 0 & 0 \\ 0 & 0 & C & \frac{sy_1}{l} - \beta_2y_1 & r_2 \\ 0 & 0 & \beta_2y_2 & D & 0 \\ t_1 & 0 & 0 & t_2 & E \end{pmatrix}$$

$A = r(1 - \frac{x_1+x_2}{k}) - \beta_1 x_2, \quad B = \beta_1 x_1 - t_1 - d_2, \quad C = s(1 - \frac{y_1+y_2}{l}) - \beta_2 y_2,$
$D = \beta_2 y_1 - t_2 - d_3, \quad E = -(d_1 + r_1 + r_2)$

Theorem 3. *The trivial equilibrium E_0 is a saddle point with unstable manifold in X-direction and stable manifold in Y-plane.*

Proof: The Jacobian matrix evaluated at E_0 is given by;

$$J = \begin{pmatrix} r & 0 & 0 & 0 & r_1 \\ 0 & -t_1 - d_2 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & r_2 \\ 0 & 0 & 0 & -t_2 - d_3 & 0 \\ t_1 & 0 & 0 & t_2 & -(d_1 + r_1 + r_2) \end{pmatrix} \tag{6}$$

$Det(JE) = (r - \lambda)\left[(-t_1 - d_2) - \lambda\right](s - \lambda)\left[(-t_2 - d_3) - \lambda\right]\left[-(d_1 + r_1 + r_2) - \lambda\right]$
$\lambda_1 > 0$, $\lambda_2 = -t_1 - d_2 < 0$, $\lambda_3 = s > 0$, $\lambda\lambda_4 = -t_2 - d_2 < 0$ and $\lambda_5 = -d_1 - r_1 - r_2 < 0$. Three eigenvalues are negative and two are positive. So the trivial equilibrium point in the absence of predator is the saddle point with unstable manifold in the X-direction and stable manifold in the Y-plane.

Theorem 4. *The axial equilibrium E_A is a saddle point if $\beta_1 k - t_1 - d_2 > 0$ and unstable manifold in X-direction if $\beta_1 k - t_1 - d_2 < 0$, then E_A is stable.*

Proof: The Jacobian matrix evaluated at E_A is given by;

$$J = \begin{pmatrix} r & r - \beta_1 k & 0 & 0 & r_1 \\ 0 & \beta_1 k - t_1 - d_2 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & r_2 \\ 0 & 0 & 0 & -t_2 - d_3 & 0 \\ t_1 & 0 & 0 & t_2 & -(d_1 + r_1 + r_2) \end{pmatrix} \tag{7}$$

$det(JE_1) = (r-\lambda)\left[(\beta_1 k - t_1 - d_2) - \lambda\right](s-\lambda)\left[-(t_2+d_3)-\lambda\right]\left[-(d_1+r_1+r_2)-\lambda\right]$

The eigenvalues are;$\lambda_1 = r > 0$, $\lambda_2 = \beta_1 k - t_1 - d_2 > 0$, $\lambda_3 = s > 0$, $\lambda_4 = -d_1 - r_1 - r_2 < 0$, $\lambda_5 = -t_2 - d_3 < 0$. Three eigenvalues are positive and two are negative. So the anxial equilibrium point is a saddle point which is unstable.

Theorem 5. *The positive equilibrium point E_1 is stable if* $r(1-\frac{x_1^*+x_2^*}{k}) - \beta_1 x_2^* > 0, \quad \beta_1 x_1^* - t_1 - d_2 > 0,$ $s(1-\frac{y_1^*+y_2^*}{l}) - \beta_2 y_2^* > 0, \quad \beta_2 y_1^* - t_2 - d_3 > 0, \quad -(d_1+r_1+r_2) > 0$, *otherwise is unstable.*

Proof: The Jacobian matrix evaluated at $E_1(x_1, x_2, y_1, y_2, T)$ is given as;

$$J = \begin{pmatrix} A^* & \frac{rx_1^*}{k} - \beta_1 x_1^* & 0 & 0 & r_1 \\ \beta_1 x_2^* & B^* & 0 & 0 & 0 \\ 0 & 0 & C^* & \frac{sy_1^*}{l} - \beta_2 y_1^* & r_2 \\ 0 & 0 & \beta_2 y_2^* & D^* & 0 \\ t_1 & 0 & 0 & t_2 & E^* \end{pmatrix} \tag{8}$$

$A^* = r(1-\frac{x_1^*+x_2^*}{k}) - \beta_1 x_2^*, \quad B^* = \beta_1 x_1^* - t_1 - d_2, \quad C^* = s(1-\frac{y_1^*+y_2^*}{l}) - \beta_2 y_2^*,$
$D = \beta_2 y_1^* - t_2 - d_3, \quad E = -(d_1+r_1+r_2)$
. $\left[r(1-\frac{x_1^*+x_2^*}{k}) - \beta_1 x_2^* - \lambda\right]\left[\beta_1 x_1^* - t_1 - d_2 - \lambda\right]\left[s(1-\frac{y_1^*+y_2^*}{l}) - \beta_2 y_2^* - \lambda\right]$
$\left[\beta_2 y_1^* - t_2 - d_3 - \lambda\right]\left[-(d_1+r_1+r_2) - \lambda\right] = 0$

Hence, the positive equilibrium point is stable when; $r(1-\frac{x_1^*+x_2^*}{k}) - \beta_1 x_2^* > 0, \quad \beta_1 x_1^* - t_1 - d_2 > 0,$ $s(1-\frac{y_1^*+y_2^*}{l}) - \beta_2 y_2^* > 0, \quad \beta_2 y_1^* - t_2 - d_3 > 0, \quad -(d_1+r_1+r_2) > 0$, otherwise is unstable.

Stability analysis in the absence of infectious disease: In the absence of infectious disease, that is when $x_2(t)$, $y_1(t)$, $y_2(t)$, $z_2(t)$ and $T(t)$ are zero, then the model(1) become;

$$\begin{aligned} \frac{dx_1}{dt} &= rx_1(1-\frac{x_1}{k}) - \frac{p_1 x_1 z_1}{1+ax_1} \\ \frac{dx_2}{dt} &= sy_1(1-\frac{y_1}{l}) - \frac{p_5 y_1 z_1}{1+dy_1} \\ \frac{dz_1}{dt} &= \frac{qp_1 x_1 z_1}{1+ax_1} + \frac{qp_5 y_1 z_1}{1+dy_1} - d_5 z_1 \end{aligned}$$

The Jacobian matrix of the given system is given as;

$$J = \begin{pmatrix} r(1-\frac{x_1}{k}) - \frac{p_1 z_1}{(1+ax_1)^2} & 0 & \frac{-p_1 x_1}{1+ax_1} \\ 0 & s(1-\frac{y_1}{l}) - \frac{p_5 y_1}{(1+dy_1)^2} & \frac{-p_5 y_1}{1+dy_1} \\ \frac{-qp_1 z_1}{1+ax_1} & \frac{qp_5 z_1}{1+dy_1} & -d_5 + \frac{qp_1 x_1}{1+ax_1} + \frac{qp_5 y_1}{1+dy_1} \end{pmatrix} \tag{9}$$

Theorem 6. *The trivial equilibrium point E_0 is a saddle point with stable manifold in the X-direction and unstable manifold in the Y-direction*

Proof: The Jacobian matrix at $E_0(0,0,0)$ is given as;

$$J = \begin{pmatrix} r & 0 & 0 \\ 0 & s & 0 \\ 0 & 0 & -d_5 \end{pmatrix} \tag{10}$$

Hence, eigenvalues are, $\lambda_1 = r > 0$, $\lambda_2 = s > 0$ and $\lambda_3 = -d_5 < 0$ which is a saddle point.

Theorem 7. *The axial equilibrium $E_A(k,0,0)$ is stable if $r < 0$, $s < 0$, $d_5 + \frac{qp_1 k}{1+ak} < 0$ otherwise unstable.*

Proof: The Jacobian matrix at E_A is given by;

$$J = \begin{pmatrix} r & 0 & \frac{-p_1 k}{1+ak} \\ 0 & s & 0 \\ 0 & 0 & -d_5 + \frac{qp_1 k}{1+ak} \end{pmatrix} \tag{11}$$

$\lambda_1 = r$, $\lambda_2 = s$, $\lambda_3 = -d_5 + \frac{qp_1 k}{1+ak}$, hence unstable.

Theorem 8. *The positive equilibrium point $E(x_1, y_1, z_1)$ is stable if $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$ otherwise unstable.*

Proof: The Jacobian matrix evaluated at $E(x_1, y_1, z_1)$ is given as;

$$J = \begin{pmatrix} r(1-\frac{x_1^*}{k}) - \frac{p_1 z_1^*}{(1+ax_1^*)^2} & 0 & \frac{-p_1 x_1^*}{1+ax_1^*} \\ 0 & s(1-\frac{y_1^*}{l}) - \frac{p_5 y_1^*}{(1+dy_1^*)^2} & \frac{-p_5 y_1^*}{1+dy_1^*} \\ \frac{-qp_1 z_1^*}{1+ax_1^*} & \frac{qp_5 z_1^*}{1+dy_1^*} & -d_5 + \frac{qp_1 x_1^*}{1+ax_1^*} + \frac{qp_5 y_1^*}{1+dy_1^*} \end{pmatrix} \tag{12}$$

$(A^* - L)\left[(B^* - L)(C^* - L) + \frac{qp_5^2 y_1^* z_1^*}{(1+dy_1^*)^2}\right] - \frac{p_1 x_1^*}{1+ax_1^*}\left[(B^* - L)(\frac{-qp_1 z_1^*}{1+ax_1^*})\right] = 0$

$L^3 - (A^* + B^* + C^*)L^2 + \left(A^*B^* + A^*C^* + B^*C^* - \frac{qp_5^2 y_1^* z_1^*}{(1+dy_1^*)^2} + \frac{qp_1^2 x_1^* z_1^*}{(1+ax_1^*)^2}\right) L$

$+\frac{qp_5^2 A^* y_1^* z_1^*}{(1+dy_1^*)^2} + \frac{qp_1^2 x_1^* z_1^* B^*}{(1+ax_1^*)^2} = 0$

The characteristic equation is given by; $L^3 + a_1 L^2 + a_2 L + a_3 = 0$

$a_1 = A^* + B^* + C^*$

$a_2 = A^*B^* + A^*C^* + B^*C^* - \frac{qp_5^2 y_1^* z_1^*}{(1+dy_1^*)^2} + \frac{qp_1^2 x_1^* z_1^*}{(1+ax_1^*)^2}$

$a_3 = \frac{qp_5^2 A^* y_1^* z_1^*}{(1+dy_1^*)^2} + \frac{qp_1^2 x_1^* z_1^* B^*}{(1+ax_1^*)^2}$

By Routh stability criterion, the equilibrium point is stable if, $a_1 > 0$, $a_2 > 0$, $a_3 > 0$ and $a_1 a_2 > a_3$, otherwise is unstable.

To study the stability analysis of equilibrium points of model(1), it is better to linearize model (1) using the variation matrix. Then the variation matrix obtained from model(1) is given as;

$$J = \begin{pmatrix} A & \frac{rx_1}{k} - \beta_1 x_1 & 0 & 0 & \frac{-p_1 x_1}{1+ax_1} & \frac{p_2 x_1}{1+ax_1} & r_1 \\ \beta_1 x_2 & B & 0 & 0 & \frac{-p_3 x_2}{1+ax_2} & \frac{-p_4 x_2}{1+dy_1} & 0 \\ 0 & 0 & C & \frac{sy_1}{l} - \beta_2 y_1 & \frac{-p_5 y_1}{1+dy_1} & \frac{-p_6 y_1}{1+dy_1} & r_2 \\ 0 & 0 & \beta_2 y_2 & D & \frac{-p_7 y_2}{1+dy_2} & \frac{-p_8 y_2}{1+dy_2} & 0 \\ \frac{qp_1 z_1}{(1+ax_1)^2} & 0 & \frac{qp_5 z_1}{(1+dy_1)^2} & 0 & E & \alpha z_1 & r_3 \\ \frac{qp_2 z_2}{(1+ax_1)^2} & \frac{qp_4 z_2}{(1+ax_1)^2} & \frac{qp_6 z_2}{(1+dy_1)^2} & \frac{qp_8 z_2}{(1+dy_2)^2} & \alpha z_2 & F & 0 \\ 0 & t_1 & 0 & t_2 & 0 & t_3 & G \end{pmatrix} \tag{13}$$

Where;

$A = r(1 - \frac{x_1+x_2}{k}) - \beta_1 x_2 - \frac{p_1 x_1 z_1}{(1+ax_1)^2} - \frac{p_2 x_2 z_2}{(1+ax_1)^2}$

$B = \beta_1 x_1 - t_1 - d_2 - \frac{p_3 x_2 z_1}{(1+ax_2)^2} - \frac{p_4 x_2 z_2}{(1+ax_2)^2}$

$C = s(1 - \frac{y_1+y_2}{l}) - \beta_2 y_2 - \frac{p_5 y_1 z_1}{(1+dy_1)^2} - \frac{p_6 y_2 z_2}{(1+dy_1)^2}$

$D = \beta_2 y_2 - t_2 - d_3 - \frac{p_7 z_1}{(1+dy_2)^2} - \frac{p_8 z_2}{(1+dy_2)^2}$

$E = \frac{qp_1 x_1}{1+ax_1} + \frac{qp_5 y_1}{1+dy_1} - \alpha z_2 - d_5$

$F = \alpha z_1 + \frac{qp_2 x_1}{1+ax_1} + \frac{qp_4 x_2}{1+ax_2} + \frac{qp_6 y_1}{1+dy_1} + \frac{qp_8 y_2}{1+dy_2} - t_3 - d_4$

$G = -d_1 - r_1 - r_2 - r_3$

Theorem 9. *The trivial equilibrium point* $E_0(0,0,0,0,0,0,0)$ *exists and is always locally asymptotically unstable.*

Proof: Consider the Jacobian matrix evaluated at E_0,

$$J = \begin{pmatrix} r & 0 & 0 & 0 & 0 & 0 & r_1 \\ 0 & -t_1-d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & -t_2-d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & -d_5 & 0 & r_3 \\ 0 & 0 & 0 & 0 & 0 & -t_3-d_4 & 0 \\ 0 & t_1 & 0 & t_2 & 0 & t_3 & -(d_1+r_1+r_2+r_3) \end{pmatrix} \tag{14}$$

$det(E_0) = (r-\lambda)\left[(-t_1-d_2)-\lambda\right](s-\lambda)\left[(-t_2-d_3)-\lambda\right](-d_5-\lambda)$
$\left[(-t_3-d_4)-\lambda\right]\left[(-d_1-r_1-r_2-r_3)-\lambda\right] = 0$

where; $\lambda_1 = r, \lambda_2 = -t_1-d_2, \lambda_3 = s, \lambda_4 = -t_2-d_3, \lambda_5 = -d_5, \lambda_6 = -t_3-d_4, \lambda_7 = -d_1-r_1-r_2-r_3$. Five eigenvalues are negative while two eigenvalues are positive. This implies that the trivial equilibrium point is a saddle point which is unstable.

Theorem 10. *The axial equilibrium point* $E_A(k,0,0,0,0,0,0)$ *exists and is always asymptotically stable in model* (1) *if and only if model parameter satisfy the condition* $\beta_1 k - t_1 - d_2 < 0$*,* $s < 0$*,* $-t_2 - d_3 < 0$*,* $\frac{qp_1k}{1+ak} - d_5 < 0$*,* $\frac{qp_2k}{1+ak} - d_4 - t_3 < 0$ *and* $-(d_1 + r_1 + r_2 + r_3) < 0$*. Otherwise is unstable*

Proof: Consider the the variation matrix below,

$$J = \begin{pmatrix} k & r-\beta_1 k & 0 & 0 & \frac{-p_1 k}{1+ak} & \frac{-p_2 k}{1+ak} & r_1 \\ 0 & \beta_1 k - t_1 - d_2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & s & 0 & 0 & 0 & r_2 \\ 0 & 0 & 0 & -t_2-d_3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{qp_1 k}{1+ak} - d_5 & 0 & r_3 \\ 0 & 0 & 0 & 0 & 0 & \frac{qp_2 k}{1+ak} - d_4 - t_3 & 0 \\ 0 & t_1 & 0 & t_2 & 0 & t_3 & -(d_1+r_1+r_2+r_3) \end{pmatrix} \tag{15}$$

it is clear that, the axial equilibrium point $E_A(k,0,0,0,0,0,0)$ exists and is always asymptotically stable in model (1) if and only if model parameter satisfy the condition $\beta_1 k - t_1 - d_2 < 0$, $s < 0$, $-t_2 - d_3 < 0$, $\frac{qp_1 k}{1+ak} - d_5 < 0$, $\frac{qp_2 k}{1+ak} - d_4 - t_3 < 0$ and $-(d_1 + r_1 + r_2 + r_3) < 0$. Otherwise is unstable

$$J = \begin{pmatrix} A_1 & \frac{rx_1^*}{\beta_1 x_1^*} & 0 & 0 & \frac{-p_1 x_1^*}{1+ax_1^*} & \frac{-p_2 x_1^*}{1+ax_1^*} & r_1 \\ 0 & B_1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & C_1 & \frac{sy_1^*}{l} - \beta_2 y_1^* & \frac{-p_5 y_1^*}{1+dy_1^*} & \frac{-p_6 y_1^*}{1+dy_1^*} & r_2 \\ 0 & 0 & 0 & D_1 & 0 & 0 & 0 \\ \frac{qp_1 z_1^*}{(1+ax_1^*)^2} & 0 & \frac{qp_5 z_1^*}{(1+dy_1^*)^2} & 0 & E_1 & \alpha z_1^* & r_3 \\ 0 & 0 & 0 & 0 & 0 & F_1 & 0 \\ 0 & t_1 & 0 & t_2 & 0 & t_3 & G_1 \end{pmatrix} \tag{16}$$

$A = r(1 - \frac{x_1^*}{k}) - \frac{p_1 z_1^*}{(1+ax_1^*)^2}(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*)(-d_1 - r_1 - r_2 - r_3)$
$B = \frac{p_1 x_1^* q}{(1+ax_1^*)^3}(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*)\left(s(1 - \frac{y_1^*}{l}) - \frac{p_5 z_1^*}{1+dy_1^*}\right)(-t_2 - d_3 - p_7 z_1^*)$
$\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^* - t_3 - d_4}\right)(-d_1 - r_1 - r_2 - r_3) + \frac{p_1 p_6 x_1^* y_1^*}{(1+ax_1^*)(1+dy_1^*}(-t_2 - d_3 - p_7 z_1^*)$
$\frac{qp_5 z_1^*}{(1+dy_1^*)^2}\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^*} - t_3 - d_4\right)$
$C = \frac{-p_2 x_1^*}{1+ax_1^*}(-t_2 - d_3 - p_7 z_1^*)(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*) - \frac{p_2 x_1^*}{1+ax_1^*}\left(\frac{qp_1 x_1^*}{1+ax_1^*} + \frac{qp_5 y_1^*}{1+dy_1^*} - d_5\right)$
$\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^*} - t_3 - d_4\right)\left[r(1 - \frac{x_1^*}{k}) - \frac{p_1 z_1^*}{(1+ax_1)^2} - t_2 - d_3 - p_7 z_1^*\right]$
$D = \frac{-p_6 y_1^*}{1+dy_1^*}\left(\frac{qp_1 x_1^*}{1+ax_1^*} + \frac{qp_5 y_1^*}{1+dy_1^*} - d_5\right)(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*) + \frac{p_1 p_6 x_1^* y_1^*}{(1+ax_1^*)(1+dy_1^*)}(-d_1 - r_1 - r_2 - r_3)$
$E = \left[r_2 r(1 - \frac{x_1^*}{k}) - \frac{p_1 z_1^*}{(1+ax_1^*)^2}\right](-d_1 - r_1 - r_2 - r_3)(-t_2 - d_3 - p_7 z_1^*)$
$+\frac{qp_2 x_1^*}{(1+dy_1^*)^2}\left(\frac{qp_1 x_1^*}{1+ax_1^*} + \frac{qp_5 y_1^*}{1+dy_1^*} - d_5\right)\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^*} - t_3 - d_4\right)$
$F = r_2(-d_1 - r_1 - r_2 - r_3) - \frac{p_1 x_1^* q}{1+ax_1^*}(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*)$
$+\frac{qp_5 z_1^*}{(!+dy_1^*)^2}\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^*} - t_3 - d_4\right) - d_1 - r_1 - r_2 - r_3$
$G = \frac{-p_6 r_2 y_1^*}{(1+dy_1^*)^2}(-d_1 - r_1 - r_2 - r_3)(-t_2 - d_3 - p_7 z_1^*) + \frac{p_1 p_6 x_1^* y_1^*}{(1+ax_1^*)(1+dy_1^*)}$
$(\beta_1 x_1^* - t_1 - d_2 - p_3 z_1^*)\left(\frac{qp_1 x_1^*}{1+ax_1^*} + \frac{qp_5 y_1^*}{1+dy_1^*} - d_5\right)\left(\alpha z_1^* + \frac{qp_2 x_1^*}{1+ax_1^*} + \frac{qp_6 y_1^*}{1+ax_1^*} - t_3 - d_4\right)$

5. Local Stability of the co-existence equilibrium point

$$J = \begin{pmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} \\ a_{21} & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} \\ a_{31} & a_{32} & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} \\ a_{41} & a_{42} & a_{43} & a_{44} & a_{45} & a_{46} & a_{47} \\ a_{51} & a_{52} & a_{53} & a_{54} & a_{55} & a_{56} & a_{57} \\ a_{61} & a_{62} & a_{63} & a_{64} & a_{65} & a_{66} & a_{67} \\ a_{71} & a_{72} & a_{73} & a_{74} & a_{75} & a_{76} & a_{77} \end{pmatrix} \tag{17}$$

Where;

$a_{11} = r(1 - \frac{x_1+x_2}{k}) - \beta_1 x_2 - \frac{p_1 x_1 z_1}{(1+ax_1)^2} - \frac{p_2 x_2 z_2}{(1+ax_1)^2}$

$a_{12} = \frac{rx_1}{k} - \beta_1 x_1$, $a_{13} = 0$, $a_{14} = 0$, $a_{15} = \frac{-p_1 x_1}{1+ax_1}$, $a_{16} = \frac{p_2 x_1}{1+ax_1}$, $a_{17} = r_1$, $a_{21} = \beta_1 x_2$,

$a_{22} = \beta_1 x_1 - t_1 - d_2 - \frac{p_3 x_2 z_1}{(1+ax_2)^2} - \frac{p_4 x_2 z_2}{(1+ax_2)^2}$

$a_{23} = 0$, $a_{24} = 0$, $a_{25} = \frac{-p_3 x_2}{1+ax_2}$, $a_{26} = \frac{-p_4 x_2}{1+dy_1}$, $a_{27} = 0$, $a_{31} = 0$, $a_{32} = 0$

$a_{33} = s(1 - \frac{y_1+y_2}{l}) - \beta_2 y_2 - \frac{p_5 y_1 z_1}{(1+dy_1)^2} - \frac{p_6 y_2 z_2}{(1+dy_1)^2}$

$a_{34} = \frac{sy_1}{l} - \beta_2 y_1$, $a_{35} = \frac{-p_5 y_1}{1+dy_1}$, $a_{36} = \frac{-p_6 y_1}{1+dy_1}$, $a_{37} = r_2$, $a_{41} = 0$, $a_{42} = 0$, $a_{43} = \beta_2 y_2$

$a_{44} = \beta_2 y_2 - t_2 - d_3 - \frac{p_7 z_1}{(1+dy_2)^2} - \frac{p_8 z_2}{(1+dy_2)^2}$

$a_{45} = \frac{-p_7 y_2}{1+dy_2}$, $a_{46} = \frac{-p_8 y_2}{1+dy_2}$, $a_{47} = 0$, $a_{51} = \frac{qp_1 z_1}{(1+ax_1)^2}$, $a_{52} = 0$, $a_{53} = \frac{qp_5 z_1}{(1+dy_1)^2}$, $a_{54} = 0$

$a_{55} = \frac{qp_1 x_1}{1+ax_1} + \frac{qp_5 y_1}{1+dy_1} - \alpha z_2 - d_5$

$a_{56} = \alpha z_1$, $a_{57} = r_3$, $a_{61} = \frac{qp_2 z_2}{(1+ax_1)^2}$, $a_{62} = \frac{qp_4 z_2}{(1+ax_1)^2}$, $a_{63} = \frac{qp_6 z_2}{(1+dy_1)^2}$, $a_{64} = \frac{qp_8 z_2}{(1+dy_2)^2}$, $a_{65} = \alpha z_2$

$a_{66} = \alpha z_1 + \frac{qp_2 x_1}{1+ax_1} + \frac{qp_4 x_2}{1+ax_2} + \frac{qp_6 y_1}{1+dy_1} + \frac{qp_8 y_2}{1+dy_2} - t_3 - d_4$

$a_{67} = 0$, $a_{71} = 0$, $a_{72} = t_1$, $a_{73} = 0$, $a_{74} = t_2$, $a_{75} = 0$, $a_{76} = t_3$

$a_{77} = -d_1 - r_1 - r_2 - r_3$

Finding the determinant of the above matrix, we obtain the matrix below;

$\lambda^7 + A\lambda^6 + B\lambda^5 + C\lambda^4 + D\lambda^3 + E\lambda^2 + F\lambda + G = 0$

1.$A > 0$

2.$AB - C > 0$

3.$ABC + AE - A^2 D - C^2 > 0$

4.$(CD - BE)(AB - C) - (AD - E)^2 > 0$

5.$D * [(CD - BE)(AB - C) - (AD - E)^2] > 0$

$A = r(1 - \frac{x_1+x_2}{k}) - \beta_1 x_2 - \frac{p_1 x_1 z_1}{(1+ax_1)^2} - \frac{p_2 x_2 z_2}{(1+ax_1)^2}$

$B = \beta_1 x_1 - t_1 - d_2 - \frac{p_3 x_2 z_1}{(1+ax_2)^2} - \frac{p_4 x_2 z_2}{(1+ax_2)^2}$

$C = s(1 - \frac{y_1+y_2}{l}) - \beta_2 y_2 - \frac{p_5 y_1 z_1}{(1+dy_1)^2} - \frac{p_6 y_2 z_2}{(1+dy_1)^2}$

$D = \beta_2 y_2 - t_2 - d_3 - \frac{p_7 z_1}{(1+dy_2)^2} - \frac{p_8 z_2}{(1+dy_2)^2}$

$E = \frac{qp_1 x_1}{1+ax_1} + \frac{qp_5 y_1}{1+dy_1} - \alpha z_2 - d_5$

$F = \alpha z_1 + \frac{qp_2 x_1}{1+ax_1} + \frac{qp_4 x_2}{1+ax_2} + \frac{qp_6 y_1}{1+dy_1} + \frac{qp_8 y_2}{1+dy_2} - t_3 - d_4$

$G = -d_1 - r_1 - r_2 - r_3$

6. Global stability of the endemic equilibrium point

Theorem 11. (*Global Stability*): *Endemic equilibrium point* $E^*(x_1, x_2, y_1, y_2, z_1, z_2, T)$ *exists and is globally asymptotically stable.*

Proof: Define the appropriate Lyapunov function:

$$(18) \qquad \begin{cases} L(x_1, x_2, y_1, y_2, z_1, z_2, T) = \frac{(x_1 - x_1^*)^2}{2} + \alpha_1 \frac{(x_2 - x_2^*)^2}{2} + \alpha_2 \frac{(y_1 - y_1^*)^2}{2} \\ + \alpha_3 \frac{(y_2 - y_2^*)^2}{2} + \alpha_4 \frac{(z_1 - z_1^*)^2}{2} + \alpha_5 \frac{(z_2 - z_2^*)^2}{2} + \alpha_6 \frac{(T - T^*)^2}{2} \end{cases}$$

Where; $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 > 0$ are chosen properly such that $\frac{dL}{dt} \leq 0$, $\forall (x_1, x_2, y_1, y_2, z_1, z_2, T) \in R^7_+$ and $\frac{dL}{dt} \leq 0$. This implies that E^* of the system is lyapunov stable and $\frac{dL}{dt} < 0$, $\forall (x_1, x_2, y_1, y_2, z_1, z_2, T) \in R^7_+$ near E^*. This implies E^* is globally asymptotically stable point. Now differentiate the lyapunov function L with respect to t.

$$(19) \qquad \begin{cases} \frac{dL}{dt} = (x_1 - x_1^*)\frac{dx_1}{dt} + \alpha_1(x_2 - x_2^*)\frac{dx_2}{dt} + \alpha_2(y_1 - y_1^*)\frac{dy_1}{dt} + \alpha_3(y_2 - y_2^*)\frac{dy_2}{dt} \\ +\alpha_4(z_1 - z_1^*)\frac{dz_1}{dt} + \alpha_5(z_2 - z_2^*)\frac{dz_2}{dt} + \alpha_6(T - T^*)\frac{dT}{dt} \end{cases}$$

Now substitute the model equation (2) into (19), we have the following equation;

$$\frac{dL}{dt} = (x_1 - x_1^*)\left[rx_1(1 - \tfrac{x_1+x_2}{k}) - \beta_1 x_1 x_2 + r_1 T - \tfrac{p_1 x_1 z_1}{a+x_1} - \tfrac{p_2 x_1 z_2}{a+x_1}\right]$$
$$+\alpha_1(x_2 - x_2^*)\left[\beta_1 x_1 x_2 - t_1 x_2 - d_2 x_2 - \tfrac{p_3 x_2 z_1}{a+x_2} - \tfrac{p_4 x_2 z_2}{a+x_2}\right]$$
$$+\alpha_2(y_1 - y_1^*)\left[sy_1(1 - \tfrac{y_1+y_2}{l}) - \beta_2 y_1 y_2 + r_2 T - \tfrac{p_5 y_1 z_1}{d+y_1} - \tfrac{p_6 y_1 z_2}{d+y_1}\right]$$
$$+\alpha_3(y_2 - y_2^*)\left[\beta_2 y_1 y_2 - t_2 y_2 - d_3 y_2 - \tfrac{p_7 y_2 z_1}{d+y_2} - \tfrac{p_8 y_2 z_2}{d+y_2}\right]$$
$$+\alpha_4(z_1 - z_1^*)\left[\tfrac{qp_1 x_1 z_1}{a+x_1} + \tfrac{qp_5 y_1 z_1}{d+y_1} + r_3 T - \alpha z_1 z_2 - d_5 z_1\right]$$
$$+\alpha_5(z_2 - z_2^*)\left[\alpha z_1 z_2 + \tfrac{qp_2 x_1 z_2}{a+x_1} + \tfrac{qp_4 x_2 z_2}{a+x_2} + \tfrac{qp_6 y_1 z_2}{d+y_1} + \tfrac{qp_8 y_2 z_2}{d+y_2} - t_3 z_2 - d_4 z_2\right]$$
$$+\alpha_6(T - T^*)\left[t_1 x_2 + t_2 y_2 + t_3 z_2 - d_1 T - r_1 T - r_2 T - r_3 T\right]$$

Now take out $x_1, x_2, y_1, y_2, z_1, z_2, T$ from each bracket and write a change as follow;

$$\frac{dL}{dt} = (x_1 - x_1^*)(x_1 - x_1^*)\left[r(1 - \tfrac{x_1+x_2}{k}) - \beta_1 x_2 + \tfrac{r_1 T}{x_1} - \tfrac{p_1 z_1}{a+x_1} - \tfrac{p_2 z_2}{a+x_1}\right]$$
$$+\alpha_1(x_2 - x_2^*)(x_2 - x_2^*)\left[\beta_1 x_1 - t_1 - d_2 - \tfrac{p_3 z_1}{a+x_2} - \tfrac{p_4 z_2}{a+x_2}\right]$$
$$+\alpha_2(y_1 - y_1^*)(y_1 - y_1^*)\left[s(1 - \tfrac{y_1+y_2}{l}) - \beta_2 y_2 + \tfrac{r_2 T}{y_1} - \tfrac{p_5 z_1}{d+y_1} - \tfrac{p_6 z_2}{d+y_1}\right]$$
$$+\alpha_3(y_2 - y_2^*)(y_2 - y_2^*)\left[\beta_2 y_1 - t_2 - d_3 - \tfrac{p_7 z_1}{d+y_2} - \tfrac{p_8 z_2}{d+y_2}\right]$$
$$+\alpha_4(z_1 - z_1^*)(z_1 - z_1^*)\left[\tfrac{qp_1 x_1}{a+x_1} + \tfrac{qp_5 y_1}{d+y_1} + \tfrac{r_3 T}{z_1} - \alpha z_2 - d_5\right]$$
$$+\alpha_5(z_2 - z_2^*)(z_2 - z_2^*)\left[\alpha z_1 + \tfrac{qp_2 x_1}{a+x_1} + \tfrac{qp_4 x_2}{a+x_2} + \tfrac{qp_6 y_1}{d+y_1} + \tfrac{qp_8 y_2}{d+y_2} - t_3 - d_4\right]$$
$$+\alpha_6(T - T^*)(T - T^*)\left[\tfrac{t_1 x_2}{T} + \tfrac{t_2 y_2}{T} + \tfrac{t_3 z_2}{T} - d_1 - r_1 - r_2 - r_3\right]$$

By rearranging, it is obtained that;

$$\frac{dL}{dt} = -(x_1 - x_1^*)^2\left[-r(1 - \tfrac{x_1+x_2}{k}) + \beta_1 x_2 - \tfrac{r_1 T}{x_1} + \tfrac{p_1 z_1}{a+x_1} + \tfrac{p_2 z_2}{a+x_1}\right]$$
$$-\alpha_1(x_2 - x_2^*)^2\left[-\beta_1 x_1 + t_1 + d_2 + \tfrac{p_3 z_1}{a+x_2} + \tfrac{p_4 z_2}{a+x_2}\right]$$
$$-\alpha_2(y_1 - y_1^*)^2\left[-s(1 - \tfrac{y_1+y_2}{l}) + \beta_2 y_2 - \tfrac{r_2 T}{y_1} + \tfrac{p_5 z_1}{d+y_1} + \tfrac{p_6 z_2}{d+y_1}\right]$$
$$-\alpha_3(y_2 - y_2^*)^2\left[-\beta_2 y_1 + t_2 + d_3 + \tfrac{p_7 z_1}{d+y_2} + \tfrac{p_8 z_2}{d+y_2}\right]$$
$$-\alpha_4(z_1 - z_1^*)^2\left[-\tfrac{qp_1 x_1}{a+x_1} - \tfrac{qp_5 y_1}{d+y_1} - \tfrac{r_3 T}{z_1} + \alpha z_2 + d_5\right]$$
$$-\alpha_5(z_2 - z_2^*)^2\left[-\alpha z_1 - \tfrac{qp_2 x_1}{a+x_1} - \tfrac{qp_4 x_2}{a+x_2} - \tfrac{qp_6 y_1}{d+y_1} - \tfrac{qp_8 y_2}{d+y_2} + t_3 + d_4\right]$$
$$-\alpha_6(T - T^*)^2\left[-\tfrac{t_1 x_2}{T} - \tfrac{t_2 y_2}{T} - \tfrac{t_3 z_2}{T} + d_1 + r_1 + r_2 + r_3\right]$$

Thus, it is possible to set $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ such that $\frac{dL}{dt} \le 0$ and endemic equilibrium point E^* is globally stable point.

7. Basic Reproduction number

The basic reproduction number denoted by R_0 and defined as the expected number of individual getting secondary infections among the whole susceptible population (Bornaa, 2016). This number shows a potential for the spread of disease within a given population. When $R_0 < 1$, each infected individual produces an average less than one new infected individual so that the disease is expected to die out. On the other hand, if $R_0 > 1$, then each individual produces more than one new infected individual so that the disease is expected to continue spreading in the population.

Theorem 12. *The basic reproduction number for infected wildebeest at a disease free equilibrium point* (*DFEP*)
$E^*(x_1, 0, y_1, 0, z_1, 0, 0) = \left(\frac{d_5 a}{qp_1 - d_5}, 0, \frac{d_5 d}{qp_5 - d_5}, 0, \frac{raqs(kqlp_1 - kd_5 - d_5 a)}{(qp_1 - d_5)(qp_5 - d_5)kl}, 0, 0\right)$ *is given as;*
$R_{01} = \frac{\beta_1 d_5 akl(qp_1 - d_5)(qp_5 - d_5)}{kl(qp_1 - d_5)^2(qp_5 - d_5)(t_1 + d_2) + p_3 raqs(kqlp_1 - kd_5 - d_5 a)}$.

Proof: Consider the infected wildebeest in (1), we have the following model equation;

$$\frac{dx_2}{dt} = \beta_1 x_1 x_2 - t_1 x_2 - d_2 x_2 - \frac{p_3 x_2 z_1}{1 + ax_2} - \frac{p_4 x_2 z_2}{1 + ax_2}$$
$$= \left(\beta_1 x_1 - t_1 - d_2 - \frac{p_3 z_1}{1 + ax_2} - \frac{p_4 z_2}{1 + ax_2}\right) x_2$$

Now, let us define the function; $F = \beta_1 x_1$, $V = t_1 + d_2 + \frac{p_3 z_1}{1 + ax_2} + \frac{p_4 z_2}{1 + ax_2}$

To evaluate F at V at DFEP $E^*(x_1, 0, y_1, 0, z_1, 0, 0)$ is given as;

$$F = \frac{\beta_1 d_5 a}{qp_1 - d_5},$$
$$V = (t_1 + d_2) + \frac{p_3 raqs(kqlp_1 - kd_5 - d_5 a}{(qp_1 - d_5)(qp_5 - d_5)kl} = \frac{kl(qp_1 - d_5)(qp_5 - d_5)(t_1 + d_2) + p_3 raqs(kqlp_1 - kd_5 - d_5 a)}{(qp_1 - d_5)(qp_5 - d_5)kl}$$
$$FV^{-1} = R_{01} = \frac{\beta_1 d_5 akl(qp_1 - d_5)(qp_5 - d_5)}{kl(qp_1 - d_5)^2(qp_5 - d_5)(t_1 + d_2) + p_3 raqs(kqlp_1 - kd_5 - d_5 a)}$$

Theorem 13. *The basic reproduction number for infected zebra at a disease free equilibrium point* (*DFEP*)
$E^*(x_1, 0, y_1, 0, z_1, 0, 0) = \left(\frac{d_5 a}{qp_1 - d_5}, 0, \frac{d_5 d}{qp_5 - d_5}, 0, \frac{raqs(kqlp_1 - kd_5 - d_5 a)}{(qp_1 - d_5)(qp_5 - d_5)kl}, 0, 0\right)$ *is given as;*
$R_{02} = \frac{\beta_2 d_5 dkl(qp_5 - d_5)(qp_1 - d_5)}{kl(qp_5 - d_5)^2(qp_1 - d_5)(t_2 + d_3) + p_7 raqs(kqlp_5 - kd_5 - d_5 d)}$

Proof: Consider the infected zebra in (1), we have the following model equation;

$$\frac{dy_2}{dt} = \beta_2 y_1 y_2 - t_2 y_2 - d_3 y_2 - \frac{p_7 y_2 z_1}{1 + dy_2} - \frac{p_8 y_2 z_2}{1 + dy_2}$$
$$= \left(\beta_2 y_1 - t_2 - d_3 - \frac{p_7 z_1}{1 + dy_2} - \frac{p_8 z_2}{1 + dy_2}\right) y_2$$

Now, let us define the function; $F=\beta_2 y_1$, $V=t_2+d_3+\frac{p_7 z_1}{1+dy_2}+\frac{p_8 z_2}{1+dy_2}$
To evaluate F at V at DFEP $E^*(x_1,0,y_1,0,z_1,0,0)$ is given as;
$F=\frac{\beta_2 d_5 d}{qp_5-d_5}$,
$V=(t_2+d_3)+\frac{p_7 raqs(kqlp_5-kd_5-d_5 d}{(qp_1-d_5)(qp_5-d_5)kl}=\frac{kl(qp_1-d_5)(qp_5-d_5)(t_2+d_3)+p_7 raqs(kqlp_1-kd_5-d_5 a)}{(qp_5-d_5)(qp_1-d_5)kl}$
$FV^{-1}=R_{02}=\frac{\beta_2 d_5 dkl(qp_5-d_5)(qp_1-d_5)}{kl(qp_5-d_5)^2(qp_1-d_5)(t_2+d_3)+p_7 raqs(kqlp_5-kd_5-d_5 d)}$

Theorem 14. *The basic reproduction number for infected lion at a disease free equilibrium point (DFEP)*
$E^*(x_1,0,y_1,0,z_1,0,0)=\left(\frac{d_5 a}{qp_1-d_5},0,\frac{d_5 d}{qp_5-d_5},0,\frac{raqs(kqlp_1-kd_5-d_5 a)}{(qp_1-d_5)(qp_5-d_5)kl},0,0\right)$ *is given as;*
$R_{03}=\frac{(t_3+d_4)\left[\alpha z_1(qp_1-d_5-d_5 d)(qp_5-d_5-d^2 d_5)-qp_2 d_5 a(qp_5-d_5-d^2 d_5)-qp_6 d_5 d(qp_1-d_5-a^2 d_5)\right]}{(qp_1-d_5-a^2 d_5)(qp_5-d_5-d_5 d)}$

Proof: Consider the infected zebra in (1), we have the following model equation;
$\frac{dz_2}{dt}=\alpha z_1 z_2+\frac{qp_2 x_1 z_2}{a+x_1}+\frac{qp_4 x_2 z_2}{a+x_2}+\frac{qp_6 y_1 z_2}{d+y_1}+\frac{qp_8 y_2 z_2}{d+y_2}-t_3 z_2-d_4 z_2$
$=\left(\alpha z_1-t_3-d_4+\frac{qp_2 x_1}{a+x_1}+\frac{qp_4 x_2}{a+x_2}+\frac{qp_6 y_1}{d+y_1}+\frac{qp_8 y_2}{d+y_2}\right)z_2$
Now, let us define the function; $F=\alpha z_1$ and
$V=t_3+d_4-\frac{qp_2 x_1}{a+x_1}-\frac{qp_4 x_2}{a+x_2}-\frac{qp_6 y_1}{d+y_1}-\frac{qp_8 y_2}{d+y_2}$. Hence F and V at DFEP is given as;
$F=\frac{\alpha z_1(t_3+d_4)}{qp_1-d_5-a^2 d_5}$
$V^{-1}=\frac{t_3+d_4(qp_5-d_5-d_5 d)}{((qp_1-d_5-d_5 d)-qp_2 d_5 a-qp_6 d_5 d)(qp_1-d_5-a^2 d_5)}$. Hence the the basic reproduction number for the infected lion is given as;
$R_{03}=\frac{(t_3+d_4)\left[\alpha z_1(qp_1-d_5-d_5 d)(qp_5-d_5-d^2 d_5)-qp_2 d_5 a(qp_5-d_5-d^2 d_5)-qp_6 d_5 d(qp_1-d_5-a^2 d_5)\right]}{(qp_1-d_5-a^2 d_5)(qp_5-d_5-d_5 d)}$

8. Numerical Simulation

Analytical studies can never be completed without numerical verification of the derived results. In this section we present a computer simulation of some solutions of the system(1) using Rung-Kutta iteration scheme. Beside verification of our analytical findings, these numerical solutions are very important from practical point of view. Parameters are chosen following realistic ecological observation, so they are hypothetical. Parameter values are; $r=11.2$, $s=10$, $k=30$, $l=20$, $\beta_1=1.2$, $\beta_2=1$, $\alpha=1.2$, $r_1=0.001$, $r_2=0.01$, $r_3=0.02$, $p_1=0.2$, $p_2=0.3$, $p_3=0.4$, $p_4=0.2$, $p_5=0.2$, $p_6=0.2$, $p_7=0.3$, $p_8=0.4$, $a=0.5$, $d=0.5$, $t_1=0.01$, $t_2=0.02$, $t_3=0.01$, $d_1=0.05$, $d_2=0.4$, $d_3=0.08$, $d_4=0.01$, $q=0.25$. Figure 1 show how the population varies with time.

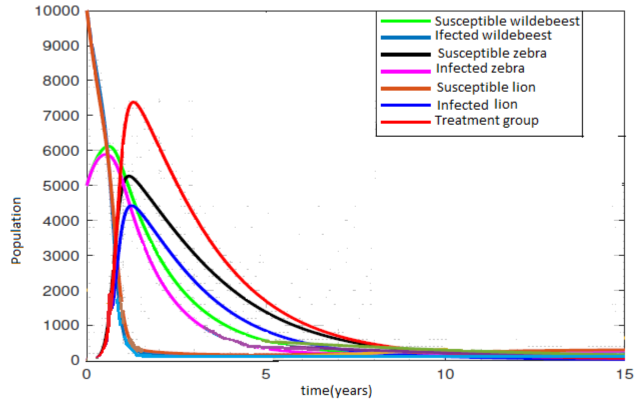

FIGURE 1. *Variation of model sub-population in relation to time*

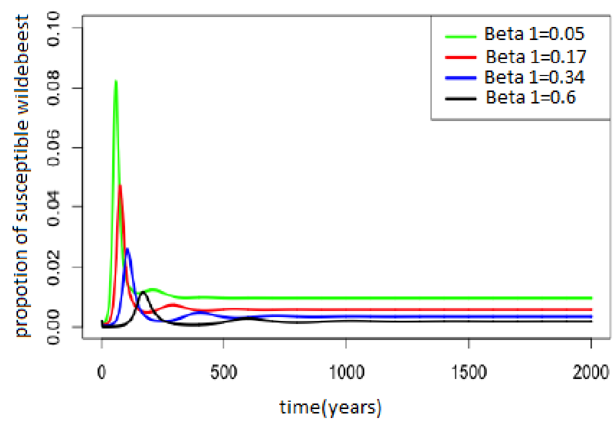

FIGURE 2. *Variation of wildebeest population with different values of* β_1

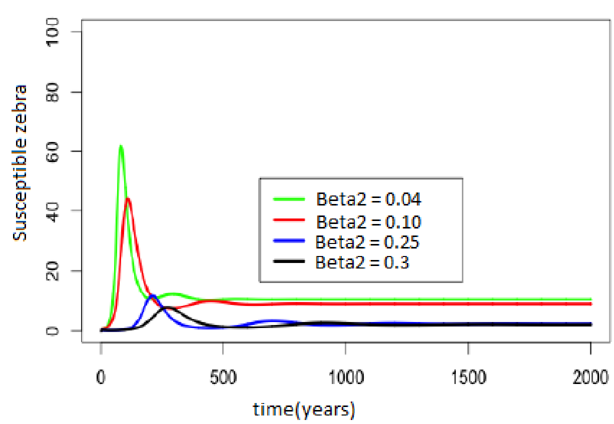

FIGURE 3. *Variation of zebra population with different values of* β_2

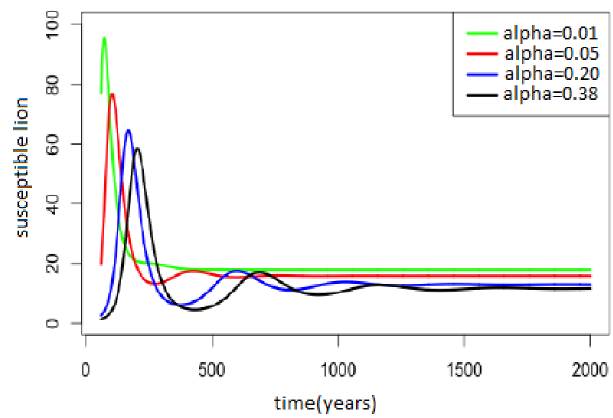

FIGURE 4. *Variation of lion population with different values of* α

Figure 1,2,3 and 4 shows that high infection results in the whole prey-predator population declining at a certain level. Therefore it is better to implement treatment mechanism to sustain stability of the ecosystem. Figure 5, 6 and 7 shows that as the treatment rate increase on infected prey-predator then the infected prey and predator is recovering and move to the susceptible classes, hence this contribute to the susceptible prey-predator population to rise in number.

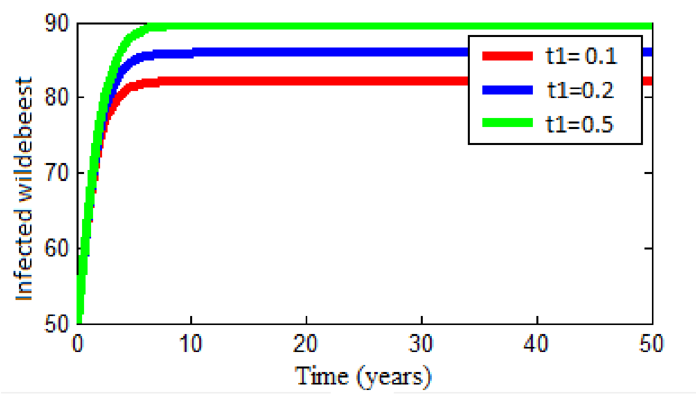

FIGURE 5. *Variation of wildebeest population with different values of* t_1

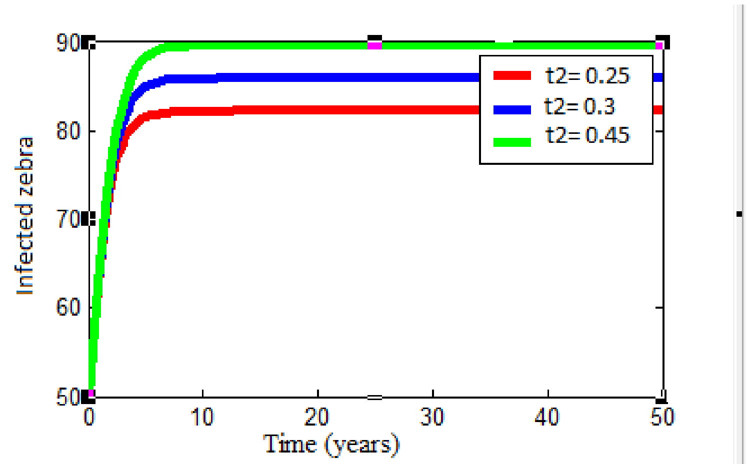

FIGURE 6. *Variation of zebra population with different values of* t_2

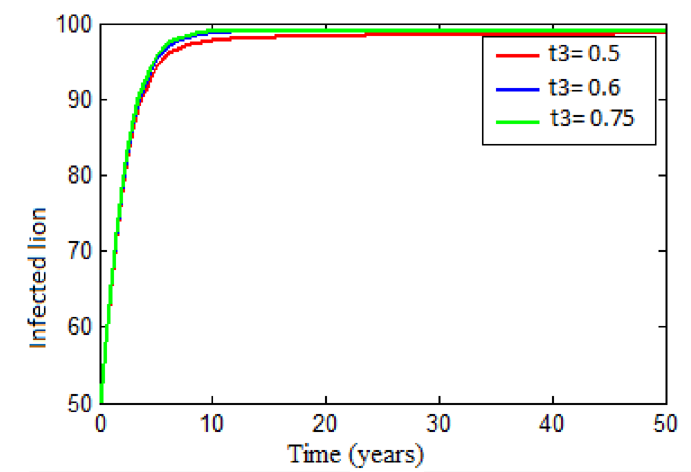

FIGURE 7. *Variation of lion population with different values of* t_3

9. Discussion and Conclusion

In this paper, it can be concluded that, the formulated model is mathematically meaningful, valid and biologically well posed by providing boundness, positivity and existence of the solutions of the model. Trivial, axial, disease free and endemic equilibrium points are investigated. Moreover in our model, it is observed that, the trivial equilibrium point is always asymptotically unstable. The axial equilibrium point is stable if and only if $\beta_1 k - t_1 - d_2 < 0$, $s < 0$, $-t_2 - d_3 < 0$, $\frac{qp_1 k}{1+ak} - d_5 < 0$, $\frac{qp_2 k}{1+ak} - d_4 - t_3 < 0$ and $-(d_1 + r_1 + r_2 + r_3) < 0$. Treatment is a helpful tool to minimize or eradicate infections in prey-predator system. Therefore, providing treatment in an infected prey-predator system creates an opportunity to recover from illness and the prey-predator population can be saved and exist in a stable situation. Thus, it is recommended to apply treatment on prey-predator to make the whole prey-predator population safe and abundant in nature. One can extend this paper by assuming the predator grows logistically or by adding parameters like death rate on prey or by including other variables like vaccination, immigration, migration on prey-predator systems and this things can be considered as limitations of this paper.

Acknowledgement

We acknowledge the Dar es salaam Institute of Technology and Serengeti National Park (SENAPA) for their support.

Conflict of interest

There is no any potential conflict of interest regarding the publication of this paper.

Data availability

All data used are obtained from different studies which have been cited in the manuscript. There is no restriction on the availability of the data.

Author's Contributions

Raymond Charles contributed to model formulation and analysis, Oluwole Daniel Makinde and Monica Kung'aro contributed to supervision, writing and review of the paper.

References

[1] A. Hugo, O.D. Makinde, S. Kumar, F.F. Chibwana, Optimal control and cost effectiveness analysis for Newcastle disease eco-epidemiological model in Tanzania, J. Biol. Dyn. 11 (2016), 190–209. https://doi.org/10.1080/17513758.2016.1258093.

[2] A. Hugo, E.S. Massawe, O.D. Makinde, An eco-epidemiological mathematical model with treatment and disease infection in both prey and predator population, J. Ecol. Nat. Environ. 4 (2012), 266-273, https://doi.org/10.5897/jene12.013.

[3] R.M. Anderson, R.M. May, The population dynamics of microparasites and their invertebrate hosts, Phil. Trans. R. Soc. Lond. B. 291 (1981), 451–524. https://doi.org/10.1098/rstb.1981.0005.

[4] P.O. Lolika, S. Mushayabasa, Dynamics and stability analysis of a Brucellosis model with two discrete delays, Discr. Dyn. Nat. Soc. 2018 (2018), 6456107. https://doi.org/10.1155/2018/6456107.

[5] J.B. Victoria, Computer modelling of serengeti-mara ecosystem, PhD dissertation, University of Leeds, 2003.

[6] T.D. Sagamiko, N. Shaban, C.L. Nahonyo, O.D. Makinde, Optimal control of a threatened wildebeest-lion prey-predator system in the Serengeti ecosystem, Open J. Ecol. 05 (2015), 110–119. https://doi.org/10.4236/oje.2015.54010.

[7] C.S. Bornaa, O.D. Makinde, I.Y. Seini, Eco-epidemiology model and optimal control of disease transmission between humans and animals, Commun. Math. Biol. Neurosci. 2015 (2015), 26.

[8] S. Sharma, G.P. Samanta, Analysis of a two prey one predator system with disease in the first prey population, Int. J. Dynam. Control. 3 (2014), 210–224. https://doi.org/10.1007/s40435-014-0107-4.

[9] A.F. Bezabih, G.K. Edessa, K.P. Rao, Ecoepidemiological model and analysis of prey-predator system, J. Appl. Math. 2021 (2021), 6679686. https://doi.org/10.1155/2021/6679686.

[10] A.F. Bezabih, G.K. Edessa, P.R. Koya, Mathematical eco-epidemiological model on prey-predator system, Math. Model. Appl. 5 (2020), 103-190.