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EXISTENCE OF SOLUTIONS FOR NONLINEAR $\psi-$CAPUTO-TYPE FRACTIONAL HYBRID DIFFERENTIAL EQUATIONS WITH PERIODIC BOUNDARY CONDITIONS

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Abstract. The main crux of this manuscript is to develop the theory of fractional hybrid differential equations with linear perturbations of second type involving $\psi-$Caputo fractional derivative of an arbitrary order $\alpha \in (0,1)$. By applying Krasnoselskii fixed point theorem and some basic concepts on fractional analysis, we prove the existence of solutions for a certain type of nonlinear fractional hybrid differential equations with periodic boundary conditions. As application, a nontrivial example is given to show the effectiveness of our theoretical results.

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1. Introduction

The theory of fractional differential equations has been of considerable interest recently, due to the evolution of fractional calculus in various scientific disciplines. This theory is an efficacious tool for modeling and describing some phenomenas in different branches of science and engineering as transport processes, earthquakes, electrochemical processes, wave propagation, signal theory, biology, electromagnetic theory, fluid flow phenomena, thermodynamics, mechanics, geology, astrophysics, economics and control theory (see, for instance [1,11,14,19,27,28]). Basic issues related to the different fractional derivatives such as Riemann-Liouville [23], Caputo [3], Hilfer [24], Erdelyi-Kober [26] and Hadamard [2].

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Almeida [6] introduced a new generalized Caputo fractional derivative with respect to another function noted ψ. some recent works on the existence and uniqueness for differential equations with $\psi-$Caputo fractional derivative can be found in [5,16,17].Due to the nonlocality nature of fractional derivatives, fractional differential equations have been of great interest recently such as functional hybrid fractional differential equations which can be employed in modeling and describing non-homogeneous physical phenomena that take place in their form. The importance of fractional hybrid differential equations lies in the fact that they include various dynamical systems as particular cases. Furthermore, hybrid differential equations arise from a variety of different areas of applied mathematics and physics, in the deflection of a curved beam having a constant or varying cross section and electromagnetic waves or gravity driven flows. Dhage et al. in [12] discussed the existence and uniqueness of solutions for the following hybrid differential equations with linear perturbations of second type:

$$\begin{cases} \frac{d}{dt}\left(x(t) - f(t, x(t))\right) = g(t, x(t)), \qquad t \in [t_0, t_0 + \delta], \\ \\ x(0) = x_0 \in \mathbb{R}. \end{cases}$$

They established the existence theorems by using some fundamental differential inequalities and comparison results. Lu et al. [25] extended the existence result for hybrid differential equations obtained in [13] involving the Riemann-Liouville fractional operator. They established an existence theorem of extremal solutions under the ϕ-Lipschitz condition. We refer the readers to the articles [4,7,8,18,20,29,30] and references therein for many works on this theory.

Motivated by the above works especially [12], we develop the theory of fractional hybrid differential equations involving ψ-Caputo fractional differential operator of order $\alpha \in (0,1)$. To be more precise, we establish the existence result of the following nonlinear fractional hybrid differential equation:

$$(1) \qquad \begin{cases} {}^{C}D_{0^+}^{\alpha,\psi}\left(x(t) - \Phi(t, x(t))\right) = \varphi(t, x(t)), \qquad t \in \Delta = [0, T], \\ \\ x(0) = x(T) = 0. \end{cases}$$

Where $T > 0$, ${}^{C}D_{0^+}^{\alpha,\psi}$ is the $\psi-$Caputo derivative and
$\Phi, \varphi \in C(\Delta \times \mathbb{R}, \mathbb{R})$.

Our manuscript is organized as follows. In Section 2, we give some basic definitions and properties of $\psi-$fractional integral and $\psi-$Caputo fractional derivative which will be used in the rest of our paper. In Section 3, we establish the existence of solutions of the $\psi-$Caputo fractional periodic boundary value problem (1) by using Krasnoselskii fixed point theorem. As application, an illustrative example is presented in Section 4 followed by conclusion in Section 5.

2. Preliminaries

In this section, we give some notations, definitions and results on ψ-fractional derivatives and ψ-fractional integrals, see the articles [5,9,22] for more details.

Notations

• We denote by $C(\Delta,\mathbb{R})$ the space of continuous real-valued functions defined on Δ provided with the topology of the supremum norm

$$\parallel x \parallel = \sup_{t\in\Delta} \mid x(t) \mid .$$

• We denote by $L^1(\Delta,\mathbb{R})$ the space of Lebesgue integrable real-valued functions on Δ equipped with the following norm

$$\parallel x \parallel_{L^1} = \int_{\Delta} \mid x(t) \mid dt.$$

Remark 1. Let $x, v \in C(\Delta,\mathbb{R})$. Clearly $C(\Delta,\mathbb{R})$ is a Banach algebra with the norm $\parallel . \parallel$ and with the multiplication $(x\times v)(t) = x(t)\times v(t)$.

Definition 1. [6] *Let $q>0$, $g\in L^1([\Delta,\mathbb{R})$ and $\psi\in C^n(\Delta,\mathbb{R})$ such that $\psi'(t)>0$ for all $t\in\Delta$. The ψ-Riemann-Liouville fractional integral at order q of the function g is given by*

$$I_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(q)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{q-1} g(s)ds. \tag{2}$$

Remark 2. Note that if $\psi(t)=t$ and $\psi(t)=log(t)$, then the equation (2) is reduced to the Riemann-Liouville and Hadamard fractional integrals respectively.

Definition 2. [6] *Let $q>0$, $g\in C^{n-1}(\Delta,\mathbb{R})$ and $\psi\in C^n(\Delta,\mathbb{R})$ such that $\psi'(t)>0$ for all $t\in\Delta$. The $\psi-$Caputo fractional derivative at order q of the function g is given by*

$${}^{C}D_{0^+}^{q,\psi} g(t) = \frac{1}{\Gamma(n-q)} \int_0^t \psi'(s)(\psi(t)-\psi(s))^{n-q-1} g_{\psi}^{[n]}(s)ds, \tag{3}$$

where

$$g_{\psi}^{[n]}(s) = \left(\frac{1}{\psi'(s)}\frac{d}{ds}\right)^n g(s) \quad and \quad n=[q]+1,$$

and $[q]$ denotes the integer part of the real number q.

Remark 3. In particular, note that if $\psi(t) = t$ and $\psi(t) = log(t)$, then the equation (3) is reduced to the the Caputo fractional derivative and Caputo-Hadamard fractional derivative respectively.

Remark 4. If $q \in (0,1)$, then the equation (3) can be written as follows

$$^{C}D_{0^+}^{q,\psi}g(t) = \frac{1}{\Gamma(q)}\int_0^t (\psi(t)-\psi(s))^{q-1}g'(s)ds.$$

In another way, we have

$$^{C}D_{0^+}^{q,\psi}g(t) = I_{0^+}^{1-q,\psi}\left(\frac{g'(t)}{\psi'(t)}\right).$$

Proposition 1. [6] Let $q > 0$, if $g \in C^{n-1}(\Delta,\mathbb{R})$, then we have

1) $^{C}D_{0^+}^{q,\psi}I_{0^+}^{q,\psi}g(t) = g(t)$.
2) $I_{0^+}^{q,\psi}\,^{C}D_{0^+}^{q,\psi}g(t) = g(t) - \sum_{k=0}^{n-1}\frac{g_{\psi}^{[k]}(0)}{k!}(\psi(t)-\psi(0))^k$.
3) $I_{0^+}^{q,\psi}$ is linear and bounded from $C(\Delta,\mathbb{R})$ to $C(\Delta,\mathbb{R})$.

Proposition 2. [6] Let $t > 0$ and $q,\alpha,\beta \geq 0$, then we have

1) $I_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta+\alpha)}(\psi(t)-\psi(0))^{\alpha+\beta-1}$.
2) $^{C}D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{\beta-1} = \dfrac{\Gamma(\beta)}{\Gamma(\beta-\alpha)}(\psi(t)-\psi(0))^{\alpha-\beta-1}$.
3) $^{C}D_{0^+}^{q,\psi}(\psi(t)-\psi(0))^{n} = 0, \quad$ for all $n \in \mathbb{N}$.

Lemma 1. (See [10]). Let Ω be a non-empty, closed convex and bounded subset of a Banach algebra X and Let $\mathcal{T}_1 : \Omega \to X$ and $\mathcal{T}_2 : \Omega \to X$ be two operators such that

(1) $\mathcal{T}_1$ is a contraction with constant $\lambda < 1$,
(2) $\mathcal{T}_2$ is compact and continuous,
(3) $x = \mathcal{T}_1 x + \mathcal{T}_2 y$ for all $x, y \in \Omega$.

Then the equation $\mathcal{T}_1 x + \mathcal{T}_2 x = x$ has a solution in Ω.

We assume the following assumptions throughout the rest of our paper.

(A_1) $\Phi(T,0) - \Phi(0,0) + \displaystyle\int_0^T \psi'(s)(\psi(T)-\psi(s))^{\alpha-1}\varphi(s,x(s))ds = 0.$

(A_2) There exists a constant $0 < \lambda < 1$ such that

$$|\,\varphi(t,x)-\varphi(t,y)\,| \leq \lambda\,|\,x-y\,| \quad for \quad all \quad t \in \Delta \quad and \quad x,y \in \mathbb{R}.$$

(A_3) There exists a function $h \in L^1(\Delta,\mathbb{R})$

$$|\,\varphi(t,x)\,| \leq h(t) \quad a.e.\ t \in \Delta, \quad for\ all\ x \in \mathbb{R}.$$

3. Main results

In this section, before we give the existence result of the fractional boundary value problem (1), we need to prove the following fundamental lemma.

Lemma 2. Suppose that hypothesis (A_1) holds, then the function $x(t) \in C(\Delta, \mathbb{R})$ is a solution of the periodic fractional boundary value problem (1) if and only if x satisfies the following fractional hybrid integral equation

$$x(t) = \Phi(t, x(t)) - \Phi(0, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds. \tag{4}$$

Proof. Let x be a solution of the problem (1), then we apply the $\psi-$fractional integral $I_{0^+}^{\alpha,\psi}$ on both sides of (1)we have

$$I_{0^+}^{q,\psi}\, {}^C D_{0^+}^{\alpha,\psi} (x(t) - \Phi(t, x(t))) = I_{0^+}^{\alpha,\psi} \varphi(t, x(t)),$$

from Proposition 1 we obtain

$$x(t) - \Phi(t, x(t)) - x(0) + \Phi(0, x(0)) = I_{0^+}^{\alpha,\psi} \varphi(t, x(t)),$$

since $x(0) = x(T) = 0$ then we obtain

$$x(t) = \Phi(t, x(t)) - \Phi(0, 0) + I_{0^+}^{\alpha,\psi} \varphi(t, x(t)),$$

thus

$$x(t) = \Phi(t, x(t)) - \Phi(0, 0) + \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds.$$

Hence equation (4) holds.
Conversely, it is clear that if $x(t)$ satisfies the equation (4), we apply the $\psi-$Caputo fractional derivative ${}^C D_{0^+}^{\alpha,\psi}$ to both sides of equation (4) and we use Proposition 1, we obtain

$${}^C D_{0^+}^{\alpha,\psi} (x(t) - \Phi(t, x)) = {}^C D_{0^+}^{\alpha,\psi} \left(\frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds - \Phi(0, 0) \right),$$
$${}^C D_{0^+}^{\alpha,\psi} (x(t) - \Phi(t, x)) = {}^C D_{0^+}^{\alpha,\psi} I_{0^+}^{\alpha,\psi} \varphi(t, x) = \varphi(t, x),$$

it follows that

$${}^C D_{0^+}^{\alpha,\psi} (x(t) - \Phi(t, x(t))) = \varphi(t, x(t)).$$

Finally, we need to verify that the condition $x(0) = x(T) = 0$ in th equation (1) also holds. For this purpose, we substitute $t = 0$ and $t = T$ in (4), we obtain

$$x(0) = \Phi(0, x(0)) - \Phi(0, 0) + \frac{1}{\Gamma(\alpha)} \int_0^0 \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds = 0,$$

and from (A_1), it follows that

$$x(T)=\Phi(T,x(T))-\Phi(0,0)+\frac{1}{\Gamma(\alpha)}\int_0^T\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,x(s))ds=0,$$

thus

$$x(0)=x(T)=0.$$

Hence, x is a solution to the problem (1). This completes the proof.

□

Theorem 1. Assume that hypotheses $(A_1)-(A_3)$ hold, then the periodic fractional boundary value problem (1) has a solution defined on Δ.

Proof. Let $E=C(\Delta,\mathbb{R})$ and let C_m be a subset of the space E defined by

$$C_m=\{x\in E:\parallel x\parallel\leq m\}.$$

where

$$m=\frac{2\Phi_0}{1-\lambda}+\frac{(\psi(T)-\psi(0))^{\alpha}}{(1-\lambda)\Gamma(\alpha+1)}\parallel h\parallel_{L^1},$$

and

$$\Phi_0=\sup_{t\in\Delta}\Phi(t,0).$$

It is easy te see that C_m is a closed, convex and bounded subset of the Banach space E. By using Lemma 2, the fractional hybrid differential equation (1) is equivalent to the following nonlinear fractional hybrid integral equation

$$x(t)=\Phi(t,x(t))-\Phi(0,0)+\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,x(s))ds. \tag{5}$$

Let $\mathcal{T}_1:E\to E$ and $\mathcal{T}_2:C_m\to E$ be two operators defined by

$$\mathcal{T}_1x(t)=\Phi(t,x(t))-\Phi(0,0),$$

and

$$\mathcal{T}_2x(t)=\frac{1}{\Gamma(\alpha)}\int_0^t\psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,x(s))ds.$$

We can transforme the fractional integral equation (5) into the operator equation as follows

$$\mathcal{T}_1x(t)+\mathcal{T}_2x(t)=x(t),\quad t\in\Delta. \tag{6}$$

Now, we will show that the operators $\mathcal{T}_1$ and $\mathcal{T}_1$ satisfy all the conditions of Lemma 1.
First, we prove that $\mathcal{T}_1$ is a contraction on E with the constant $\lambda < 1$.
Let $x, v \in E$, then by hypothesis (A_2)

$$\mid \mathcal{T}_1 x(t) - \mathcal{T}_1 y(t) \mid = \mid \Phi(t, x(t)) - \Phi(t, y(t)) \mid \leq \lambda \mid x(t) - y(t) \mid \quad for \quad all \quad t \in \Delta,$$

Taking supremum over t, we obtain

$$\parallel \mathcal{T}_1 x - \mathcal{T}_1 y \parallel \leq \lambda \parallel x - y \parallel, \quad for \quad all \quad x, y \in E.$$

Therefore, $\mathcal{T}_1$ is a contractive mapping with constant $\lambda < 1$.
Secondly, we show the operator $\mathcal{T}_2$ is completely continuous.
For this purpose, it is enough to prove that the operator $\mathcal{T}_2$ is continuous and $\mathcal{T}_2(C_m)$ is uniformly bounded and equicontinuous.
Let us show that the operator $\mathcal{T}_2$ is continuous.
Let x_n be a sequence in C_m converging to $x \in C_m$, then by the Lebesgue dominated convergence theorem, we have

$$\lim_{n \mapsto +\infty} \mathcal{T}_2 x_n(t) = \lim_{n \mapsto +\infty} \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x_n(s)) ds,$$
$$\lim_{n \mapsto +\infty} \mathcal{T}_2 x_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \lim_{n \mapsto +\infty} \varphi(s, x_n(s)) ds,$$
$$\lim_{n \mapsto +\infty} \mathcal{T}_2 x_n(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds,$$
$$\lim_{n \mapsto +\infty} \mathcal{T}_2 x_n(t) = \mathcal{T}_2 x(t), \quad for \quad all \quad t \in \Delta.$$

Wich shows that $\mathcal{T}_2$ is a continuous operator on C_m.

Next we show that $\mathcal{T}_2(C_m)$ is a uniformly bounded.
Let $x \in C_m$, then we have

$$|\mathcal{T}_2 x(t)| = \left| \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \varphi(s, x(s)) ds \right|,$$
$$|\mathcal{T}_2 x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \times |\varphi(s, x(s))| \, ds,$$

by using (A_3) we obtain

$$|\mathcal{T}_2 x(t)| \leq \frac{1}{\Gamma(\alpha)} \int_0^t \psi'(s)(\psi(t) - \psi(s))^{\alpha-1} \times |h(s))| \, ds,$$
$$|\mathcal{T}_2 x(t)| \leq \frac{(\psi(T) - \psi(0))^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel_{L^1} \quad for \quad all \quad t \in \Delta,$$

taking supremum over t, we obtain

$$\| \mathcal{T}_2 x \| \leq \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \| h \|_{L^1} \quad for \quad all \quad x \in C_m.$$

This shows that $\mathcal{T}_2$ is uniformly bounded on C_m.

Now, let us also show that $\mathcal{T}_2(C_m)$ is equicontinuous on Δ.

Let $x \in \mathcal{B}(C_m)$ and $t_1, t_2 \in \Delta$ such that $t_1 < t_2$, then we have

$$|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)| = \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds \Big|,$$

$$|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)| \leq \Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_1)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds \Big|$$

$$+\Big| \frac{1}{\Gamma(\alpha)} \int_0^{t_1} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds$$

$$-\frac{1}{\Gamma(\alpha)} \int_0^{t_2} \psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1} \varphi(s,x(s))ds \Big|,$$

it follows that

$$|\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)| \leq \frac{\| h \|_{L^1}}{\Gamma(\alpha+1)} \left(|\psi^{\alpha}(t_2) - \psi^{\alpha}(t_1) - (\psi(t_2)-\psi(t_1))^{\alpha}| - (\psi(t_2)-\psi(t_1))^{\alpha} \right),$$

since ψ is a continuous function, then we obatin

$$\lim_{t_1 \to t_2} |\mathcal{T}_2 x(t_1) - \mathcal{T}_2 x(t_2)| = 0.$$

Which shows that $\mathcal{T}_2(C_m)$ is equicontinuous.
Now the set $\mathcal{T}_2(C_m)$ is uniformly bounded and equicontinuous and by using Arzelà–Ascoli Theorem [21] we deduce that $\mathcal{T}_2(C_m)$ is compact, wich implies that the operator $\mathcal{T}_2$ is completely continuous.
Now it remains to show that the assumption (3) in Lemma 1 is verified.

Let $x \in E$ and $y \in C_m$ be arbitrary such that $x = \mathcal{T}_1 x + \mathcal{T}_2 y$, then by hypothesis (A_2), we have

$$\begin{aligned}
|x(t)| &= |\mathcal{T}_1 x(t) + \mathcal{T}_2 y(t)| \\
&= \left|\Phi(t,x(t)) - \Phi(0,0) + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\varphi(s,y(s))ds\right|, \\
&\leq |\Phi(t,x(t)) - \Phi(0,0)| + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,|\varphi(s,y(s))|\,ds, \\
&\leq |\Phi(t,x(t)) - \Phi(t,0) + \Phi(t,0) - \Phi(0,0)| \\
&+ \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,|\varphi(s,y(s))|\,ds,
\end{aligned}$$

by using (A_3) we get

$$\begin{aligned}
|x(t)| &\leq \lambda \mid x(t) \mid + \mid \Phi(t,0) \mid + \mid \Phi(0,0) \mid + \frac{1}{\Gamma(\alpha)}\int_0^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1}\,|h(s)|\,ds, \\
&\leq \lambda\,|x(t)| + 2\Phi_0 + \frac{(\psi(T)-\psi(0))^{\alpha}}{\Gamma(\alpha+1)} \parallel h \parallel_{L^1},
\end{aligned}$$

wich implies that

$$|x(t)| \leq \frac{2\Phi_0}{1-\lambda} + \frac{(\psi(T)-\psi(0))^{\alpha}}{(1-\lambda)\Gamma(\alpha+1)} \parallel h \parallel_{L^1},$$

taking supremum over t, we obtain

$$\parallel x \parallel \leq \frac{2\Phi_0}{1-\lambda} + \frac{(\psi(T)-\psi(0))^{\alpha}}{(1-\lambda)\Gamma(\alpha+1)} \parallel h \parallel_{L^1} := m.$$

Therefore, the condition (3) in Lemma 1 holds.

Finally, all conditions of Lemma 1 are satisfied for the operators $\mathcal{T}_1$ and $\mathcal{T}_1$. Hence the fractional hybrid differential equation (1) has a solution defined on Δ.

□

4. An illustrative example

In this section we give an nontrivial example to illustrate our main result.

Consider the following periodic fractional boundary value problem:

$$\begin{cases} {}^{C}D_{0^+}^{\frac{1}{2},t}\,(x(t) - \Phi(t,x(t))) = \varphi(t,x(t)), \qquad t \in \Delta = [0,1], \\ \\ x(0) = x(1) = 0. \end{cases} \tag{7}$$

Where $\frac{1}{3} + \frac{1}{\sqrt{\pi}} \int_0^1 \psi'(s)(\psi(1) - \psi(s))^{-1/2} \varphi(s, x(s)) ds = 0$.

Choose $\alpha = \frac{1}{2}$, $T = 1$, $\psi(t) = t$, $\varphi(t, x(t)) = \dfrac{t^2}{3} sin(x(t))$ and

$\Phi(t, x(t)) = \dfrac{t}{3}\sqrt{x^2(t) + 1}$. It is clear that the assumption (A_2) is satisfied with $\lambda = \frac{1}{3}$. Indeed, let $t \in \Delta$ and $x, y \in \mathbb{R}$, then we have

$$|\Phi(t, x(t)) - \Phi(t, y(t))| = \left|\frac{t}{3}\sqrt{x^2(t) + 1} - \frac{t}{3}\sqrt{y^2(t) + 1}\right|,$$

$$|\Phi(t, x(t)) - \Phi(t, y(t))| \leq \frac{t}{3} |x(t) - y(t)| \frac{|x(t) + y(t)|}{\sqrt{x^2(t) + 1} + \sqrt{y^2(t) + 1}},$$

$$|\Phi(t, x(t)) - \Phi(t, y(t))| \leq \frac{1}{3} |x(t) - y(t)|,$$

thus, the assumption (A_2) in holds true with $\lambda = \frac{1}{3}$.

It remains to verify the assumption (A_3). Let $t \in \Delta$ and $x \in \mathbb{R}$, then we have

$$|\varphi(t, x(t))| = \left|\frac{t^2}{3} sin(x(t))\right|,$$

$$|\varphi(t, x(t))| \leq \frac{t^2}{3} |sin(x(t))|,$$

$$|\varphi(t, x(t))| \leq \frac{t^2}{3}.$$

Wich implies that the assumption (A_3) is verified with $h(t) = \frac{t^2}{3}$.

Finally, all the conditions of Theorem 1 are satisfied, thus the periodic fractional hybrid problem (7) has a solution defined on $[0, 1]$.

5. Conclusion

In the present paper, we gave the definition of solutions for periodic fractional hybrid boundary value problem by using the $\psi-$Caputo fractional derivative of order $\alpha \in (0, 1)$. In addition, by employing Krasnoselskii fixed point theorem, the existence of at least one solution for this problem is discussed. Finally, as application, a nontrivial example is presented to illustrate our theoretical results.

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