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INTERVAL-VALUED NEUTROSOPHIC SUBALGEBRAS OF HILBERT ALGEBRAS

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Abstract. The concept of interval-valued neutrosophic sets (IVNSs) was first introduced by Wang et al. (H. Wang, F. Smarandache, Y. Q. Zhang, R. Sunderraman, Interval neutrosophic sets and logic: Theory and applications in computing, Hexis, Phoenix, Ariz, USA, 2005.). In this paper, the concept of IVNSs to subalgebras of Hilbert algebras is introduced. The homomorphic image and inverse image of interval-valued neutrosophic subalgebras (IVN subalgebras) in Hilbert algebras are also studied and some related properties are investigated.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [17]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1, 3, 6]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multi-criteria decision-making [10–12]. The

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concept of Hilbert algebras was introduced in early 50-ties by Henkin and Skolem for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Horn and Diego from algebraic point of view. Diego proved (cf. [7]) that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag (cf. [4], [5]) and Jun (cf. [13]) and some of their filters forming deductive systems were recognized. Dudek (cf. [8,9]) considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras. The concept of IVNSs was first considered by Wang et al. [16] in 2005.

In this paper, the concept of IVNSs to subalgebras of Hilbert algebras is introduced. The homomorphic image and inverse image of IVN subalgebras in Hilbert algebras are also studied and some related properties are investigated.

2. Preliminaries

Before we begin the study, let's review the definition of Hilbert algebras, which was defined by Diego [7] in 1966.

Definition 2.1. [7] A *Hilbert algebra* is a triplet $X = (X, \cdot, 1)$, where H is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed element of X such that the following axioms hold:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,
(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,
(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

The following result was proved in [8].

Lemma 2.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,
(2) $(\forall x \in X)(1 \cdot x = x)$,
(3) $(\forall x \in X)(x \cdot 1 = 1)$,
(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

A *fuzzy set* [17] in a nonempty set X is defined to be a function $\mu : X \to [0,1]$, where $[0,1]$ is the unit closed interval of real numbers.

Definition 2.3. [8] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}),$$

and an *anti fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\}).$$

An *interval number* we mean a close subinterval $\tilde{a} = [a^l, a^u]$ of $[0,1]$, where $0 \leq a^l \leq a^u \leq 1$. The interval number $\tilde{a} = [a^l, a^u]$ with $a^l = a^u$ is denoted by **a**. Denote by $D[0,1]$ the set of all interval numbers. In particular, if $\tilde{a_1}$ and $\tilde{a_2}$ are interval numbers, we define the *refined minimum* and the *refined maximum* of $\tilde{a_1}$ and $\tilde{a_2}$, denoted by $\mathrm{rmin}\{\tilde{a_1}, \tilde{a_2}\}$ and $\mathrm{rmax}\{\tilde{a_1}, \tilde{a_2}\}$, respectively, as follows:

$$\mathrm{rmin}\{\tilde{a_1}, \tilde{a_2}\} = [\min\{a_1^l, a_2^l\}, \min\{a_1^u, a_2^u\}],$$

$$\mathrm{rmax}\{\tilde{a_1}, \tilde{a_2}\} = [\max\{a_1^l, a_2^l\}, \max\{a_1^u, a_2^u\}].$$

If $\tilde{a_i} = [a_i^l, a_i^u] \in D[0,1]$ for $i = 1, 2, \ldots$, then we define

$$\mathrm{rsup}_i\{\tilde{a_i}\} = [\sup_i\{a_i^l\}, \sup_i\{a_i^u\}].$$

Similarly, we define

$$\mathrm{rinf}_i\{\tilde{a_i}\} = [\inf_i\{a_i^l\}, \inf_i\{a_i^u\}].$$

Definition 2.4. [14] Let $\tilde{a_1}$ and $\tilde{a_2}$ be interval numbers. We define the symbols $\succeq$, $\preceq$, and $=$ in case of $\tilde{a_1}$ and $\tilde{a_2}$ as follows:

$$\tilde{a_1} \succeq \tilde{a_2} \Leftrightarrow a_1^l \geq a_2^l \text{ and } a_1^u \geq a_2^u,$$

and similarly we may have $\tilde{a_1} \preceq \tilde{a_2}$ and $\tilde{a_1} = \tilde{a_2}$.

In the $D[0,1]$, the following assertions are valid (see [15]).

$$(\forall \tilde{a} \in D[0,1]) \begin{pmatrix} \mathrm{rmax}\{\tilde{a}, \tilde{a}\} = \tilde{a} \\ \mathrm{rmin}\{\tilde{a}, \tilde{a}\} = \tilde{a} \end{pmatrix}. \tag{2.1}$$

$$(2.2)\qquad (\forall \tilde{a}_1, \tilde{a}_2 \in D[0,1]) \left(\begin{array}{l} \mathrm{rmax}\{\tilde{a}_1, \tilde{a}_2\} \succeq \tilde{a}_1 \\ \tilde{a}_1 \succeq \mathrm{rmin}\{\tilde{a}_1, \tilde{a}_2\} \end{array} \right).$$

$$(2.3)\qquad (\forall \tilde{a}_1, \tilde{a}_2, \tilde{a}_3 \in D[0,1]) \left(\tilde{a}_1 \succeq \tilde{a}_2, \tilde{a}_3 \succeq \tilde{a}_2 \;\Leftrightarrow\; \mathrm{rmin}\{\tilde{a}_1, \tilde{a}_3\} \succeq \tilde{a}_2 \right).$$

Definition 2.5. [16] An interval-valued neutrosophic set (IVNS) A in a nonempty set X is defined to be a structure

$$(2.4)\qquad A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\},$$

where $T_A : X \to D[0,1], I_A : X \to D[0,1]$, and $F_A : X \to D[0,1]$, which are called a truth membership function, an indeterminacy membership function, and a falsity membership function, respectively. The intervals $T_A(x), I_A(x)$, and $F_A(x)$ denote the intervals of the degree of membership, indeterminacy, and non-membership of the element x to the set $D[0,1]$, respectively, where $T_A(x) = [T_A^l(x), T_A^u(x)], I_A(x) = [I_A^l(x), I_A^u(x)]$, and $F_A(x) = [F_A^l(x), F_A^u(x)]$ for all $x \in X$. Also note that $\overline{T_A}(x) = \mathbf{1} - T_A(x) = [1 - T_A^u(x), 1 - T_A^l(x)], \overline{I_A}(x) = \mathbf{1} - I_A(x) = [1 - I_A^u(x), 1 - I_A^l(x)]$, and $\overline{F_A}(x) = \mathbf{1} - F_A(x) = [1 - F_A^u(x), 1 - F_A^l(x)]$ for all $x \in X$, where $(x, \overline{T_A}(x), \overline{I_A}(x), \overline{F_A}(x))$ represents the complement of x in A. For the sake of simplicity, we shall use the symbol $A = (T_A, I_A, F_A)$ for the IVNS set $A = \{(x, T_A(x), I_A(x), F_A(x)) \mid x \in X\}$.

3. Interval-valued neutrosophic subalgebras

In this section, we introduce the concept of IVN subalgebras of Hilbert algebras and investigate some related properties.

Definition 3.1. An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$ is called an interval-valued neutrosophic subalgebra (IVN subalgebra) of X if

$$(3.1)\qquad (\forall x, y \in X) \left(\begin{array}{l} T_A(x \cdot y) \succeq \mathrm{rmin}\{T_A(x), T_A(y)\} \\ I_A(x \cdot y) \preceq \mathrm{rmax}\{I_A(x), I_A(y)\} \\ F_A(x \cdot y) \succeq \mathrm{rmin}\{F_A(x), F_A(y)\} \end{array} \right).$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an IVNS $A = (T_A, I_A, F_A)$ as follows:

$$T_A(x) = \begin{cases} [0.5, 0.6] & \text{if } x \in \{1, x, y, z\} \\ [0.1, 0.2] & \text{if } x = 0, \end{cases}$$

$$I_A(x) = \begin{cases} [0.3, 0.4] & \text{if } x \in \{1, x, y, z\} \\ [0.4, 0.5] & \text{if } x = 0, \end{cases}$$

$$F_A(x) = \begin{cases} [0.1, 0.2] & \text{if } x \in \{1, x, y, z\} \\ [0.2, 0.3] & \text{if } x = 0. \end{cases}$$

Hence, A is an IVN subalgebra of X.

Proposition 3.3. *Every IVN subalgebra $A = (T_A, I_A, F_A)$ of a Hilbert algebra $X = (X, \cdot, 1)$ satisfies*

$$(\forall x \in X) \begin{pmatrix} T_A(1) \succeq T_A(x) \\ I_A(1) \preceq I_A(x) \\ F_A(1) \succeq F_A(x) \end{pmatrix}. \tag{3.2}$$

Proof. For any $x \in X$, we have

$$\begin{aligned} T_A(1) &= T_A(x \cdot x) \\ &\succeq \text{rmin}\{T_A(x), T_A(x)\} \\ &= T_A(x), \end{aligned}$$

$$\begin{aligned} I_A(1) &= I_A(x \cdot x) \\ &\preceq \text{rmax}\{I_A(x), I_A(x)\} \\ &= I_A(x), \end{aligned}$$

$$\begin{aligned} F_A(1) &= F_A(x \cdot x) \\ &\succeq \text{rmin}\{F_A(x), F_A(x)\} \\ &= F_A(x). \end{aligned}$$

□

Proposition 3.4. *If* $A = (T_A, I_A, F_A)$ *is an IVN subalgebra of a Hilbert algebra* $X = (X, \cdot, 1)$*, then*

$$(\forall x \in X) \left(\begin{array}{l} T_A(1 \cdot x) \succeq T_A(x) \\ I_A(1 \cdot x) \preceq I_A(x) \\ F_A(1 \cdot x) \succeq F_A(x) \end{array} \right). \tag{3.3}$$

Proof. For any $x \in X$, we have

$$\begin{aligned} T_A(1 \cdot x) &\succeq \operatorname{rmin}\{T_A(1), T_A(x)\} \\ &= \operatorname{rmin}\{T_A(x \cdot x), T_A(x)\} \\ &\succeq \operatorname{rmin}\{\operatorname{rmin}\{T_A(x), T_A(x)\}, T_A(x)\} \\ &= T_A(x), \end{aligned}$$

$$\begin{aligned} I_A(1 \cdot x) &\preceq \operatorname{rmax}\{I_A(1), I_A(x)\} \\ &= \operatorname{rmax}\{I_A(x \cdot x), I_A(x)\} \\ &\preceq \operatorname{rmax}\{\operatorname{rmax}\{I_A(x), I_A(x)\}, I_A(x)\} \\ &= I_A(x), \end{aligned}$$

$$\begin{aligned} F_A(1 \cdot x) &\succeq \operatorname{rmin}\{F_A(1), F_A(x)\} \\ &= \operatorname{rmin}\{F_A(x \cdot x), F_A(x)\} \\ &\succeq \operatorname{rmin}\{\operatorname{rmin}\{F_A(x), F_A(x)\}, F_A(x)\} \\ &= F_A(x). \end{aligned}$$

□

Theorem 3.5. *An IVNS* $A = (T_A, I_A, F_A)$ *in a Hilbert algebra* $X = (X, \cdot, 1)$ *is an IVN subalgebra of* X *if and only if* T_A^l, T_A^u, F_A^l*, and* F_A^u *are fuzzy subalgebras of* X *and* I_A^l *and* I_A^u *are anti fuzzy subalgebras of* X.

Proof. Let T_A^l and T_A^u be fuzzy subalgebras of X and let $x, y \in X$. Then $T_A^l(x \cdot y) \geq \min\{T_A^l(x), T_A^l(y)\}$ and $T_A^u(x \cdot y) \geq \min\{T_A^u(x), T_A^u(y)\}$. Now,

$$\begin{aligned} T_A(x \cdot y) &= [T_A^l(x \cdot y), T_A^u(x \cdot y)] \\ &\succeq [\min\{T_A^l(x), T_A^l(y)\}, \min\{T_A^u(x), T_A^u(y)\}] \\ &= \operatorname{rmin}\{[T_A^l, (x), T_A^u(x)], [T_A^l(y), T_A^u(y)]\} \\ &= \operatorname{rmin}\{T_A(x), T_A(y)\}. \end{aligned}$$

Again, let I_A^l and I_A^u be anti fuzzy subalgebras of X and $x, y \in X$. Then $I_A^l(x \cdot y) \leq \max\{I_A^l(x), I_A^l(y)\}$ and $I_A^u(x \cdot y) \leq \max\{I_A^u(x), I_A^u(y)\}$. Now,

$$\begin{aligned} I_A(x \cdot y) &= [I_A^l(x \cdot y), I_A^u(x \cdot y)] \\ &\preceq [\max\{I_A^l(x), I_A^l(y)\}, \max\{I_A^u(x), I_A^u(y)\}] \\ &= \text{rmax}\{[I_A^l,(x), I_A^u(x)], [I_A^l(y), I_A^u(y)]\} \\ &= \text{rmax}\{I_A(x), I_A(y)\}. \end{aligned}$$

Also, let F_A^l and F_A^u be fuzzy subalgebras of X and $x, y \in X$. Then $F_A^l(x \cdot y) \geq \min\{F_A^l(x), F_A^l(y)\}$ and $F_A^u(x \cdot y) \geq \min\{F_A^u(x), F_A^u(y)\}$. Now,

$$\begin{aligned} F_A(x \cdot y) &= [F_A^l(x \cdot y), F_A^u(x \cdot y)] \\ &\succeq [\min\{F_A^l(x), F_A^l(y)\}, \min\{F_A^u(x), F_A^u(y)\}] \\ &= \text{rmin}\{[F_A^l,(x), F_A^u(x)], [F_A^l(y), F_A^u(y)]\} \\ &= \text{rmin}\{F_A(x), F_A(y)\}. \end{aligned}$$

Hence, A is an IVN subalgebra of X.

Conversely, assume that A is an IVN subalgebra of X. For any $x, y \in X$,

$$\begin{aligned} [T_A^l(x \cdot y), T_A^u(x \cdot y)] &= T_A(x \cdot y) \\ &\succeq \text{rmin}\{T_A(x), T_A(y)\} \\ &= \text{rmin}\{[T_A^l(x), T_A^u(x)], [T_A^l(y), T_A^u(y)]\} \\ &= [\min\{T_A^l(x), T_A^l(y)\}, \min\{T_A^u(x), T_A^u(y)\}], \end{aligned}$$

$$\begin{aligned} [I_A^l(x \cdot y), I_A^u(x \cdot y)] &= I_A(x \cdot y) \\ &\preceq \text{rmax}\{I_A(x), I_A(y)\} \\ &= \text{rmax}\{[I_A^l(x), I_A^u(x)], [I_A^l(y), I_A^u(y)]\} \\ &= [\max\{I_A^l(x), I_A^l(y)\}, \max\{I_A^u(x), I_A^u(y)\}], \end{aligned}$$

$$\begin{aligned} [F_A^l(x \cdot y), F_A^u(x \cdot y)] &= F_A(x \cdot y) \\ &\succeq \text{rmin}\{F_A(x), F_A(y)\} \\ &= \text{rmin}\{[F_A^l(x), F_A^u(x)], [F_A^l(y), F_A^u(y)]\} \\ &= [\min\{F_A^l(x), F_A^l(y)\}, \min\{F_A^u(x), F_A^u(y)\}]. \end{aligned}$$

Thus $T_A^l(x \cdot y) \geq \min\{T_A^l(x), T_A^l(y)\}, T_A^u(x \cdot y) \geq \min\{T_A^u(x), T_A^u(y)\}$, $I_A^l(x \cdot y) \leq \max\{I_A^l(x), I_A^l(y)\}$, $I_A^u(x \cdot y) \leq \max\{I_A^u(x), I_A^u(y)\}$, $F_A^l(x \cdot y) \geq \min\{F_A^l(x), F_A^l(y)\}$, and $F_A^u(x \cdot y) \geq \min\{F_A^u(x), F_A^u(y)\}$. Hence, T_A^l, T_A^u, F_A^l, and F_A^u are fuzzy subalgebras of X and I_A^l and I_A^u are anti fuzzy subalgebras of X. □

Theorem 3.6. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are two IVN subalgebras of a Hilbert algebra $X = (X, \cdot, 1)$, then $A \cap B = (T_{A\cap B}, I_{A\cup B}, F_{A\cap B})$ is an IVN subalgebra of X, where*

$$(\forall x \in X) \begin{pmatrix} T_{A\cap B}(x) = [T^l_{A\cap B}(x), T^u_{A\cap B}(x)] \\ I_{A\cup B}(x) = [I^l_{A\cup B}(x), I^u_{A\cup B}(x)] \\ F_{A\cap B}(x) = [F^l_{A\cap B}(x), F^u_{A\cap B}(x)] \end{pmatrix}. \tag{3.4}$$

Proof. Let $x, y \in X$. Then

$$\begin{aligned} & T_{A\cap B}(x \cdot y) \\ & = [T^l_{A\cap B}(x \cdot y), T^u_{A\cap B}(x \cdot y)] \\ & = [\min\{T^l_A(x \cdot y), T^l_B(x \cdot y)\}, \min\{T^u_A(x \cdot y), T^u_B(x \cdot y)\}] \\ & \succeq [\min\{T^l_{A\cap B}(x), T^l_{A\cap B}(y)\}, \min\{T^u_{A\cap B}(x), T^u_{A\cap B}(y)\}] \\ & = \text{rmin}\{T_{A\cap B}(x), T_{A\cap B}(y)\}, \end{aligned}$$

$$\begin{aligned} & I_{A\cup B}(x \cdot y) \\ & = [I^l_{A\cup B}(x \cdot y), I^u_{A\cup B}(x \cdot y)] \\ & = [\max\{I^l_A(x \cdot y), I^l_B(x \cdot y)\}, \max\{I^u_A(x \cdot y), I^u_B(x \cdot y)\}] \\ & \preceq [\max\{I^l_{A\cup B}(x), I^l_{A\cup B}(y)\}, \max\{I^u_{A\cup B}(x), I^u_{A\cup B}(y)\}] \\ & = \text{rmax}\{I_{A\cup B}(x), I_{A\cup B}(y)\}, \end{aligned}$$

$$\begin{aligned} & F_{A\cap B}(x \cdot y) \\ & = [F^l_{A\cap B}(x \cdot y), F^u_{A\cap B}(x \cdot y)] \\ & = [\min\{F^l_A(x \cdot y), F^l_B(x \cdot y)\}, \min\{F^u_A(x \cdot y), F^u_B(x \cdot y)\}] \\ & \succeq [\min\{F^l_{A\cap B}(x), F^l_{A\cap B}(y)\}, \min\{F^u_{A\cap B}(x), F^u_{A\cap B}(y)\}] \\ & = \text{rmin}\{F_{A\cap B}(x), F_{A\cap B}(y)\}. \end{aligned}$$

Hence, $A \cap B$ is an IVN subalgebra of X. □

For any elements x and y of a Hilbert algebra X, let $\prod^n x \cdot y$ denotes the expression $x \cdot (\cdots (x \cdot (x \cdot y)))$, where x occurred n times.

Theorem 3.7. *Let $A = (T_A, I_A, F_A)$ be an IVN subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$ and let $n \in \mathbb{N}$. Then*

(1) $T_A\left(\prod^n x \cdot x\right) \succeq T_A(x)$ *for any odd number n,*

(2) $I_A\left(\prod^n x \cdot x\right) \preceq I_A(x)$ *for any odd number n,*

(3) $F_A\left(\prod^n x \cdot x\right) \succeq F_A(x)$ *for any odd number n,*

(4) $T_A\left(\prod^{n} x\cdot x\right) = T_A(x)$ *for any even number* n,

(5) $I_A\left(\prod^{n} x\cdot x\right) = I_A(x)$ *for any even number* n,

(6) $F_A\left(\prod^{n} x\cdot x\right) = F_A(x)$ *for any even number* n.

Proof. Let $x \in X$ and assume that n is odd. Then $n = 2p-1$ for some positive integer p. Now, $T_A(x\cdot x) = T_A(1) \succeq T_A(x), I_A(x\cdot x) = I_A(1) \preceq I_A(x)$, and $F_A(x\cdot x) = F_A(1) \succeq F_A(x)$. Suppose that $T_A\left(\prod^{2p-1} x\cdot x\right) \succeq T_A(x), I_A\left(\prod^{2p-1} x\cdot x\right) \preceq I_A(x)$, and $F_A\left(\prod^{2p-1} x\cdot x\right) \succeq F_A(x)$. Then by assumption, we have

$$\begin{aligned}
T_A\left(\prod^{2(p+1)-1} x\cdot x\right) &= T_A\left(\prod^{2p+1} x\cdot x\right)\\
&= T_A\left(\prod^{2p-1} x\cdot (x\cdot (x\cdot x))\right)\\
&= T_A\left(\prod^{2p-1} x\cdot x\right)\\
&\succeq T_A(x),\\
I_A\left(\prod^{2(p+1)-1} x\cdot x\right) &= I_A\left(\prod^{2p+1} x\cdot x\right)\\
&= I_A\left(\prod^{2p-1} x\cdot (x\cdot (x\cdot x))\right)\\
&= I_A\left(\prod^{2p-1} x\cdot x\right)\\
&\preceq I_A(x),\\
F_A\left(\prod^{2(p+1)-1} x\cdot x\right) &= F_A\left(\prod^{2p+1} x\cdot x\right)\\
&= F_A\left(\prod^{2p-1} x\cdot (x\cdot (x\cdot x))\right)\\
&= F_A\left(\prod^{2p-1} x\cdot x\right)\\
&\succeq F_A(x),
\end{aligned}$$

which proves (1), (2), and (3). Proofs are similar for the cases (4), (5), and (6). □

Definition 3.8. Let $A = (T_A, I_A, F_A)$ be an IVNS in a Hilbert algebra $X = (X, \cdot, 1)$. The IVNSs $\oplus A, \otimes A$, and $\odot A$ in X are defined as follows: $\oplus A = (T_A, \overline{T_A}, F_A), \otimes A = (\overline{I_A}, I_A, F_A)$, and $\odot A = (\overline{I_A}, I_A, \overline{I_A})$.

Theorem 3.9. *If* $A = (T_A, I_A, F_A)$ *is an IVN subalgebra of a Hilbert algebra* $X = (X, \cdot, 1)$, *then* $\oplus A, \otimes A$, *and* $\odot A$ *are IVN subalgebras of* X.

Proof. Let $x, y \in X$. Then $\overline{T_A}(x \cdot y) = \mathbf{1} - T_A(x \cdot y) \preceq \mathbf{1} - \text{rmin}\{T_A(x), T_A(y)\} = \text{rmax}\{\mathbf{1} - T_A(x), \mathbf{1} - T_A(y)\} = \text{rmax}\{\overline{T_A}(x), \overline{T_A}(y)\}$. Hence, $\oplus A$ is an IVN subalgebra of X. Let $x, y \in X$. Then $\overline{I_A}(x \cdot y) = \mathbf{1} - I_A(x \cdot y) \succeq \mathbf{1} - \text{rmax}\{I_A(x), I_A(y)\} = \text{rmin}\{\mathbf{1} - I_A(x), \mathbf{1} - I_A(y)\} = \text{rmin}\{\overline{I_A}(x), \overline{I_A}(y)\}$. Hence, $\otimes A$ is an IVN subalgebra of X. The proof for $\odot A$ is similar. □

For an IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra $X = (X, \cdot, 1)$, the sets $\{x \in X \mid T_A(x) = T_A(1)\}$, $\{x \in X \mid I_A(x) = I_A(1)\}$, and $\{x \in X \mid F_A(x) = F_A(1)\}$ are denoted by T_A^1, I_A^1, and F_A^1, respectively. These three sets are also subalgebra of X.

Theorem 3.10. *If $A = (T_A, I_A, F_A)$ is an IVN subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then the sets T_A^1, I_A^1, and F_A^1 are subalgebras of X.*

Proof. Let $x, y \in T_A^1$. Then $T_A(x) = T_A(1) = T_A(y)$ and so $T_A(x \cdot y) \preceq \text{rmin}\{T_A(x), T_A(y)\} = T_A(1)$. By using Proposition 3.3, we have $T_A(x \cdot y) = T_A(1)$. Thus $x \cdot y \in T_A^1$. Again, let $x, y \in I_A^1$. Then $I_A(x) = I_A(1) = I_A(y)$ and so $I_A(x \cdot y) \preceq \text{rmax}\{I_A(x), I_A(y)\} = I_A(1)$. By using Proposition 3.3, we have $I_A(x \cdot y) = I_A(1)$. Thus $x \cdot y \in I_A^1$. Let $x, y \in F_A^1$. Then $F_A(x) = F_A(1) = T_A(y)$ and so $F_A(x \cdot y) \preceq \text{rmin}\{F_A(x), F_A(y)\} = F_A(1)$. By using Proposition 3.3, we have $F_A(x \cdot y) = F_A(1)$. Thus $x \cdot y \in F_A^1$. Hence, the sets T_A^1, I_A^1, and F_A^1 are subalgebras of X. □

Theorem 3.11. *Let B be a nonempty subset of a Hilbert algebra $X = (X, \cdot, 1)$ and $A = (T_A, I_A, F_A)$ be an IVNS in X defined by*

$$T_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ [\beta_1, \beta_2] & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} [\theta_1, \theta_2] & \text{if } x \in B \\ [\delta_1, \delta_2] & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} [\sigma_1, \sigma_2] & \text{if } x \in B \\ [\omega_1, \omega_2] & \text{otherwise} \end{cases}$$

for $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\theta_1, \theta_2], [\delta_1, \delta_2], [\sigma_1, \sigma_2], [\omega_1, \omega_2] \in D[0, 1]$ with $[\alpha_1, \alpha_2] \succeq [\beta_1, \beta_2]$, $[\theta_1, \theta_2] \preceq [\delta_1, \delta_2]$, and $[\sigma_1, \sigma_2] \succeq [\omega_1, \omega_2]$ and $\alpha_2 + \theta_2 \leq 1$, $\beta_2 + \delta_2 \leq 1$, and $\sigma_2 + \omega_2 \leq 1$. Then A is an IVN subalgebra of X if and only if B is a subalgebra of X. Moreover, $T_A^1 = I_A^1 = F_A^1 = B$.

Proof. Let A be an IVN subalgebra of X. Let $x, y \in B$. Then $T_A(x \cdot y) \succeq \text{rmin}\{T_A(x), T_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2]$, $I_A(x \cdot y) \preceq \text{rmax}\{I_A(x), I_A(y)\} = \text{rmax}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} =$

$[\alpha_1, \alpha_2]$, and $F_A(x \cdot y) \succeq \text{rmin}\{F_A(x), F_A(y)\} = \text{rmin}\{[\sigma_1, \sigma_2], [\sigma_1, \sigma_2]\} = [\sigma_1, \sigma_2]$. Thus $x \cdot y \in B$. Hence, B is a subalgebra of X.

Conversely, suppose that B is a subalgebra of X. Let $x, y \in X$. Consider two cases:

Case (i): If $x, y \in B$, then $x \cdot y \in B$. Thus $T_A(x \cdot y) = [\alpha_1, \alpha_2] = \text{rmin}\{T_A(x), T_A(y)\}$, $I_A(x \cdot y) = [\theta_1, \theta_2] = \text{rmax}\{I_A(x), I_A(y)\}$, and $F_A(x \cdot y) = [\sigma_1, \sigma_2] = \text{rmin}\{F_A(x), F_A(y)\}$.

Case (ii): If $x \notin B$ or $y \notin B$, then $T_A(x \cdot y) \succeq [\beta_1, \beta_2] = \text{rmin}\{T_A(x), T_A(y)\}$, $I_A(x \cdot y) \preceq [\theta_1, \theta_2] = \text{rmax}\{I_A(x), I_A(y)\}$, and $F_A(x \cdot y) \succeq [\omega_1, \omega_2] = \text{rmin}\{F_A(x), F_A(y)\}$.

Hence, A is an IVN subalgebra of X. Now, $T_A^1 = \{x \in X \mid T_A(x) = T_A(1)\} = \{x \in X \mid T_A(x) = [\alpha_1, \alpha_2]\} = B$, $I_A^1 = \{x \in X \mid I_A(x) = I_A(1)\} = \{x \in X \mid I_A(x) = [\theta_1, \theta_2]\} = B$, and $F_A^1 = \{x \in X \mid F_A(x) = F_A(1)\} = \{x \in X \mid F_A(x) = [\sigma_1, \sigma_2]\} = B$. □

Definition 3.12. Let $A = (T_A, I_A, F_A)$ be an IVN subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$. For $[s_1, s_2] \in D[0,1]$, the sets $U(T_A : [s_1, s_2]) = \{x \in X \mid T_A(x) \succeq [s_1, s_2]\}$ is called an *upper* $[s_1, s_2]$*-level cut* of A, $L(I_A : [s_1, s_2]) = \{x \in X \mid I_A(x) \preceq [s_1, s_2]\}$ is called a *lower* $[s_1, s_2]$*-level cut* of A, and $U(F_A : [s_1, s_2]) = \{x \in X \mid F_A(x) \succeq [s_1, s_2]\}$ is called an *upper* $[s_1, s_2]$*-level cut* of A.

Theorem 3.13. *If $A = (T_A, I_A, F_A)$ is an IVN subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$, then the nonempty upper $[s_1, s_2]$-level cut, the nonempty lower $[t_1, t_2]$-level cut, and the nonempty upper $[u_1, u_2]$-level cut of A are subalgebras of X.*

Proof. Let $x, y \in U(T_A : [s_1, s_2])$. Then $T_A(x) \succeq [s_1, s_2]$ and $T_A(y) \succeq [s_1, s_2]$. It follows that $T_A(x \cdot y) \succeq \text{rmin}\{T_A(x), T_A(y)\} \succeq [s_1, s_2]$ so that $x \cdot y \in U(T_A : [s_1, s_2])$. Hence, $U(T_A : [s_1, s_2])$ is a subalgebra of X. Let $x, y \in L(I_A : [t_1, t_2])$. Then $I_A(x) \preceq [t_1, t_2]$ and $I_A(y) \preceq [t_1, t_2]$. It follows that $I_A(x \cdot y) \preceq \text{rmax}\{I_A(x), I_A(y)\} \preceq [t_1, t_2]$ so that $x \cdot y \in L(I_A : [t_1, t_2])$. Hence, $L(I_A : [t_1, t_2])$ is a subalgebra of X. Let $x, y \in U(F_A : [u_1, u_2])$. Then $F_A(x) \succeq [u_1, u_2]$ and $F_A(y) \succeq [u_1, u_2]$. It follows that $F_A(x \cdot y) \succeq \text{rmin}\{F_A(x), F_A(y)\} \succeq [u_1, u_2]$ so that $x \cdot y \in U(F_A : [u_1, u_2])$. Hence, $U(F_A : [u_1, u_2])$ is a subalgebra of X. □

Theorem 3.14. *Let $A = (T_A, I_A, F_A)$ be an IVNS in a Hilbert algebra $X = (X, \cdot, 1)$ such that the nonempty sets $U(T_A : [s_1, s_2])$, $L(I_A : [t_1, t_2])$, and $U(F_A : [u_1, u_2])$ are subalgebras of X for every $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0,1]$. Then A is an IVN subalgebra of X.*

Proof. Let for every $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0,1]$, $U(T_A : [s_1, s_2])$, $L(I_A : [t_1, t_2])$, and $U(F_A : [u_1, u_2])$ are subalgebras of X. In contrary, let $x_0, y_0 \in X$ be such that $T_A(x_0 \cdot y_0) \prec \text{rmin}\{T_A(x_0), T_A(y_0)\}$. Let $T_A(x_0) = [\theta_1, \theta_2], T_A(y_0) = [\theta_3, \theta_4]$, and $T_A(x_0 \cdot y_0) = [s_1, s_2]$.

Then $[s_1, s_2] \prec \text{rmin}\{[\theta_1, \theta_2], [\theta_3, \theta_4]\} = [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]$. So, $s_1 < \min\{\theta_1, \theta_3\}$ and $s_2 < \min\{\theta_2, \theta_4\}$. Consider,

$$\begin{aligned}
[\rho_1, \rho_2] &= \tfrac{1}{2}[T_A(x_0 \cdot y_0) + \text{rmin}\{T_A(x_0), T_A(y_0)\}] \\
&= \tfrac{1}{2}[[s_1, s_2] + [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}]] \\
&= [\tfrac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}), \tfrac{1}{2}(s_2 + \min\{\theta_2, \theta_4\})].
\end{aligned}$$

Therefore, $\min\{\theta_1, \theta_3\} > \rho_1 = \frac{1}{2}(s_1 + \min\{\theta_1, \theta_3\}) > s_1$ and $\min\{\theta_2, \theta_4\} > \rho_2 = \frac{1}{2}(s_2 + \min\{\theta_2, \theta_4\}) > s_2$. Hence, $[\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1, \rho_2] \succ [s_1, s_2]$, so that $x_0 \cdot y_0 \notin U(T_A : [s_1, s_2])$, which is a contradiction because $T_A(x_0) = [\theta_1, \theta_2] \succeq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1, \rho_2]$ and $T_A(y_0) = [\theta_3, \theta_4] \succeq [\min\{\theta_1, \theta_3\}, \min\{\theta_2, \theta_4\}] \succ [\rho_1, \rho_2]$. This implies $x_0 \cdot y_0 \in U(T_A : [s_1, s_2])$. Thus $T_A(x \cdot y) \succeq \text{rmin}\{T_A(x), T_A(y)\}$ for all $x, y \in X$. Again, in contrary, let $x_0, y_0 \in X$ be such that $I_A(x_0 \cdot y_0) \succ \text{rmax}\{I_A(x_0), I_A(y_0)\}$. Let $I_A(x_0) = [\eta_1, \eta_2], I_A(y_0) = [\eta_3, \eta_4]$, and $I_A(x_0 \cdot y_0) = [t1, t_2]$. Then $[t_1, t_2] \succ \text{rmax}\{[\eta_1, \eta_2], [\eta_3, \eta_4]\} = [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}]$. So, $t_1 > \max\{\eta_1, \eta_3\}$ and $t_2 > \max\{\eta_2, \eta_4\}$. Let us consider,

$$\begin{aligned}
[\beta_1, \beta_2] &= \tfrac{1}{2}[I_A(x_0 \cdot y_0) + \text{rmax}\{I_A(x_0), I_A(y_0)\}] \\
&= \tfrac{1}{2}[[t_1, t_2] + [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \\
&= \tfrac{1}{2}(t_1 + \max\{\eta_1, \eta_3\}), \tfrac{1}{2}(t_2 + \max\{\eta_2, \eta_4\}).
\end{aligned}$$

Therefore, $\max\{\eta_1, \eta_3\} < \beta_1 = \frac{1}{2}[(t_1 + \max\{\eta_1, \eta_3\})] < t_1$ and $\max\{\eta_2, \eta_4\} < \beta_2 = \frac{1}{2}[(t_2 + \max\{\eta_2, \eta_4\})] < t_2$. Hence, $[\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \prec [\beta_1, \beta_2] < [t_1, t_2]$, so that $x_0 \cdot y_0 \notin L(I_A : [t_1, t_2])$, which is a contradiction because $I_A(x_0) = [\eta_1, \eta_2] \preceq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \prec [\beta_1, \beta_2]$, and $I_A(y_0) = [\eta_3, \eta_4] \succeq [\max\{\eta_1, \eta_3\}, \max\{\eta_2, \eta_4\}] \succ [\beta_1, \beta_2]$. Hence, $x_0 \cdot y_0 \in L(I_A : [t_1, t_2])$. Thus $I_A(x \cdot y) \succeq \text{rmax}\{I_A(x), I_A(y)\}$ for all $x, y \in X$. In contrary, let $x_0, y_0 \in X$ be such that $F_A(x_0 \cdot y_0) \prec \text{rmin}\{F_A(x_0), F_A(y_0)\}$. Let $F_A(x_0) = [\omega_1, \omega_2], F_A(y_0) = [\omega_3, \omega_4]$, and $F_A(x_0 \cdot y_0) = [u_1, u_2]$. Then $[u_1, u_2] \prec \text{rmin}\{[\omega_1, \omega_2], [\omega_3, \omega_4]\} = [\min\{\omega_1, \omega_3\}, \min\{\omega_2, \omega_4\}]$. So, $u_1 < \min\{\omega_1, \omega_3\}$ and $u_2 < \min\{\omega_2, \omega_4\}$. Consider,

$$\begin{aligned}
[\sigma_1, \sigma_2] &= \tfrac{1}{2}[F_A(x_0 \cdot y_0) + \text{rmin}\{F_A(x_0), F_A(y_0)\}] \\
&= \tfrac{1}{2}[[u_1, u_2] + [\min\{\omega_1, \omega_3\}, \min\{\omega_2, \omega_4\}]] \\
&= [\tfrac{1}{2}(u_1 + \min\{\omega_1, \omega_3\}), \tfrac{1}{2}(u_2 + \min\{\omega_2, \omega_4\})].
\end{aligned}$$

Therefore, $\min\{\omega_1, \omega_3\} > \sigma_1 = \frac{1}{2}(u_1 + \min\{\omega_1, \omega_3\}) > u_1$ and $\min\{\omega_2, \omega_4\} > \sigma_2 = \frac{1}{2}(u_2 + \min\{\omega_2, \omega_4\}) > u_2$. Hence, $[\min\{\omega_1, \omega_3\}, \min\{\omega_2, \omega_4\}] \succ [\sigma_1, \sigma_2] \succ [u_1, u_2]$, so that $x_0 \cdot y_0 \notin U(F_A : [u_1, u_2])$, which is a contradiction because $F_A(x_0) = [\omega_1, \omega_2] \succeq [\min\{\omega_1, \omega_3\}, \min\{\omega_2, \omega_4\}] \succ [\sigma_1, \sigma_2]$, and $F_A(y_0) = [\omega_3, \omega_4] \succeq [\min\{\omega_1, \omega_3\}, \min\{\omega_2, \omega_4\}] \succ$

$[\sigma_1, \sigma_2]$. This implies $x_0 \cdot y_0 \in U(F_A : [u_1, u_2])$. Thus $F_A(x \cdot y) \succeq \text{rmin}\{F_A(x), F_A(y)\}$ for all $x, y \in X$. Hence, A is an IVN subalgebra of X. □

Theorem 3.15. *Any subalgebra of a Hilbert algebra $X = (X, \cdot, 1)$ can be realized as the upper level cut, lower level cut, and upper level cut of some IVN subalgebra of X.*

Proof. Let B be a subalgebra of X and A be an IVNS in X defined by

$$T_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ \mathbf{0} & \text{otherwise,} \end{cases}$$

$$I_A(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in B \\ \mathbf{1} & \text{otherwise,} \end{cases}$$

$$F_A(x) = \begin{cases} [\omega_1, \omega_2] & \text{if } x \in B \\ \mathbf{0} & \text{otherwise} \end{cases}$$

for $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\omega_1, \omega_2] \in D[0, 1]$ and $\alpha_2 + \beta_2 + \omega_2 \leq 1$. We consider the following cases:

Case (i): If $x, y \in B$, then $T_A(x) = [\alpha_1, \alpha_2]$, $I_A(x) = [\beta_1, \beta_2]$, $F_A(x) = [\omega_1, \omega_2]$, $T_A(y) = [\alpha_1, \alpha_2]$, $I_A(y) = [\beta_1, \beta_2]$, and $F_A(y) = [\omega_1, \omega_2]$. Thus

$$T_A(x \cdot y) = [\alpha_1, \alpha_2] = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = \text{rmin}\{T_A(x), T_A(y)\},$$

$$I_A(x \cdot y) = [\beta_1, \beta_2] = \text{rmax}\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = \text{rmax}\{I_A(x), I_A(y)\},$$

$$F_A(x \cdot y) = [\omega_1, \omega_2] = \text{rmin}\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = \text{rmin}\{F_A(x), F_A(y)\}.$$

Case (ii): If $x \in B$ and $y \notin B$, then $T_A(x) = [\alpha_1, \alpha_2]$, $I_A(x) = [\beta_1, \beta_2]$, $F_A(x) = [\omega_1, \omega_2]$, $T_A(y) = \mathbf{0}$, $I_A(y) = \mathbf{1}$, and $F_A(y) = \mathbf{0}$. Thus

$$T_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{[\alpha_1, \alpha_2], \mathbf{0}\} = \text{rmin}\{T_A(x), T_A(y)\},$$

$$I_A(x \cdot y) \succeq \mathbf{1} = \text{rmax}\{[\beta_1, \beta_2], \mathbf{1}\} = \text{rmax}\{I_A(x), I_A(y)\},$$

$$F_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{[\omega_1, \omega_2], \mathbf{0}\} = \text{rmin}\{F_A(x), F_A(y)\}.$$

Case (iii): If $x \notin B$ and $y \in B$, then $T_A(x) = \mathbf{0}$, $I_A(x) = \mathbf{1}$, $F_A(x) = \mathbf{0}$, $T_A(y) = [\alpha_1, \alpha_2]$, $I_A(y) = [\beta_1, \beta_2]$, and $F_A(y) = [\omega_1, \omega_2]$. Thus

$$T_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{\mathbf{0}, [\alpha_1, \alpha_2]\} = \text{rmin}\{T_A(x), T_A(y)\},$$

$$I_A(x \cdot y) \preceq \mathbf{1} = \text{rmax}\{\mathbf{1}, [\beta_1, \beta_2]\} = \text{rmax}\{I_A(x), I_A(y)\},$$

$$F_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{\mathbf{0}, [\omega_1, \omega_2]\} = \text{rmin}\{F_A(x), F_A(y)\}.$$

Case (iv): If $x \notin B$ and $y \notin B$, then $T_A(x) = \mathbf{0}$, $I_A(x) = \mathbf{1}$, $F_A(x) = \mathbf{0}$, $T_A(y) = \mathbf{0}$, $I_A(y) = \mathbf{1}$, and $F_A(y) = \mathbf{0}$. Thus

$$T_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{\mathbf{0}, \mathbf{0}\} = \text{rmin}\{T_A(x), T_A(y)\},$$

$$I_A(x \cdot y) \preceq \mathbf{1} = \text{rmax}\{\mathbf{1}, \mathbf{1}\} = \text{rmax}\{I_A(x), I_A(y)\},$$

$$F_A(x \cdot y) \succeq \mathbf{0} = \text{rmin}\{\mathbf{0}, \mathbf{0}\} = \text{rmin}\{F_A(x), F_A(y)\}.$$

Therefore, A is an IVN subalgebra of X. □

Theorem 3.16. *Let B be a nonempty subset of a Hilbert algebra $X = (X, \cdot, 1)$ and A be an IVNS in X defined by*

$$T_A(x) = \begin{cases} [\alpha_1, \alpha_2] & \text{if } x \in B \\ \mathbf{0} & \text{otherwise}, \end{cases}$$

$$I_A(x) = \begin{cases} [\beta_1, \beta_2] & \text{if } x \in B \\ \mathbf{1} & \text{otherwise}, \end{cases}$$

$$F_A(x) = \begin{cases} [\omega_1, \omega_2] & \text{if } x \in B \\ \mathbf{0} & \text{otherwise} \end{cases}$$

for $[\alpha_1, \alpha_2], [\beta_1, \beta_2], [\omega_1, \omega_2] \in D[0, 1]$ and $\alpha_2 + \beta_2 + \omega_2 \leq 1$. If A is realized as a lower level subalgebra and an upper level subalgebra of some IVN subalgebra of X, then B is a subalgebra of X.

Proof. Let A be an IVN subalgebra of X and let $x, y \in B$. Then $T_A(x) = [\alpha_1, \alpha_2] = T_A(y)$, $I_A(x) = [\beta_1, \beta_2] = I_A(y)$, and $F_A(x) = [\omega_1, \omega_2] = F_A(y)$. Thus

$$T_A(x \cdot y) \succeq \text{rmin}\{T_A(x), T_A(y)\} = \text{rmin}\{[\alpha_1, \alpha_2], [\alpha_1, \alpha_2]\} = [\alpha_1, \alpha_2],$$

$$I_A(x \cdot y) \preceq \text{rmax}\{I_A(x), I_A(y)\} = \text{rmax}\{[\beta_1, \beta_2], [\beta_1, \beta_2]\} = [\beta_1, \beta_2],$$

$$F_A(x \cdot y) \succeq \text{rmin}\{F_A(x), F_A(y)\} = \text{rmin}\{[\omega_1, \omega_2], [\omega_1, \omega_2]\} = [\omega_1, \omega_2],$$

which implies that $x \cdot y \in B$. Hence, B is a subalgebra of X. □

Definition 3.17. Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be IVNSs in Hilbert algebras X and Y, respectively. The cartesian product $A \times B = \{((x, y), (T_A \times T_B)(x, y), (I_A \times I_B)(x, y), (F_A \times F_B)(x, y)) \mid x \in X, y \in Y\}$ is defined by

$$(\forall (x, y) \in X \times Y) \begin{pmatrix} (T_A \times T_B)(x, y) = \text{rmin}\{T_A(x), T_B(y)\} \\ (I_A \times I_B)(x, y) = \text{rmax}\{I_A(x), I_B(y)\} \\ (F_A \times F_B)(x, y) = \text{rmin}\{F_A(x), F_B(y)\} \end{pmatrix},$$

where $T_A \times T_B : X \times Y \to D[0,1]$, $I_A \times I_B : X \times Y \to D[0,1]$, and $F_A \times F_B : X \times Y \to D[0,1]$.

Remark 3.18. Let X and Y be Hilbert algebras. We define the binary operation $\cdot$ on $X \times Y$ by $(x,y)\cdot(u,v) = (x\cdot u, y\cdot v)$ for every $(x,y),(u,v) \in X \times Y$. Then clearly $(X \times Y, \cdot, (1,1))$ is a Hilbert algebra.

Proposition 3.19. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are IVN subalgebras of Hilbert algebras X and Y, respectively, then the cartesian product $A \times B$ is also an IVN subalgebra of $X \times Y$.*

Proof. Let $(x_1,y_1),(x_2,y_2) \in X \times Y$. Then

$$\begin{aligned}
&(T_A \times T_B)((x_1,y_1)\cdot(x_2,y_2))\\
&= (T_A \times T_B)((x_1\cdot x_2),(y_1\cdot y_2))\\
&= \mathrm{rmin}\{T_A(x_1\cdot x_2), T_B(y_1\cdot y_2)\}\\
&\succeq \mathrm{rmin}\{\mathrm{rmin}\{T_A(x_1),T_A(x_2)\},\mathrm{rmin}\{T_B(y_1),T_B(y_2)\}\}\\
&= \mathrm{rmin}\{\mathrm{rmin}\{T_A(x_1),T_B(y_1)\},\mathrm{rmin}\{T_A(x_2),T_B(y_2)\}\}\\
&= \mathrm{rmin}\{(T_A \times T_B)(x_1,y_1),(T_A \times T_B)(x_2,y_2)\},
\end{aligned}$$

$$\begin{aligned}
&(I_A \times I_B)((x_1,y_1)\cdot(x_2,y_2))\\
&= (I_A \times I_B)((x_1\cdot x_2),(y_1\cdot y_2))\\
&= \mathrm{rmin}\{I_A(x_1\cdot x_2), I_B(y_1\cdot y_2)\}\\
&\preceq \mathrm{rmin}\{\mathrm{rmax}\{I_A(x_1),I_A(x_2)\},\mathrm{rmax}\{I_B(y_1),I_B(y_2)\}\}\\
&= \mathrm{rmax}\{\mathrm{rmin}\{I_A(x_1),I_B(y_1)\},\mathrm{rmin}\{I_A(x_2),I_B(y_2)\}\}\\
&= \mathrm{rmax}\{(T_A \times T_B)(x_1,y_1),(T_A \times T_B)(x_2,y_2)\},
\end{aligned}$$

$$\begin{aligned}
&(F_A \times F_B)((x_1,y_1)\cdot(x_2,y_2))\\
&= (F_A \times F_B)((x_1\cdot x_2),(y_1\cdot y_2))\\
&= \mathrm{rmin}\{F_A(x_1\cdot x_2), F_B(y_1\cdot y_2)\}\\
&\succeq \mathrm{rmin}\{\mathrm{rmin}\{F_A(x_1),F_A(x_2)\},\mathrm{rmin}\{F_B(y_1),F_B(y_2)\}\}\\
&= \mathrm{rmin}\{\mathrm{rmin}\{F_A(x_1),F_B(y_1)\},\mathrm{rmin}\{F_A(x_2),F_B(y_2)\}\}\\
&= \mathrm{rmin}\{(F_A \times F_B)(x_1,y_1),(F_A \times F_B)(x_2,y_2)\}.
\end{aligned}$$

Hence, $A \times B$ is an IVN subalgebra of $X \times Y$. □

Lemma 3.20. *If $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ are IVN subalgebras of Hilbert algebras X and Y, respectively, then $\oplus(A \times B)$, $\otimes(A \times B)$, and $\odot(A \times B)$ are IVN subalgebras of $X \times Y$.*

Proof. It follows from Theorem 3.9 and Proposition 3.19. □

Theorem 3.21. *Let $A = (T_A, I_A, F_A)$ and $B = (T_B, I_B, F_B)$ be any two IVNSs in Hilbert algebras X and Y, respectively. Then $A \times B$ is an IVN subalgebra of $X \times Y$ if and only if the nonempty upper $[s_1, s_2]$-level cut $U(T_A \times T_B : [s_1, s_2])$, the nonempty lower $[t_1, t_2]$-level cut $L(I_A \times I_B : [t_1, t_2])$, and the nonempty upper $[u_1, u_2]$-level cut $U(F_A \times F_B : [u_1, u_2])$ are subalgebras of $X \times Y$ for all $[s_1, s_2], [t_1, t_2], [u_1, u_2] \in D[0, 1]$.*

Proof. It follows from Theorems 3.13 and 3.14. □

For any fixed interval numbers $\tilde{a}^+, \tilde{a}^-, \tilde{b}^+, \tilde{b}^-, \tilde{c}^+, \tilde{c}^- \in D[0,1]$ such that $\tilde{a}^+ \succ \tilde{a}^-$, $\tilde{b}^+ \succ \tilde{b}^-$, $\tilde{c}^+ \succ \tilde{c}^-$ and a nonempty subset G of a Hilbert algebra X, the IVNS

$$A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix} = \left(T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix}, I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix}, F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} \right)$$

in X is defined by for all $x \in X$,

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x) = \begin{cases} \tilde{a}^+ & \text{if } x \in G \\ \tilde{a}^- & \text{otherwise,} \end{cases}$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (x) = \begin{cases} \tilde{b}^- & \text{if } x \in G \\ \tilde{b}^+ & \text{otherwise,} \end{cases}$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x) = \begin{cases} \tilde{c}^+ & \text{if } x \in G \\ \tilde{c}^- & \text{otherwise.} \end{cases}$$

Lemma 3.22. *If the constant 1 of a Hilbert algebra $X = (X, \cdot, 1)$ is in a nonempty subset G of X, then the IVNS $A^G \begin{bmatrix} \tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+ \\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^- \end{bmatrix}$ in X satisfies the condition (3.2).*

Proof. If $1 \in G$, then $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (1) = \tilde{a}^+$, $I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (1) = \tilde{b}^-$, and $F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (1) = \tilde{c}^+$. Thus

$$(\forall x \in X) \left(\begin{array}{l} T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (1) = \tilde{a}^+ \succeq T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x) \\ I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (1) = \tilde{b}^- \preceq I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (x) \\ F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (1) = \tilde{c}^+ \succeq F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x) \end{array} \right). \tag{3.5}$$

Hence, $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ satisfies the condition (3.2). □

Lemma 3.23. *If the IVNS* $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ *in a Hilbert algebra* $X = (X, \cdot, 1)$ *satisfies the condition* (3.2)*, then the constant 1 of* X *is in a nonempty subset* G *of* X*.*

Proof. Assume that $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ satisfies the condition (3.2). Then $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(1) \succeq T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x)$ for all $x \in X$. Since G is nonempty, there exists $g \in G$. Thus $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(g) = \tilde{a}^+$ and so $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(1) \succeq T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(g) = \tilde{a}^+ \succeq T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(1)$, that is, $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(1) = \tilde{a}^+$. Hence, $1 \in G$. □

Theorem 3.24. *The IVNS* $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ *in a Hilbert algebra* $X = (X, \cdot, 1)$ *is an IVN subalgebra of* X *if and only if a nonempty subset* G *of* X *is a subalgebra of* X*.*

Proof. Assume that $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\ \tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ is an IVN subalgebra of X. Let $x, y \in G$. Then $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x) = \tilde{a}^+ = T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(y)$. Thus

$$\begin{aligned}
T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x \cdot y) &\succeq \operatorname{rmin}\left\{T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x), T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(y)\right\}\\
&= \operatorname{rmin}\{\tilde{a}^+, \tilde{a}^+\}\\
&= \tilde{a}^+\\
&\succeq T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x \cdot y)
\end{aligned}$$

and so $T_A^G\begin{bmatrix}\tilde{a}^+\\ \tilde{a}^-\end{bmatrix}(x \cdot y) = \tilde{a}^+$. Thus $x \cdot y \in G$. Hence, G is a subalgebra of X.

Conversely, assume that G is a subalgebra of X. Let $x, y \in X$. We consider the following cases:

Case (1): Let $x, y \in G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x) = \tilde{a}^+ = T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y),$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (x) = \tilde{b}^- = I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y),$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x) = \tilde{c}^+ = F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y).$$

Since G is a subalgebra of X, we have $x \cdot y \in G$ and so $T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x \cdot y) = \tilde{a}^+$, $I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (x \cdot y) = \tilde{b}^-$, and $F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x \cdot y) = \tilde{c}^+$. It follows from (2.1) that

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x \cdot y) = \tilde{a}^+ \succeq \tilde{a}^+ = \text{rmin}\{\tilde{a}^+, \tilde{a}^+\} = \text{rmin} \left\{ T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y) \right\},$$

$$I_A^G \begin{bmatrix} \tilde{b}^+ \\ \tilde{b}^- \end{bmatrix} (x \cdot y) = \tilde{b}^- \preceq \tilde{b}^- = \text{rmax}\{\tilde{b}^-, \tilde{b}^-\} = \text{rmax} \left\{ I_A^G \begin{bmatrix} \tilde{b}^+ \\ \tilde{b}^- \end{bmatrix} (x), T_A^G \begin{bmatrix} \tilde{b}^+ \\ \tilde{b}^- \end{bmatrix} (y) \right\},$$

$$F_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x \cdot y) = \tilde{c}^+ \succeq \tilde{c}^+ = \text{rmin}\left\{\tilde{c}^+, \tilde{c}^+\right\} = \text{rmin} \left\{ T_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x), T_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y) \right\}.$$

Case 2: Let $x \notin G$ or $y \notin G$. Then

$$T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x) = \tilde{a}^- \text{ or } T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y) = \tilde{a}^-,$$

$$I_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (x) = \tilde{b}^+ \text{ or } T_A^G \begin{bmatrix} \tilde{b}^- \\ \tilde{b}^+ \end{bmatrix} (y) = \tilde{b}^+,$$

$$T_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (x) = \tilde{c}^- \text{ or } T_A^G \begin{bmatrix} \tilde{c}^+ \\ \tilde{c}^- \end{bmatrix} (y) = \tilde{c}^-.$$

It follows from (2.1) that

$$\text{rmin} \left\{ T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (x), T_A^G \begin{bmatrix} \tilde{a}^+ \\ \tilde{a}^- \end{bmatrix} (y) \right\} = \text{rmin}\left\{\tilde{a}^+, \tilde{a}^-\right\} = \tilde{a}^-,$$

$$\text{rmax}\left\{I_A^G\begin{bmatrix}\tilde{b}^-\\\tilde{a}^+\end{bmatrix}(x), I_A^G\begin{bmatrix}\tilde{b}^-\\\tilde{b}^+\end{bmatrix}(y)\right\} = \text{rmax}\{\tilde{b}^+,\tilde{b}^+\} = \tilde{b}^+,$$

$$\text{rmin}\left\{F_A^G\begin{bmatrix}\tilde{c}^+\\\tilde{c}^-\end{bmatrix}(x), F_A^G\begin{bmatrix}\tilde{c}^+\\\tilde{c}^-\end{bmatrix}(y)\right\} = \text{rmin}\left\{\tilde{c}^-,\tilde{c}^-\right\} = \tilde{c}^-.$$

Therefore,

$$T_A^G\begin{bmatrix}\tilde{a}^+\\\tilde{a}^-\end{bmatrix}(x\cdot y) \succeq \tilde{a}^- = \text{rmin}\left\{T_A^G\begin{bmatrix}\tilde{a}^+\\\tilde{a}^-\end{bmatrix}(x), T_A^G\begin{bmatrix}\tilde{a}^+\\\tilde{a}^-\end{bmatrix}(y)\right\},$$

$$I_A^G\begin{bmatrix}\tilde{b}^-\\\tilde{b}^+\end{bmatrix}(x\cdot y) \preceq \tilde{b}^+ = \text{rmax}\left\{I_A^G\begin{bmatrix}\tilde{b}^-\\\tilde{b}^+\end{bmatrix}(x), I_A^G\begin{bmatrix}\tilde{b}^-\\\tilde{b}^+\end{bmatrix}(y)\right\},$$

$$F_A^G\begin{bmatrix}\tilde{c}^+\\\tilde{c}^-\end{bmatrix}(x\cdot y) \succeq \tilde{c}^- = \text{rmin}\left\{F_A^G\begin{bmatrix}\tilde{c}^+\\\tilde{c}^-\end{bmatrix}(x), F_A^G\begin{bmatrix}\tilde{c}^+\\\tilde{c}^-\end{bmatrix}(y)\right\}.$$

Hence, $A^G\begin{bmatrix}\tilde{a}^+, & \tilde{b}^-, & \tilde{c}^+\\\tilde{a}^-, & \tilde{b}^+, & \tilde{c}^-\end{bmatrix}$ is an IVN subalgebra of X. □

4. Images and inverse images of interval-valued neutrosophic subalgebras

A mapping $f : X \to Y$ of Hilbert algebras X and Y is called a *homomorphism* if $f(x\cdot y) = f(x)\cdot f(y)$ for all $x, y \in X$. Note that if $f : X \to Y$ is a homomorphism of Hilbert algebras X and Y, then $f(1) = 1$. Let $f : X \to Y$ be a homomorphism of Hilbert algebras X and Y. For any IVNS $A = (T_A, I_A, F_A)$ in Y, we define the IVNS $f^{-1}(A) = (T_{f^{-1}(A)}, I_{f^{-1}(A)}, F_{f^{-1}(A)})$ in X by

$$(\forall x\in X)\begin{pmatrix}T_{f^{-1}(A)}(x) = T_A(f(x))\\ I_{f^{-1}(A)}(x) = I_A(f(x))\\ F_{f^{-1}(A)}(x) = F_A(f(x))\end{pmatrix}.$$

Proposition 4.1. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X into a Hilbert algebra Y. If $A = (T_A, I_A, F_A)$ is an IVN subalgebra of Y, then the inverse image $f^{-1}(A)$ of A is an IVN subalgebra of X.*

Proof. Let $x, y \in X$. Then

$$\begin{aligned}T_{f^{-1}(A)}(x\cdot y) &= T_A(f(x\cdot y))\\ &= T_A(f(x)\cdot f(y))\\ &\succeq \text{rmin}\{T_A(f(x)), T_A(f(y))\}\\ &= \text{rmin}\{T_{f^{-1}(A)}(x), T_{f^{-1}(A)}(y)\},\end{aligned}$$

$$\begin{aligned} I_{f^{-1}(A)}(x\cdot y) &= I_A(f(x\cdot y)) \\ &= I_A(f(x)\cdot f(y)) \\ &\preceq \operatorname{rmax}\{I_A(f(x)), I_A(f(y))\} \\ &= \operatorname{rmax}\{I_{f^{-1}(A)}(x), I_{f^{-1}(A)}(y)\}, \end{aligned}$$

$$\begin{aligned} F_{f^{-1}(A)}(x\cdot y) &= F_A(f(x\cdot y)) \\ &= F_A(f(x)\cdot f(y)) \\ &\succeq \operatorname{rmin}\{F_A(f(x)), F_A(f(y))\} \\ &= \operatorname{rmin}\{F_{f^{-1}(A)}(x), F_{f^{-1}(A)}(y)\}. \end{aligned}$$

Hence, $f^{-1}(A)$ of A is an IVN subalgebra of X. □

Let $f : X \to Y$ be an onto homomorphism of Hilbert algebras X and Y. For any IVNS $A = (T_A, I_A, F_A)$ in X, we define the IVNS $f(A) = (T_{f(A)}, I_{f(A)}, F_{f(A)})$ in Y by

$$(\forall y \in Y) \begin{pmatrix} T_{f(A)}(y) = \operatorname{rsup}\{T_A(x) \mid x \in f^{-1}(y)\} \\ I_{f(A)}(y) = \operatorname{rinf}\{I_A(x) \mid x \in f^{-1}(y)\} \\ F_{f(A)}(y) = \operatorname{rsup}\{F_A(x) \mid x \in f^{-1}(y)\} \end{pmatrix}.$$

Definition 4.2. An IVNS $A = (T_A, I_A, F_A)$ in a Hilbert algebra X is said to have the rsup-property and rinf-property if for any nonempty subset T of X, there exists $t_0 \in T$ such that $T_A(t_0) = \operatorname{rsup}_{t\in T} T_A(t)$, $I_A(t_0) = \operatorname{rinf}_{t\in T} T_A(t)$, and $F_A(t_0) = \operatorname{rsup}_{t\in T} F_A(t)$.

Proposition 4.3. *Let $f : X \to Y$ be a homomorphism of a Hilbert algebra X onto a Hilbert algebra Y. If $A = (T_A, I_A, F_A)$ is an IVN subalgebra of X that has the* rsup*-property and* rinf*-property, then the image $f(A)$ of A is an IVN subalgebra of Y.*

Proof. Let $A = (T_A, I_A, F_A)$ be an IVN subalgebra of X and $y_1, y_2 \in Y$. Then

$$\begin{aligned} &T_{f(A)}(y_1\cdot y_2) \\ &= \operatorname{rsup}\{T_A(x) \mid x \in f^{-1}(y_1\cdot y_2)\} \\ &\succeq \operatorname{rsup}\{T_A(x_1\cdot x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &\succeq \operatorname{rsup}\{\operatorname{rmin}\{T_A(x_1, T_A(x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\} \\ &= \operatorname{rmin}\{\operatorname{rsup}\{T_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \operatorname{rsup}\{T_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\} \\ &= \operatorname{rmin}\{T_{f(A)}(y_1), T_{f(A)}(y_2)\}, \end{aligned}$$

$$\begin{aligned}
&I_{f(A)}(y_1 \cdot y_2)\\
&= \text{rinf}\{I_A(x) \mid x \in f^{-1}(y_1 \cdot y_2)\}\\
&\preceq \text{rinf}\{I_A(x_1 \cdot x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}\\
&\preceq \text{rinf}\{\text{rmax}\{I_A(x_1, I_A(x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}\\
&= \text{rmax}\{\text{rinf}\{I_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \text{rinf}\{I_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\}\\
&= \text{rmax}\{I_{f(A)}(y_1), I_{f(A)}(y_2)\},
\end{aligned}$$

$$\begin{aligned}
&F_{f(A)}(y_1 \cdot y_2)\\
&= \text{rsup}\{F_A(x) \mid x \in f^{-1}(y_1 \cdot y_2)\}\\
&\succeq \text{rsup}\{F_A(x_1 \cdot x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}\\
&\succeq \text{rsup}\{\text{rmin}\{F_A(x_1, T_A(x_2)\} \mid x_1 \in f^{-1}(y_1), x_2 \in f^{-1}(y_2)\}\\
&= \text{rmin}\{\text{rsup}\{F_A(x_1) \mid x_1 \in f^{-1}(y_1)\}, \text{rsup}\{F_A(x_2) \mid x_2 \in f^{-1}(y_2)\}\}\\
&= \text{rmin}\{F_{f(A)}(y_1), F_{f(A)}(y_2)\}.
\end{aligned}$$

Hence, $f(A)$ is an IVN subalgebra of Y. □

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