Asia Pac. J. Math. 2022 **9**:17

ON LEFT AND RIGHT WEAK-INTERIOR BASES OF SEMIGROUPS

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Received May 4, 2022

Abstract. In this paper, we introduce the notions of left and right weak-interior bases of a semigroup and present some examples. We focus only the results for right weak-interior bases of a semigroup. For left weak-interior bases of a semigroup, we can show dually. We define the quasi-order using principal left (right) weak-interior ideals of a semigroup and give a characterization when a non-empty subset of a semigroup is a right weak-interior base of a semigroup. Finally, we give a characterization when a right weak-interior base of a semigroup is a subsemigroup.

2010 Mathematics Subject Classification. 20M10; 20M12.

Key words and phrases. semigroup; left (right) weak-interior ideal; left (right) weak-interior base; quasi-order.

1. Introduction

Ideal theory and base theory play an important role in studying semigroup and other algebraic structures. The study on generalizations of ideals in algebraic structures is necessary for further study of algebraic structures. We know that the notion of a one-sided ideal of any algebraic structure is a generalization of the notion of an ideal. The quasi ideals are generalizations of left ideals and right ideals. The weak-interior ideals are generalizations of quasi-ideals and interior ideals. The notion of weak-interior ideals of semigroups was introduced in [18,19] by Rao, who gave the definitions of left and right weak-interior ideals and studied their properties in [18]. A one-sided base is the smallest set generated one-sided ideal

DOI: 10.28924/APJM/9-17

under some condition. The notion of one-sided bases of a semigroup was first introduced by Tamura in [20]. Later, Fabrici introduced and studied the structure of semigroups containing one-sided bases and two sided-bases in [3,4]. The notions of one-sided bases and two-sided bases have been studied in other algebraic structures, for example [1,2,5,7,9–11,15,21,22]. In 2017, Kummoon and Changphas introduced the notion of bi-bases of a semigroup in [13]. Furthermore, the notion of bi-bases was introduced and studied in other algebraic structures, for example [6,8,14,23]. Recently, Panate, Chatthong and Nakkhasen introduced the notion of interior bases of semigroups in [17]. These are motivated to research in this paper. In this paper, the notions of left and right weak-interior bases of a semigroup are introduced and the structure of a semigroup containing right weak-interior bases will be studied. For the structure of a semigroup containing left weak-interior bases can be considered similarly.

2. Preliminaries

In this section, we provide definitions and results that used throughout this paper. Those can be found in [13,16–19].

Definition 2.1. A semigroup is a set S together with a binary operation $\cdot : S \times S \to S$ that satisfies the associative property $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in S$.

For simplicity, we write ab for $a \cdot b$. Let A and B be non-empty subsets of a semigroup S, we denote $AB := \{ab \mid a \in A \text{ and } b \in B\}$. For $a \in S$, we write aB for $\{a\}B$.

Definition 2.2. A non-empty subset A of a semigroup S is called a subsemigroup of S if $ab \in A$ for all $a, b \in A$ or $AA \subseteq A$.

Definition 2.3. A non-empty subset A of a semigroup S is called a left (right) ideal of S if $SA \subseteq A$ $(AS \subseteq A)$.

Definition 2.4. [18] A non-empty subset A of a semigroup S is called a left (right) weak-interior ideal of S if A is a subsemigroup of S and $SAA \subseteq A$ $(AAS \subseteq A)$.

From Theorem 4.5 of [18], we know that every left (right) ideal of a semigroup S is a left (right) weak-interior ideal of S, but the converse is not generally true. The following example shows that a left (right) weak-interior ideal of S is not a left (right) ideal of S.

Example 2.5. Let $S = \{a, b, c, d\}$ be a semigroup ([16]) with the binary operation defined by:

$\cdot$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	a	a
d	a	a	b	a

We have that $A = \{a, c\}$ is a left weak-interior ideal and $B = \{a, d\}$ is a right weak-interior ideal of S. But A is not a left ideal of S since $SA = \{a, b\} \nsubseteq A$, and B is not a right ideal of S since $BS = \{a, b\} \nsubseteq B$.

Lemma 2.6. Let S be a semigroup and A_i be a subsemigroup of S for each i in an indexed set I. If $\bigcap_{i\in I} A_i \neq \varnothing$, then $\bigcap_{i\in I} A_i$ is a subsemigroup of S.

Proof. Assume that $\bigcap_{i\in I} A_i \neq \varnothing$. Let $a, b \in \bigcap_{i\in I} A_i$. Then $a, b \in A_i$ for all $i \in I$. Since A_i is a subsemigroup of S for all $i \in I$, we obtain that $ab \in A_i$ for all $i \in I$. Thus, $ab \in \bigcap_{i\in I} A_i$. This shows that $\bigcap_{i\in I} A_i$ is a subsemigroup of S. □

Lemma 2.7. Let S be a semigroup. Then the following statements hold.

(1) For any A_i is a left weak-interior ideal of S for all $i \in I$, if $\bigcap_{i\in I} A_i \neq \varnothing$, then $\bigcap_{i\in I} A_i$ is a left weak-interior ideal of S.

(2) For any A_i is a right weak-interior ideal of S for all $i \in I$, if $\bigcap_{i\in I} A_i \neq \varnothing$, then $\bigcap_{i\in I} A_i$ is a right weak-interior ideal of S.

Proof. We will prove the statement (1). Let A_i be a left weak-interior ideal of S for all $i \in I$. Suppose that $\bigcap_{i\in I} A_i \neq \varnothing$. By Lemma 2.6, we obtain that $\bigcap_{i\in I} A_i$ is a subsemigroup of S. Next, to show that $S(\bigcap_{i\in I} A_i)(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} A_i$, we let $x \in S(\bigcap_{i\in I} A_i)(\bigcap_{i\in I} A_i)$. Then $x = sab$ for some $s \in S$ and $a, b \in \bigcap_{i\in I} A_i$. Since $a, b \in \bigcap_{i\in I} A_i$, we have $a, b \in A_i$ for all $i \in I$. So, we obtain that $x = sab \in SA_iA_i \subseteq A_i$ for all $i \in I$. Thus, $x \in \bigcap_{i\in I} A_i$. Hence, $S(\bigcap_{i\in I} A_i)(\bigcap_{i\in I} A_i) \subseteq \bigcap_{i\in I} A_i$. This shows that $\bigcap_{i\in I} A_i$ is a left weak-interior ideal of S. The statement (2) can be proved similarly. □

Let A be a non-empty subset of a semigroup S. Then the intersection of all left (right) weak-interior ideals of S containing A, denoted by $L_{wi}(A)$ ($R_{wi}(A)$) is the smallest left (right) weak-interior ideal of S containing A. For any $a \in S$, we write $L_{wi}(\{a\})$ and $R_{wi}(\{a\})$ by $L_{wi}(a)$ and $R_{wi}(a)$, respectively.

Lemma 2.8. Let A be a non-empty subset of a semigroup S. Then the following statements hold.

(1) $L_{wi}(A) = A \cup AA \cup SAA$.

(2) $R_{wi}(A) = A \cup AA \cup AAS$.

Proof. We will prove the statement (1). Let $L = A \cup AA \cup SAA$. Obviously, $A \subseteq L$. First, we consider

$$\begin{aligned} LL &= (A \cup AA \cup SAA)(A \cup AA \cup SAA) \\ &\subseteq AA \cup SAA \\ &\subseteq A \cup AA \cup SAA = L \end{aligned}$$

and

$$\begin{aligned} SLL &= S(A \cup AA \cup SAA)(A \cup AA \cup SAA) \\ &\subseteq SAA \\ &\subseteq A \cup AA \cup SAA = L. \end{aligned}$$

Thus, L is a left weak-interior ideal of S. Next, let L' be a left weak-interior ideal of S containing A. So, we have $A \subseteq L'$, $AA \subseteq L'L' \subseteq L'$ and $SAA \subseteq SL'L' \subseteq L'$. It follows that $L = A \cup AA \cup SAA \subseteq L'$. Hence, L is the smallest left weak-interior ideal of S containing A. Therefore, $L_{wi}(A) = A \cup AA \cup SAA$. The statement (2) can be proved similarly. □

3. Main Results

We begin this section with the definition of left and right weak-interior bases of a semigroup as follows.

Definition 3.1. A non-empty subset A of a semigroup S is called a right (resp. left) weak-interior base of S if it satisfies the following two conditions:

(1) $S = L_{wi}(A)$ (resp. $S = R_{wi}(A)$);

(2) if B is a subset of A such that $S = L_{wi}(B)$ (resp. $S = R_{wi}(B)$), then $B = A$.

Example 3.2. Let $S = \{a, b, c, d, e\}$ be a semigroup ([12]) with the binary operation defined by:

$\cdot$	a	b	c	d	e
a	a	a	a	a	a
b	a	a	a	a	a
c	a	a	c	a	c
d	d	d	d	d	d
d	d	d	e	d	e

We have that the right weak-interior bases of S are $\{b, c\}$ and $\{b, e\}$. And the left weak-interior base of S is $\{b, c, e\}$.

Beside the right and left weak-interior base of a semigroup S, we define the quasi-order on S as follows:

Definition 3.3. Let S be a semigroup. Define a quasi-order on S by, for any $a, b \in S$,

$$a \leqslant b :\Leftrightarrow L_{wi}(a) \subseteq L_{wi}(b).$$

The following example shows that the relation $\leqslant$ defined above is not, in general, a partial order.

Example 3.4. Let $S = \{a, b, c, d\}$ be a semigroup ([17]) with the binary operation defined by:

$\cdot$	a	b	c	d
a	a	b	a	a
b	b	a	b	b
c	a	b	c	d
d	a	b	d	c

We have that $\{c\}$ and $\{d\}$ are right weak-interior bases of S. Moreover, we obtain that $L_{wi}(c) \subseteq L_{wi}(d)$ (i.e., $c \leqslant d$) and $L_{wi}(d) \subseteq L_{wi}(c)$ (i.e., $d \leqslant c$), but $c \neq d$. Thus, $\leqslant$ is not a partial order on S.

We shall consider only right weak-interior base, since for left weak-interior base analogous statements hold.

Lemma 3.5. Let A be a right weak-interior base of a semigroup S. For any a, b in A, if $a \in bb \cup Sbb$, then $a = b$.

Proof. Let a, b be any elements of A such that $a \in bb \cup Sbb$ and $a \neq b$. We set $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$, we have $b \in B$. Now, to show that $L_{wi}(B) = S$. Clearly, $L_{wi}(B) \subseteq S$. Let $x \in S$. Since A is a right weak-interior base of S, we have $S = L_{wi}(A)$ and so $x \in L_{wi}(A) = A \cup AA \cup SAA$. Then there are three cases to consider:

Case 1: $x \in A$.

Subcase 1.1: $x \neq a$. Then $x \in A \setminus \{a\} = B \subseteq L_{wi}(B)$.

Subcase 1.2: $x = a$. Since $a \in bb \cup Sbb$, it follows that

$$x = a \in bb \cup Sbb \subseteq BB \cup SBB \subseteq L_{wi}(B).$$

Case 2: $x \in AA$. Then $x = a_1a_2$ for some $a_1, a_2 \in A$. We have following four subcases:

Subcase 2.1: $a_1 = a$ and $a_2 = a$. Since $a \in bb \cup Sbb$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (bb \cup Sbb)(bb \cup Sbb) \\ &= bbbb \cup bbSbb \cup Sbbbb \cup SbbSbb \\ &\subseteq BBBB \cup BBSBB \cup SBBBB \cup SBBSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.2: $a_1 \neq a$ and $a_2 = a$. Since $B = A \setminus \{a\}$ and $a \in bb \cup Sbb$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (A \setminus \{a\})(bb \cup Sbb) \\ &= (A \setminus \{a\})bb \cup (A \setminus \{a\})Sbb \\ &\subseteq BBB \cup BSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.3: $a_1 = a$ and $a_2 \neq a$. Since $a \in bb \cup Sbb$ and $B = A \setminus \{a\}$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (bb \cup Sbb)(A \setminus \{a\}) \\ &= bb(A \setminus \{a\}) \cup Sbb(A \setminus \{a\}) \\ &\subseteq BBB \cup SBBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.4: $a_1 \neq a$ and $a_2 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$x = a_1a_2 \in (A \setminus \{a\})(A \setminus \{a\}) = BB \subseteq L_{wi}(B).$$

Case 3: $x \in SAA$. Then $x = sa_3a_4$ for some $s \in S$ and $a_3, a_4 \in A$. We have following four subcases:

Subcase 3.1: $a_3 = a$ and $a_4 = a$. Since $a \in bb \cup Sbb$, it follows that

$$
\begin{aligned}
x = sa_3a_4 &\in S(bb \cup Sbb)(bb \cup Sbb) \\
&= Sbbbb \cup SbbSbb \cup SSbbbb \cup SSbbSbb \\
&\subseteq SBBBB \cup SBBSBB \cup SSBBBB \cup SSBBSBB \\
&\subseteq SBB \subseteq L_{wi}(B).
\end{aligned}
$$

Subcase 3.2: $a_3 \neq a$ and $a_4 = a$. Since $B = A \setminus \{a\}$ and $a \in bb \cup Sbb$, it follows that

$$
\begin{aligned}
x = sa_3a_4 &\in S(A \setminus \{a\})(bb \cup Sbb) \\
&= S(A \setminus \{a\})bb \cup S(A \setminus \{a\})Sbb \\
&\subseteq SBBB \cup SBSBB \\
&\subseteq SBB \subseteq L_{wi}(B).
\end{aligned}
$$

Subcase 3.3: $a_3 = a$ and $a_4 \neq a$. Since $a \in bb \cup Sbb$ and $B = A \setminus \{a\}$, it follows that

$$
\begin{aligned}
x = sa_3a_4 &\in S(bb \cup Sbb)(A \setminus \{a\}) \\
&= Sbb(A \setminus \{a\}) \cup SSbb(A \setminus \{a\}) \\
&\subseteq SBBB \cup SSBBB \\
&\subseteq SBB \subseteq L_{wi}(B).
\end{aligned}
$$

Subcase 3.4: $a_3 \neq a$ and $a_4 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$x = sa_3a_4 \in S(A \setminus \{a\})(A \setminus \{a\}) = SBB \subseteq L_{wi}(B).$$

In all this cases, we obtain that $x \in L_{wi}(B)$, and so $S \subseteq L_{wi}(B)$. Hence, $L_{wi}(B) = S$. This is a contradiction. Therefore, $a = b$ as required. $\square$

Lemma 3.6. Let A be a right weak-interior base of a semigroup S. For any a, b, c in A, if $a \in bc \cup Sbc$, then $a = b$ or $a = c$.

Proof. Let a, b, c be any elements of A such that $a \in bc \cup Sbc$. Suppose that $a \neq b$ and $a \neq c$. We set $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$ and $a \neq c$, we have $b, c \in B$. We will show that $L_{wi}(A) \subseteq L_{wi}(B)$. Let $x \in L_{wi}(A) = A \cup AA \cup SAA$. Then there are three cases to consider:

Case 1: $x \in A$.

Subcase 1.1: $x \neq a$. Then $x \in A \setminus \{a\} = B \subseteq L_{wi}(B)$.

Subcase 1.2: $x = a$. Since $a \in bc \cup Sbc$, it follows that

$$x = a \in bc \cup Sbc \subseteq BB \cup SBB \subseteq L_{wi}(B).$$

Case 2: $x \in AA$. Then $x = a_1a_2$ for some $a_1, a_2 \in A$. We have following four subcases:

Subcase 2.1: $a_1 = a$ and $a_2 = a$. Since $a \in bc \cup Sbc$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (bc \cup Sbc)(bc \cup Sbc) \\ &= bcbc \cup bcSbc \cup Sbcbc \cup SbcSbc \\ &\subseteq BBBB \cup BBSBB \cup SBBBB \cup SBBSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.2: $a_1 \neq a$ and $a_2 = a$. Since $B = A \setminus \{a\}$ and $a \in bc \cup Sbc$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (A \setminus \{a\})(bc \cup Sbc) \\ &= (A \setminus \{a\})bc \cup (A \setminus \{a\})Sbc \\ &\subseteq BBB \cup BSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.3: $a_1 = a$ and $a_2 \neq a$. Since $a \in bc \cup Sbc$ and $B = A \setminus \{a\}$, it follows that

$$\begin{aligned} x = a_1a_2 &\in (bc \cup Sbc)(A \setminus \{a\}) \\ &= bc(A \setminus \{a\}) \cup Sbc(A \setminus \{a\}) \\ &\subseteq BBB \cup SBBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 2.4: $a_1 \neq a$ and $a_2 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$x = a_1a_2 \in (A \setminus \{a\})(A \setminus \{a\}) = BB \subseteq L_{wi}(B).$$

Case 3: $x \in SAA$. Then $x = sa_3a_4$ for some $s \in S$ and $a_3, a_4 \in A$. We have following four subcases:

Subcase 3.1: $a_3 = a$ and $a_4 = a$. Since $a \in bc \cup Sbc$, it follows that

$$\begin{aligned} x = sa_3a_4 &\in S(bc \cup Sbc)(bc \cup Sbc) \\ &= Sbcbc \cup SbcSbc \cup SSbcbc \cup SSbcSbc \\ &\subseteq SBBBB \cup SBBSBB \cup SSBBBB \cup SSBBSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 3.2: $a_3 \neq a$ and $a_4 = a$. Since $B = A \setminus \{a\}$ and $a \in bc \cup Sbc$, it follows that

$$\begin{aligned} x = sa_3a_4 &\in S(A \setminus \{a\})(bc \cup Sbc) \\ &= S(A \setminus \{a\})bc \cup S(A \setminus \{a\})Sbc \\ &\subseteq SBBB \cup SBSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 3.3: $a_3 = a$ and $a_4 \neq a$. Since $a \in bc \cup Sbc$ and $B = A \setminus \{a\}$, it follows that

$$\begin{aligned} x = sa_3a_4 &\in S(bc \cup Sbc)(A \setminus \{a\}) \\ &= Sbc(A \setminus \{a\}) \cup SSbc(A \setminus \{a\}) \\ &\subseteq SBBB \cup SSBBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Subcase 3.4: $a_3 \neq a$ and $a_4 \neq a$. Since $B = A \setminus \{a\}$, it follows that

$$x = sa_3a_4 \in S(A \setminus \{a\})(A \setminus \{a\}) = SBB \subseteq L_{wi}(B).$$

In all this cases, we obtain that $x \in L_{wi}(B)$, and so $L_{wi}(A) \subseteq L_{wi}(B)$. Since A is a right weak-interior base of S, it follows that

$$S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S.$$

Thus, $L_{wi}(B) = S$. This is a contradiction. Hence, $a = b$ or $a = c$. □

Lemma 3.7. Let A be a right weak-interior base of a semigroup S. For any a, b in A, if $a \neq b$, then neither $a \leqslant b$ nor $b \leqslant a$.

Proof. Let a, b be any elements of A such that $a \neq b$. Suppose that $a \leqslant b$. We have $a \in L_{wi}(a) \subseteq L_{wi}(b)$. Since $a \neq b$ and

$$a \in L_{wi}(b) = b \cup bb \cup Sbb,$$

we obtain $a \in bb \cup Sbb$. By Lemma 3.5, $a = b$. This is a contradiction. Similarly, if $b \leqslant a$, it follows that $b = a$, which is a contradiction. □

Lemma 3.8. Let A be a right weak-interior base of a semigroup S. Let $a, b, c \in A$ and $s \in S$.

(1) If $a \in bc \cup bcbc \cup Sbcbc$, then $a = b$ or $a = c$.

(2) If $a \in sbc \cup sbcsbc \cup Ssbcsbc$, then $a = b$ or $a = c$.

Proof. (1). Assume that $a \in bc \cup bcbc \cup Sbcbc$. Suppose that $a \neq b$ and $a \neq c$. Setting $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$ and $a \neq c$, we have $b, c \in B$. We will show that $L_{wi}(A) \subseteq L_{wi}(B)$, it suffices to show that $A \subseteq L_{wi}(B)$. Let $x \in A$. If $x \neq a$, then $x \in B$, and so $x \in L_{wi}(B)$. If $x = a$, and $a \in bc \cup bcbc \cup Sbcbc$, we obtain

$$\begin{aligned} x = a &\in bc \cup bcbc \cup Sbcbc \\ &\subseteq BB \cup BBBB \cup SBBBB \\ &\subseteq BB \cup SBB \subseteq L_{wi}(B). \end{aligned}$$

Thus, $A \subseteq L_{wi}(B)$. Since $A \subseteq L_{wi}(B)$ and $L_{wi}(B)$ is a left weak-interior ideal of S, we obtain

$$AA \subseteq (L_{wi}(B))(L_{wi}(B)) \subseteq L_{wi}(B)$$

and

$$SAA \subseteq S(L_{wi}(B))(L_{wi}(B)) \subseteq L_{wi}(B).$$

It follows that $L_{wi}(A) = A \cup AA \cup SAA \subseteq L_{wi}(B)$. Since A is a right weak-interior base of S, it implies that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. Hence, $L_{wi}(B) = S$. This is a contradiction.

(2). Assume that $a \in sbc \cup sbcsbc \cup Ssbcsbc$. Suppose that $a \neq b$ and $a \neq c$. Setting $B = A \setminus \{a\}$. Then $B \subset A$. Since $a \neq b$ and $a \neq c$, we have $b, c \in B$. We will show that $L_{wi}(A) \subseteq L_{wi}(B)$. It suffices to show that $A \subseteq L_{wi}(B)$. Let $x \in A$. If $x \neq a$, then $x \in B$, and so $x \in L_{wi}(B)$. If $x = a$, and $a \in sbc \cup sbcsbc \cup Ssbcsbc$, it implies that

$$\begin{aligned} x = a &\in sbc \cup sbcsbc \cup Ssbcsbc \\ &\subseteq SBB \cup SBBSBB \cup SSBBSBB \\ &\subseteq SBB \subseteq L_{wi}(B). \end{aligned}$$

Thus, $A \subseteq L_{wi}(B)$. Since $A \subseteq L_{wi}(B)$ and $L_{wi}(B)$ is a left weak-interior ideal of S, we obtain

$$AA \subseteq (L_{wi}(B))(L_{wi}(B)) \subseteq L_{wi}(B)$$

and

$$SAA \subseteq S(L_{wi}(B))(L_{wi}(B)) \subseteq L_{wi}(B).$$

It follows that $L_{wi}(A) = A \cup AA \cup SAA \subseteq L_{wi}(B)$. Since A is a right weak-interior base of S, we have that $S = L_{wi}(A) \subseteq L_{wi}(B) \subseteq S$. Hence, $L_{wi}(B) = S$. This is a contradiction. □

Lemma 3.9. Let A be a right weak-interior base of a semigroup S.

(1) For any $a, b, c \in A$, if $a \neq b$ and $a \neq c$, then $a \nleq bc$.

(2) For any $a, b, c \in A$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleq sbc$.

Proof. (1). Let $a, b, c \in A$ such that $a \neq b$ and $a \neq c$. If $a \leqslant bc$, it follows that

$$a \in L_{wi}(a) \subseteq L_{wi}(bc) = bc \cup bcbc \cup Sbcbc.$$

By Lemma 3.8(1), we obtain $a = b$ or $a = c$. This is a contradiction. Hence, $a \nleqslant bc$.

(2). Let $a, b, c \in A$ and $s \in S$ such that $a \neq b$ and $a \neq c$. If $a \leqslant sbc$, it follows that

$$a \in L_{wi}(a) \subseteq L_{wi}(sbc) = sbc \cup sbcsbc \cup Ssbcsbc.$$

By Lemma 3.8(2), we obtain $a = b$ or $a = c$. This is a contradiction. Hence, $a \nleqslant sbc$. □

We now give the main result of this paper by characterizing when a non-empty subset of a semigroup S is a right weak-interior base of S.

Theorem 3.10. A non-empty subset A of a semigroup S is a right weak-interior base of S if and only if A satisfies the following conditions:

(1) For any $x \in S$,

(1.1) there exists $a \in A$ such that $x \leqslant a$; or

(1.2) there exist $a_1, a_2 \in A$ such that $x \leqslant a_1a_2$; or

(1.3) there exist $a_3, a_4 \in A,\ s \in S$ such that $x \leqslant sa_3a_4$.

(2) For any $a, b, c \in A$, if $a \neq b$ and $a \neq c$, then $a \nleqslant bc$.

(3) For any $a, b, c \in A$ and $s \in S$, if $a \neq b$ and $a \neq c$, then $a \nleqslant sbc$.

Proof. Assume that A is a right weak-interior base of S. So, we have $S = L_{wi}(A)$. First, we will show that (1) holds. Let $x \in S$. Then $x \in L_{wi}(A) = A \cup AA \cup SAA$. We have following three cases:

Case 1: $x \in A$. Then $x = a$ for some $a \in A$. It implies $L_{wi}(x) = L_{wi}(a)$. Hence, $x \leqslant a$.

Case 2: $x \in AA$. Then $x = a_1a_2$ for some $a_1, a_2 \in A$. It implies $L_{wi}(x) = L_{wi}(a_1a_2)$. Hence, $x \leqslant a_1a_2$.

Case 3: $x \in SAA$. Then $x = sa_3a_4$ for some $a_3, a_4 \in A$ and $s \in S$. It implies $L_{wi}(x) = L_{wi}(sa_3a_4)$. Hence, $x \leqslant sa_3a_4$.

The conditions (2) and (3) hold from Lemma 3.9(1) and Lemma 3.9(2), respectively.

Conversely, assume that the conditions (1), (2) and (3) hold. We will show that A is a right weak-interior base of S. First, we will claim that $S = L_{wi}(A)$. Clearly, $L_{wi}(A) \subseteq S$. Let $x \in S$. By (1), we have following three cases:

Case 1: Suppose that (1.1) holds. Then there exists $a \in A$ such that $x \leqslant a$. So, we obtain

$$x \in L_{wi}(x) \subseteq L_{wi}(a) = a \cup aa \cup Saa \subseteq A \cup AA \cup SAA = L_{wi}(A).$$

Case 2: Suppose that (1.2) holds. Then there exist $a_1, a_2 \in A$ such that $x \leqslant a_1a_2$. So, we obtain

$$\begin{aligned} x \in L_{wi}(x) &\subseteq L_{wi}(a_1a_2) \\ &= a_1a_2 \cup a_1a_2a_1a_2 \cup Sa_1a_2a_1a_2 \\ &\subseteq AA \cup AAAA \cup SAAAA \\ &\subseteq AA \cup SAA \subseteq L_{wi}(A). \end{aligned}$$

Case 3: Suppose that (1.3) holds. Then there exist $a_3, a_4 \in A$ and $s \in S$, such that $x \leqslant sa_3a_4$. So, we obtain

$$\begin{aligned} x \in L_{wi}(x) &\subseteq L_{wi}(sa_3a_4) \\ &= sa_3a_4 \cup sa_3a_4sa_3a_4 \cup Ssa_3a_4sa_3a_4 \\ &\subseteq SAA \cup SAASAA \cup SSAASAA \\ &\subseteq SAA \subseteq L_{wi}(A). \end{aligned}$$

In all this cases, we infer that $x \in L_{wi}(A)$. Thus, $S \subseteq L_{wi}(A)$, and so $S = L_{wi}(A)$. Next, to show that A is a minimal subset of S with the property $S = L_{wi}(A)$. Suppose that B is a proper subset of A such that $S = L_{wi}(B)$. Since $B \subset A$, we have $a \in A$ such that $a \notin B$. Since $a \notin B$ and $a \in A \subseteq S = L_{wi}(B) = B \cup BB \cup SBB$, it implies that $a \in BB \cup SBB$. Thus, $a \in BB$ or $a \in SBB$. If $a \in BB$, then $a = a_1a_2$ for some $a_1, a_2 \in B \subset A$. Since $a \notin B$, we have $a \neq a_1$ and $a \neq a_2$. Since $a = a_1a_2$, it implies that $L_{wi}(a) \subseteq L_{wi}(a_1a_2)$, and so $a \leqslant a_1a_2$. This contradicts to (2). If $a \in SBB$, then $a = sa_3a_4$ for some $a_3, a_4 \in B \subset A$ and $s \in S$. Since $a \notin B$, we have $a \neq a_3$ and $a \neq a_4$. Since $a = sa_3a_4$, it implies that $L_{wi}(a) \subseteq L_{wi}(sa_3a_4)$, and so $a \leqslant sa_3a_4$. This contradicts to (3). Hence, there is no a proper subset B of A such that $S = L_{wi}(B)$. Therefore, A is a right weak-interior base of S. □

In Example 3.2, we have that $\{b, c\}$ is a right weak-interior base of a semigroup S, but it is not a subsemigroup. This shows that a right weak-interior base of a semigroup S need not to be a subsemigroup in general. In the following theorem we shall find a condition for a right weak-interior base to be a subsemigroup.

Theorem 3.11. Let A be a right weak-interior base of a semigroup S. Then A is a subsemigroup of S if and only if A satisfies the following conditions: for any $a, b \in A$, $ab = a$ or $ab = b$.

Proof. Assume that A is a subsemigroup of S. Let $a, b \in A$. So, we have $ab \in A$. Since $ab \in ab \cup Sab$, it follows by Lemma 3.6, we obtain that $ab = a$ or $ab = b$.

The converse statement is obvious. □

Finally, we present the example of a right weak-interior base of a semigroup to be a subsemigroup.

Example 3.12. Let $S = \{a, b, c, d, e\}$ be a semigroup ([12]) with the binary operation defined by:

$\cdot$	a	b	c	d	e
a	a	b	c	b	b
b	b	b	b	b	b
c	a	b	c	b	b
d	d	b	d	b	b
e	e	e	e	e	e

We have that the right weak-interior base of S is $A = \{a, c\}$. Moreover, it is easy to see that A is a subsemigroup of S, and for $a, c \in A$, we obtain that $ac = c$.

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