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ON KINEMATICS FOR THE PMI DURING THE HOMOTHETIC MOTIONS IN $\mathbb{C}_p$

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Abstract. In this paper, the polar moments of inertia for the trajectories of the points have been obtained during the one parameter planar homothetic motions in the generalized complex plane $\mathbb{C}_p$. Then, Holditch-type theorem that express the relationship among the polar moments of inertia of points has been given for homothetic motion in $\mathbb{C}_p$. Moreover, the some geometric interpretations of the polar moment of inertia in physics have been expressed. So, this study is thought to be an interdisciplinary study that establishes a link between geometry and physics.
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1. Introduction

Mechanics is a sub-branch of physics that studies the motion of objects. Kinematics is also a branch of mechanics that examines the motion of objects in space time (regardless of the factors that cause it). In other words, kinematics is a field of science in which motion is studied mathematically. If the position of an object changes over time, it is said the object is moving. Objects can move in many different ways. In order to understand complex motion models, the location of the motion, i.e. being in one dimension (linear), two dimensions (planar) and high dimension (in space), and what motion or motions the object makes (translation, rotation, vibration, etc.) are important. It is one of the aims of kinematics to describe the motion of an object and to be able to predict the change of this motion over time. In this case, some concepts (such as position vector, path, velocity, acceleration) should be

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defined in order to understand the motion of an object. After defining these concepts, the area or the moment of inertia of the trajectory drawn by any point along the motion can be obtained. The moment of inertia is one of the quantities calculated depending on the geometry of the sections. Consider an axis perpendicular to the plane of an area drawn by any point. The moment of inertia of this area with respect to this perpendicular axis is called the polar moment of inertia of that area.

Holditch gave a theorem in a study as follows; "Once the end points of a fixed-length line segment in the plane are entangled along an oval, a point detected on the line segment also usually draws a closed curve that does not need to be convex. The surface area of the "Holditch Ring" between this oval and the curve depends only on the distance of the selected point from the end points of the line segment and is independent of its curves and motion." [15]. Then, Steiner calculated the area of the trajectory under the closed one-parameter planar motions [22]. This classical Holditch Theorem and Steiner area formula were later generalized by many scientists [1, 5, 20, 21, 29]. Based on the Holditch's theorem, which expresses the relationship among the areas of the trajectories drawn by the points on the plane given by Holditch, the theorem that tell the relationship among the polar moments of inertia of the trajectories drawn by these points has been named as Holditch-type theorem [7,9,24,27,28].

The equations of motion are given by transformations that maintain distance. Homothetic motions, on the other hand, are given by transformations that preserve the angles while changing the distances at the same rate. Therefore, motions are a special case of homothetic motions. Based on this situation, thestudies given for one-parameter planar motions given in many studies have been expanded for homothetic motions. These studies are as follows; [2, 13, 16–18, 23, 26, 28]. Generally, moment of inertia is the resistance per unit surface area resists rotation. In other words, it is the reaction of the surface to the force trying to change its shape. The sum of the moments of inertia defined in the plane with respect to the two axes is also called the polar moment of inertia. Müller calculated the polar moment of inertia of the closed trajectory under the closed planar motions [19]. Then, the polar moment of inertia for homothetic motions was calculated [6, 24]. Then, many studies have been done by many scientists about polar moment of inertia and homothetic motions [27, 28].

2. Preliminaries

The generalized complex number system is isomorphic to dual, ordinary, and double complex numbers ($p + q^2/4$ is zero, negative, and positive, respectively) and is defined as $Z = x + iy$ where $i^2 = iq + p$ and $x, y, p, q \in \mathbb{R}$ [25]. In this paper, the one parameter family $\mathbb{C}_p = \{x + iy : x, y \in \mathbb{R}, i^2 = p \in \mathbb{R}\}$ of this system has been studied. So, the addition, subtraction and product on $\mathbb{C}_p$ can be defined

$$Z_1 \pm Z_2 = (x_1 + iy_1) \pm (x_2 + iy_2) = x_1 \pm x_2 + i(y_1 \pm y_2)$$

and

$$M^p(Z_1, Z_2) = (x_1x_2 + py_1y_2) + i(x_1y_2 + x_2y_1)$$

for two generalized complex numbers $Z_1 = (x_1 + iy_1), Z_2 = (x_2 + iy_2)$ [14,25]. Moreover, for the $p-$magnitude of $Z = x + iy$, the equation

$$|Z|_p = \sqrt{\left|M^p(Z, \bar{Z})\right|} = \sqrt{|x^2 - py^2|} \tag{1}$$

is hold where "$-$" is the complex conjugate. In addition that, the scalar product on generalized complex plane $\mathbb{C}_p$ is given by

$$\langle \mathbf{z}_1, \mathbf{z}_2 \rangle_p = \mathrm{Re}\left(M^p\left(\mathbf{z}_1, \overline{\mathbf{z}_2}\right)\right) = \mathrm{Re}\left(M^p\left(\overline{\mathbf{z}_1}, \mathbf{z}_2\right)\right) = x_1y_1 - px_2y_2$$

for two generalized complex vectors $\mathbf{z}_1 = (x_1 + iy_1), \mathbf{z}_2 = (x_2 + iy_2)$ [14]. Moreover, the unit circle in $\mathbb{C}_p$ is characterized by $|Z|_p = 1$. So, it can be given Figure 1 for the unit circle in $\mathbb{C}_p$ for the special cases of p. [14]. Considering the above-mentioned description of circle for cases

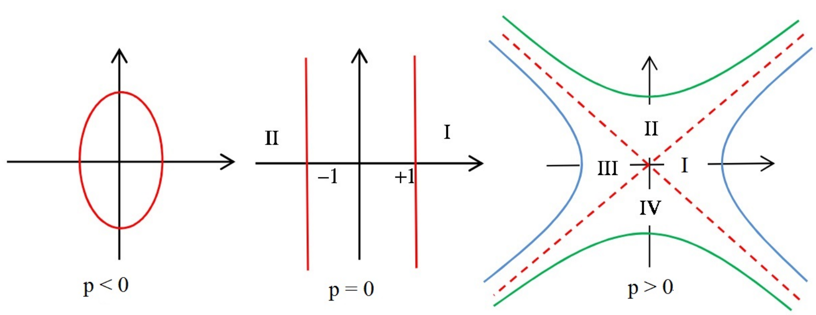

Figure 1. The Unit Circle in $\mathbb{C}_\mathrm{p}$

of p, the circle in $\mathbb{C}_p$ can be defined as follows.

Definition 1. *Consider the circle with the radius r and the center $M(a,b)$. Thus, this circle is written by equation*

$$\left|(x-a)^2 - p(y-b)^2\right| = r^2$$

[14].

Let a number in $\mathbb{C}_p$ be $Z = x + iy$ which is symbolize $\overrightarrow{OT}$ and Figure 2 be as follows.

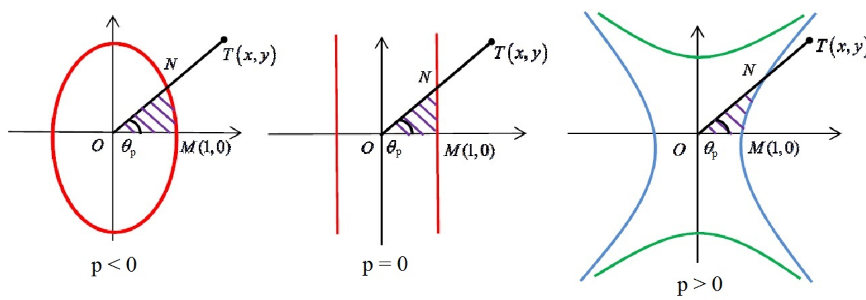

Figure 2. Elliptic, Parabolic and Hyperbolic Angles

So, consider $\sigma \equiv y/x$, the angle θ_p formed by inverse tangent functions can be defined as

$$\theta_p = \begin{cases} \frac{1}{\sqrt{|\mathrm{p}|}}\tan^{-1}\left(\sigma\sqrt{|p|}\right), & p<0 \\ \sigma, & p=0 \\ \frac{1}{\sqrt{\mathrm{p}}}\tan^{-1}\left(\sigma\sqrt{p}\right), & p>0 \ (branch\ I, III) \end{cases}$$

Let the point N be the intersection point of $\overrightarrow{OT}$ with unit circle in $\mathbb{C}_p$. Moreover, the orthogonal projection on the OM of the point N is the point L and the line QM is also the tangent at the point M of the unit circle (see Figure 3). Thus, $p-$trigonometric functions can be obtained by

$$\cos p\theta_p = \begin{cases} \cos\left(\theta_p\sqrt{|p|}\right), & p<0 \\ 1, & p=0 \quad (branch\ \ I) \\ \cosh\left(\theta_p\sqrt{p}\right), & p>0 \quad (branch\ \ I) \end{cases}$$

$$\sin p\theta_p = \begin{cases} \frac{1}{\sqrt{|\mathrm{p}|}}\sin\left(\theta_p\sqrt{|p|}\right), & p<0 \\ \theta_p, & p=0 \quad (branch\ \ I) \\ \frac{1}{\sqrt{\mathrm{p}}}\sinh\left(\theta_p\sqrt{p}\right), & p>0 \quad (branch\ \ I) \end{cases}$$

and

$$\tan p\theta_p = \frac{\sin \mathrm{p}\theta_\mathrm{p}}{\cos \mathrm{p}\theta_\mathrm{p}}.$$

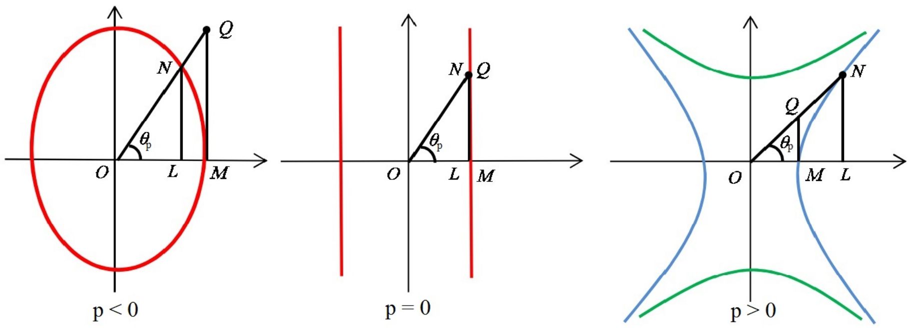

FIGURE 3. θ_{p} for the special cases of p

Thus, the Maclaurin expansions of the p-trigonometric function on the $branch\ I$ is

$$\cos p\theta_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n)!}\theta_p^{2n} \text{ and } \sin p\theta_p = \sum_{n=0}^{\infty} \frac{\mathrm{p}^n}{(2n+1)!}\theta_p^{2n+1}.$$

By the help of the Maclaurin series, the generalized Euler Formula in $\mathbb{C}_p$ is

$$e^{i\theta_p} = \cos p\theta_p + i \sin p\theta_p \tag{2}$$

where $i^2 = p$. On the other hand, the exponential forms of Z in $\mathbb{C}_p$ is

$$Z = r_p(cosp\theta_p + i \sin p\theta_p) = r_p e^{i\theta_p}$$

where $r_p = |Z|_p$ [14]. Moreover, the $p-$rotation matrix given by the equation (2) is

$$A(\theta_p) = \begin{bmatrix} \cos p\theta_p & p \sin p\theta_p \\ \sin p\theta_p & \cos p\theta_p \end{bmatrix}.$$

[14]. Moreover, the derivatives of $\cos p$ and $\sin p$ is obtained that

$$\frac{d}{d\alpha}(\cos p\alpha) = p \sin p\alpha, \qquad \frac{d}{d\alpha}(\sin p\alpha) = \cos p\alpha,$$

[14].

Proposition 2. *Consider two arbitrary generalized complex vectors* $\boldsymbol{a} = (a_1, a_2)$ *and* $\boldsymbol{b} = (b_1, b_2)$ *in* $\mathbb{C}_p$. *Thus, the following equation is satisfied*

$$\left\langle \boldsymbol{a}e^{i\theta_p}, \boldsymbol{b}e^{i\theta_p} \right\rangle_p = \langle \boldsymbol{a}, \boldsymbol{b} \rangle_p \tag{3}$$

where h *is homothetic scale.*

Considering this plane, many studies have been done in planar motions [3,8–12]

The one parameter homothetic motions in the p-complex plane

$$\mathbb{C}_j = \left\{x + Jy : x, y \in \mathbb{R}, J^2 = p, p \in \{-1, 0, 1\}\right\}$$

which is the subset of the generalized complex plane $\mathbb{C}_p$ was studied by Gürses et. al [4]. Similar to that study the one parameter homothetic motions in $\mathbb{C}_p$ have been given as follows briefly.

Let $\mathbb{K}_p'$, $\mathbb{K}_p$ be the fixed and moving planes in $\mathbb{C}_p$, respectively and $\mathbf{x} = x_1 + ix_2$ and $\mathbf{x}' = x_1' + ix_2'$ be the position vectors of a point X, and $\overrightarrow{\mathrm{OO}'} = \mathbf{u}$. So, the equation of the one-parameter planar homothetic motion in $\mathbb{C}_p$ is written by

$$\mathbf{x}' = (h\mathbf{x} - \mathbf{u})\, e^{i\theta_p} \tag{4}$$

where θ_p is the $p-$rotation angle of the motion $\mathbb{K}_p/\mathbb{K}'_p$, $\mathbf{u}' = -\mathbf{u}e^{i\theta_p}$ and h is the homothetic scale in $\mathbb{C}_p$. So, the relative and absolute velocity vectors of X in $\mathbb{K}_p \subset \mathbb{C}_p$ are

$$\mathbf{V}_r{}' = \mathbf{V}_r e^{i\theta_p} = h\dot{\mathbf{x}}e^{i\theta_p} \tag{5}$$

and

$$\mathbf{V}_a{}' = \mathbf{V}_a e^{i\theta_p} = \left(\dot{h} + i\dot{\theta}_p h\right)\mathbf{x}e^{i\theta_p} - \left(\dot{\mathbf{u}} + i\dot{\theta}_p\mathbf{u}\right)e^{i\theta_p} + h\dot{\mathbf{x}}e^{i\theta_p}, \tag{6}$$

respectively. Using the equations (5) and (6) the guide velocity vector is

$$\mathbf{V}_f{}' = \mathbf{V}_f e^{i\theta_p} = \left(\dot{h} + i\dot{\theta}_p h\right)\mathbf{x}e^{i\theta_p} + \dot{\mathbf{u}}'.$$

Theorem 3. *Let $\mathbb{K}_p/\mathbb{K}_p'$ be the one-parameter planar homothetic motion in $\mathbb{C}_p$. So, the relationship among velocity vectors is given by*

$$\boldsymbol{V}_a = \boldsymbol{V}_f + \boldsymbol{V}_r.$$

There are some points that remain fixed in both the fixed plane $\mathbb{K}_p'$ and the moving plane $\mathbb{K}_p$ in $\mathbb{C}_p$. These points are called pole points. Thus, let the pole points of homothetic motions $\mathbb{K}_p/\mathbb{K}_p'$ be $Q = (q_1, q_2) \in \mathbb{C}_p$. So, the components of pole points $Q = (q_1, q_2)$ are

$$\begin{aligned} q_1 &= \tfrac{dh(du_1 + \mathrm{p}u_2 d\theta_\mathrm{p}) - \mathrm{p}h(du_2 + u_1 d\theta_\mathrm{p})d\theta_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\theta_\mathrm{p}{}^2}, \\ q_2 &= \tfrac{dh(du_2 + u_1 d\theta_\mathrm{p}) - h(du_1 + \mathrm{p}u_2 d\theta_\mathrm{p})d\theta_\mathrm{p}}{dh^2 - \mathrm{p}h^2 d\theta_\mathrm{p}{}^2}. \end{aligned} \tag{7}$$

where $\mathbf{V}_f = 0$.

In this paper, the open motions restricted to time interval $[t_1, t_2]$ on the ***branch I*** of $\mathbb{C}_p$ are considered.

3. Main Theorems and Proofs

In this section, the polar moments of inertia of the trajectories drawn by points taken during the homothetic motions in $\mathbb{C}_p$ have been calculated. In addition, by using the relationship among the points taken, some theorems and conclusions about the polar moments of inertia of these points and geometric interpretations have been given. As a result, the Holditch type theorem has been expressed for the moments that give the basic relationship among the polar moments of inertia of the points. For this, the motion mentioned here is restricted to the time interval $I = [t_1, t_2]$. Then, any fixed point X in $\mathbb{K}_p$ is considered. So, the following theorem can be given for the polar moment of inertia of this point for homothetic motions in $\mathbb{C}_p$. (It should be noted that; since the expression polar moment of inertia is used a lot in this study, abbreviation PMI will be used for this expression from now on.)

Theorem 4. *Consider that $\mathbb{K}_p/\mathbb{K}'_p$ is the homothetic motion in $\mathbb{C}_p$ and $X = (x_1, x_2)$ is any fixed point in $\mathbb{K}_p$. So, the PMI of the trajectory drawn by $X = (x_1, x_2)$ is given by*

$$T_X = T_O + h^2(t_0)\,\delta_p\left(\boldsymbol{x}\bar{\boldsymbol{x}} - \boldsymbol{x}\bar{\boldsymbol{s}} - \bar{\boldsymbol{x}}\boldsymbol{s}\right) + \mu_1 x_1 + \mu_2 x_2 \tag{8}$$

where $\mu_1 = -2\int_{t_1}^{t_2} hq_2 dh$, $\mu_2 = 2\int_{t_1}^{t_2} hq_1 dh$, T_0 is the PMI of the trajectory drawn by the origin O and h is the homothetic scale.

Proof. The point $X = (x_1, x_2)$ is considered any fixed point in $\mathbb{K}_p \subset \mathbb{C}_p$. So, for the homothetic motions the PMI of the trajectory drawn by this point is calculated

$$T_X = \int_{t_1}^{t_2} |\mathbf{x}'|_p^2\, d\theta_p \tag{9}$$

where $\overrightarrow{O'X} = \mathbf{x}'$ and O' is the origin of $\mathbb{K}'_p$, and $\frac{d\theta_p}{dt} = \dot{\theta}_p \neq 0$ is a continuous function. So, if the equations (1), (3), (4) and (9), the moment is obtained

$$T_X = \mathbf{x}\bar{\mathbf{x}}\int_{t_1}^{t_2} h^2 d\theta_p - 2\int_{t_1}^{t_2} h\left(x_1u_1 - px_2u_2\right) d\theta_p + \int_{t_1}^{t_2} u\bar{u}\, d\theta_p.$$

Now, the point $X \in \mathbb{K}_p$ is chosen as $X = 0$. So, from the last equation the PMI of trajectory drawn by this point is

$$T_0 = \int_{t_1}^{t_2} \mathbf{u}\bar{\mathbf{u}} d\theta_p. \tag{10}$$

On the other hand, the mean value theorem for integrals states that there is at least one point t_0 in the interval $I = [t_1, t_2]$ so that the following equation can be written

$$\int_{t_1}^{t_2} h^2 d\theta_p = \int_{t_1}^{t_2} h^2(t)\dot{\theta}_p(t)dt = h^2(t_0)\,\delta_p$$

where $\delta_p = \theta_p(t_2) - \theta_p(t_1)$ is the total rotation angle. Then, considering all these operations and pole points in (7) for homothetic motions in $\mathbb{C}_p$, the PMI is found as

$$\begin{aligned} T_X = T_O + \mathbf{x}\bar{\mathbf{x}}h^2(t_0)\,\delta_p - 2x_1\left(\int_{t_1}^{t_2} h^2 q_1 d\theta_p - \int_{t_1}^{t_2} h du_2\right) + 2x_2\left(p\int_{t_1}^{t_2} h^2 q_2 d\theta_p - \int_{t_1}^{t_2} h du_1\right) \\ - 2x_1\int_{t_1}^{t_2} hq_2 dh + 2x_2\int_{t_1}^{t_2} hq_1 dh \end{aligned}.$$

Moreover, since the center of gravity of the pole curve is called Steiner point $S = (s_1, s_2)$, this point for the PMI during homothetic motions in $\mathbb{C}_p$ is given by

$$h^2(t_0)\,\delta_p s_1 = \int_{t_1}^{t_2} h^2 q_1 d\theta_p - \int_{t_1}^{t_2} h du_2, \quad ph^2(t_0)\,\delta_p s_2 = p\int_{t_1}^{t_2} h^2 q_2 d\theta_p - \int_{t_1}^{t_2} h du_1.$$

So, the PMI of trajectory drawn by X is

$$T_X = T_O + h^2(t_0)\,\delta_p\left(\mathbf{x}\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{s}} - \bar{x}\mathbf{s}\right) + \mu_1 x_1 + \mu_2 x_2$$

where $\mu_1 = -2\int_{t_1}^{t_2} hq_2 dh$, $\mu_2 = 2\int_{t_1}^{t_2} hq_1 dh$. □

Now, as a special selection, consider $h = 1$. So, the equation (8) is obtained as $T_X = T_O + \delta_p\left(\mathbf{x}\bar{\mathbf{x}} - \mathbf{x}\bar{\mathbf{s}} - \bar{\mathbf{x}}\mathbf{s}\right)$. This equation is the same as the moment given in [9]. Therefore, this study is the generalized version of the study given in [9].

Let the PMI T_X be considered as constant in the equation (8). In this case, as a result of the Theorem 4, we can give the following corollary without proof.

Corollary 1. *Consider that all points X have the same PMI T_X during homothetic motions in $\mathbb{C}_p$. So, the geometric location of these points X is a circle with center*

$$\left(s_1 - \frac{\mu_1}{2h^2\left(t_0\right)\delta_p}, s_2 + \frac{\mu_2}{2ph^2\left(t_0\right)\delta_p}\right)$$

in the plane $\mathbb{C}_p$.

It should be emphasized here; if $h = 1$, the center of the circle mentioned above becomes the Steiner point $S = (s_1, s_2)$ [9].

Until now, the PMI of trajectory drawn by a fixed point on the moving plane $\mathbb{K}_p$ has been calculated. Now, the PMI calculation be expanded one more step by taking three linear points. So, the following theorem can be given.

Theorem 5. *Suppose that the points X, Y and Z are the linear points in the moving plane $\mathbb{K}_p$ during homothetic motion in $\mathbb{C}_p$. Let the point Z be on the line segment XY, and each of these points draw a curve during the homothetic motion. In this case, if the PMIs of the trajectories drawn by the X, Y and Z are T_X, T_Y and T_Z, respectively, then, there is the relationship*

$$T_Z = \alpha T_X + \beta T_Y - \alpha\beta h^2(t_0)\delta_p d^2 \tag{11}$$

among these moments where α and β are barycentric coordinates $(\alpha + \beta = 1)$, h is homothetic scale, δ_p is rotation angle, and d is the distance of Z to X and Y in $\mathbb{C}_p$.

Proof. Consider that X, Y and Z are the linear points and Z be on the line segment XY in $\mathbb{K}_p$ during homothetic motion in $\mathbb{C}_p$. Moreover, suppose that $\overrightarrow{OX'} = \mathbf{x}'$, $\overrightarrow{OY'} = \mathbf{y}'$, $\overrightarrow{OZ'} = \mathbf{z}'$ are the position vectors of the points X, Y and Z with respect to the fixed plane $\mathbb{K}'_p \subset \mathbb{C}_p$, respectively. So, there is a relationship

$$\mathbf{z}' = \alpha\mathbf{x}' + \beta\mathbf{y}' \tag{12}$$

where $\alpha, \beta \in \mathbb{R}$ $(\alpha + \beta = 1)$ are barycentric coordinates. Now, the PMI T_Z of the trajectory drawn by Z is calculated. In this case, similar to the equation (9), the PMI T_Z with the aid of (12) can be written by

$$T_Z = \int_{t_1}^{t_2} |\mathbf{z}'|_p^2 \, d\theta_p = \int_{t_1}^{t_2} \langle \alpha\mathbf{x}' + \beta\mathbf{y}', \alpha\mathbf{x}' + \beta\mathbf{y}' \rangle_p d\theta_p,$$

and if the equations $T_X = \int_{t_1}^{t_2} \langle \mathbf{x}', \mathbf{x}' \rangle_p d\theta_p$, $T_Y = \int_{t_1}^{t_2} \langle \mathbf{y}', \mathbf{y}' \rangle_p d\theta_p$, $T_{XY} = \int_{t_1}^{t_2} \langle \mathbf{x}', \mathbf{y}' \rangle_p d\theta_p$ are considered, the PMI is written by

$$T_Z = \alpha^2 T_X + 2\alpha\beta T_{XY} + \beta^2 T_Y \tag{13}$$

where T_X and T_Y are the PMIs of the trajectories drawn by the X and Y, respectively. It is unknown what the moment T_{XY} is in equation (13) in here. In this case, this moment can be calculated as follows. If the necessary arrangements in the equation $T_{XY} = \int_{t_1}^{t_2} \langle \mathbf{x}', \mathbf{y}' \rangle_p d\theta_p$ are made, the moment

$$T_{XY} = T_O + h^2(t_0)\,\delta_p\,(x_1y_1 - px_2y_2 - (x_1+y_1)s_1 + p(x_2+y_2)s_2) + \mu_1 x_1 + \mu_2 x_2 \tag{14}$$

is hold where T_O is the PMI of trajectory drawn by the origin point on $\mathbb{K}_p$ and same formulae in the equation (10), $\mu_1 = -2\int_{t_1}^{t_2} hq_2 dh$, $\mu_2 = 2\int_{t_1}^{t_2} hq_1 dh$, and the point $S = (s_1, s_2)$ is Steiner point for homothetic motions in $\mathbb{C}_p$. There is another point that needs to be emphasized here. If consider $X = Y$ in the equation (14), then, the equation (8) is obtained. This means that; the equation (14) is an extended version of the equation (8). Now, by using the equations (8) and (14) the equation $T_X - 2T_{XY} + T_Y$ is calculated. So, the equation

$$T_X - 2T_{XY} + T_Y = h^2(t_0)\,\delta_p\,(\mathbf{x}\bar{\mathbf{x}} + \mathbf{y}\bar{\mathbf{y}} - \mathbf{x}\bar{\mathbf{y}} - \bar{\mathbf{x}}\mathbf{y})$$

is hold where $T_{XY} = T_{YX}$. It is known very easily from here

$$\mathbf{x}\bar{\mathbf{x}} + \mathbf{y}\bar{\mathbf{y}} - \mathbf{x}\bar{\mathbf{y}} - \bar{\mathbf{x}}\mathbf{y} = (x_1 - y_1)^2 - p(x_2 - y_2)^2 = d^2$$

where the distance between X and Y for branch I of $\mathbb{C}_p$ is d. In this case, the PMI T_{XY} can be written by the PMIs T_X and T_Y

$$T_{XY} = \frac{1}{2}\left(T_X + T_Y - h^2(t_0)\delta_p d^2\right). \tag{15}$$

Finally, from the equations (13) and (15), it is obtained

$$T_Z = \alpha T_X + \beta T_Y - \alpha\beta h^2(t_0)\delta_p d^2$$

where $\alpha + \beta = 1$. □

Now, consider that a special case of the expression given in Theorem 5. This means that the points X and Y mentioned in Theorem 5 move on the same trajectory. In this case, the PMIs of the trajectories drawn by X and Y are same ($T_X = T_Y$). So, the equation (11) is obtained by $T_X - T_Z = \alpha\beta h^2(t_0)\delta_p d^2$ where $\alpha + \beta = 1$. In addition, if the distances are considered $|YZ| = \lambda d$ and $|XZ| = \mu d$, then the relationship among PMIs T_X (or T_Y) and T_Z is obtained

$$T_X - T_Z = h^2(t_0)\delta_p |XZ| |YZ| .$$

So, the main theorem (generalization of Holditch-type theorem) during homothetic motions in $\mathbb{C}_p$ can be given as follows.

Theorem 6. **(Main Theorem):** *Consider that X, Y and Z are the linear and fixed points on the moving plane $\mathbb{K}_p \subset \mathbb{C}_p$ and X and Y draw the same trajectory at the time interval $[t_1, t_2]$. Moreover, Z on XY with constant length draws a different trajectory at the same interval. So, the difference between the PMIs of these two trajectories is independent of the curves drawn by these trajectories. This difference of moments depends only on the distance of the Z from X and Y, the rotation angle and the homothetic scale in the generalized complex plane $\mathbb{C}_p$.*

4. Conclusion

This study has been done about PMIs for homothetic planar motions in the generalized complex plane $\mathbb{C}_p$. The generalized complex plane; it is the most general case of complex $(p = -1)$, dual $(p = 0)$ and hyperbolic $(p = +1)$ planes, as well as planes given for other special situations of p. A lot of study has been done for one-parameter plane motions in this plane. The motion are given by transformations that maintain distance. The homothetic motions are given by transformations that preserve the angles while changing the distances at the same rate. Therefore, motions are a special case (homothetic scale=1) of homothetic motions. Thus, this study is the most generalized form of PMI studies done so far, both because it is in the generalized complex plane and because it is made for homothetic movements. In addition, the fact that this study has been studied about moment in plane geometry has created a bridge between physics and geometry sciences.

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