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USING QUASI-SUBORDINATION PRINCIPLE TO ESTABLISH THE COEFFICIENT ESTIMATES OF A SAKAGUCHI TYPE CLASS

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Abstract. The initial coefficient bounds for a class $M_q^{\lambda}(\gamma, t, b)$ of an analytic function involving modified q-sigmoid associated with quasi-subordination were obtained. The Fekete-Szegö functional and Hankel determinant for the class were investigated.

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1. Introduction

Let A represent the class of functions of the form:

(1) $$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ (z \in \mathbb{U})$$

which are analytic in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ normalised by $f(0) = 0$ and $f'(0) = 1$. Also S represent the class of analytic univalent and normalised function in $\mathbb{U}$. The function l is said to be subordinate to L, written $l(z) \prec L(z)$, if there exists a function ω analytic in $\mathbb{U}$, with $\omega(0) = 0$ and $|\omega(z)| < 1$ and such that $l(z) = L(\omega(z))$.
Let $\varphi(z) \leq 1\ (z \in \mathbb{U})$ be an analytic function such that $\dfrac{l(z)}{\varphi(z)}$ is analytic in $\mathbb{U}$ and

(2) $$\frac{l(z)}{\varphi(z)} \prec L(z)\ (z \in \mathbb{U})$$

which means there exist a Schwartz function $w(z)$ such that $l(z) = \varphi(z).L(w(z)),\ z \in \mathbb{U}$ then $l(z) \prec_q L(z)$ implies l is said to be quasi-subordinate to $L \in \mathbb{U}$.

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Hence, the normal subordination $\prec$ can be gotten from the quasi-subordination $\prec_q$ if $\varphi(z) = 1,\ (z \in \mathbb{U})$.

Frasin [6] introduced and studied coefficient inequalities for certain classes of Sakaguchi type functions $f \in A$ when $s, b \in \mathbb{C}$ with $s \neq b$ and for some $\alpha,\ (0 \leq \alpha < 1)$ which satisfies

$$Re\left\{\frac{(s-b)zf'(z)}{f(sz)-f(bz)}\right\} > \alpha,\ z \in \mathbb{U} \tag{3}$$

Recently, Olatunji [7] introduced a class

$$\left[f'(z)\left(\frac{(s-b)z}{f(sz)-f(bz)}\right)^{\lambda} - 1\right] \prec_q \left[\Phi(z) - 1\right],\ z \in \mathbb{U} \tag{4}$$

when $s, b \in \mathbb{C}$ with $s \neq b$ and $\Phi(z) = 1 + \frac{z}{2} - \frac{z^3}{24} + \frac{z^4}{240} - \frac{z^6}{64} + \frac{779z^7}{20160} - ...$, the series form of modified sigmoid function.

P is also denoted as the class of functions $p(z)$ analytic in $\mathbb{U}$ such that $p(0) = 1$ and $Re(p(z)) > 0$.

For the purpose of our results, the following lemmas and definitions are employed.

Lemma 1.1. [2] *If* $\omega(z) = b_1 z + b_2 z^2 + ...,\ b_1 \neq 0$ *is analytic and satisfy* $|\omega(z)| < 1$ *in the unit disk* $\mathbb{U}$*, then for each* $0 < r < 1$, $|\omega'(z)| < 1$ *and* $|\omega(re^{i\theta})| < 1$ *unless* $\omega(z) = e^{i\theta}z$ *for some real number* σ.

Lemma 1.2. [3] *Let* $\omega \in \Omega = \{\omega \in A : |\omega(z)| \leq |z|, z \in \mathbb{U}\}$
If $\omega \in \Omega$, $\omega(z) = \sum_{n=1}^{\infty} c_n z^n\ (z \in \mathbb{U})$, *then*

$$|c_n| \leq 1\ n = 1, 2, ...\ |c_2| \leq 1 - |c_1|^2$$

and

$$|c_2 - \mu c_1^2| \leq max\{1, |\mu|\}\ (\mu \in \mathbb{C})$$

The result is sharp. The functions

$$\omega(z) = z, \omega_a(z) = z\frac{z+a}{1+\bar{a}z}\ (z \in \mathbb{U}, |a| < 1).$$

are extremal functions.

Lemma 1.3. [5] *Let* $h(z)$ *be a sigmoid function and*

$$G(z) = 2h(z) = 1 + \sum_{m=1}^{\infty} \frac{(-1)^m}{2^m}\left[\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]!} z^n\right]^m$$

then $G(z) \in P$, *where* $G(z)$ *is the modified sigmoid function.*

Definition 1.1. [1] *A q-analogue of the ordinary exponential function* $e^z = \sum_{k=0}^{\infty} \frac{z^k}{k!}$ *is defined as*

$$e_q^z = \sum_{k=0}^{\infty} \frac{z^k}{[k]q!}$$

Definition 1.2. [1] *A q-Sigmoid function is defined as*

$$G_q(z) = \frac{1}{1+e_q^{-z}} \tag{5}$$

Definition 1.3. [4] *A modified q-sigmoid is defined as*

$$\gamma_{q,m,n}(z) = \frac{2}{1+e_q^{-z}} = 1 + \left(\sum_{k=1}^{\infty} \frac{(-1)^k}{2^k} \left[\sum_{n=1}^{\infty} \frac{(-1)^n}{[n]_q!} z^n \right]^k \right) \tag{6}$$

Definition 1.4. *Let* $\gamma(z) \in P$ *be univalent, for* $t \neq b$*,* $t, b \in \mathbb{C}$ *and* $\lambda \geq 0$*, a function* $f \in A$ *is said to be in the class* $M_q^\lambda(\gamma, t, b)$ *if*

$$\left[f'(z) \left(\frac{(t-b)z}{f(tz)-f(bz)} \right)^\lambda - 1 \right] \prec_q \left[\gamma_{q,m,n}(z) - 1 \right], \ (z \in \mathbb{U}) \tag{7}$$

for the powers take principal values only.
If there exist an analytic function $\varphi(z)$ *with* $|\varphi(z)| \leq 1, \ (z \in \mathbb{U})$*, then it follows that* $f \in M_q^\lambda(\gamma, t, b)$ *as defined in* (7).

$$\frac{f'(z) \left(\frac{(t-b)z}{f(tz)-f(bz)} \right)^\lambda - 1}{\varphi(z)} \prec \gamma_{q,m,n}(z) - 1, \ (z \in \mathbb{U}) \tag{8}$$

If $|\varphi(z)| = 1, \ (z \in \mathbb{U})$ *in* (8)*, then*

$$f'(z) \left(\frac{(t-b)z}{f(tz)-f(bz)} \right)^\lambda \prec \gamma_{q,m,n}(z), \ (z \in \mathbb{U}) \tag{9}$$

In this work, quasi-subordination is used to establish the coefficient estimates of a Sakaguchi type of univalent function.

2. Main Results

Theorem 2.1. *If* $f(z)$ *belongs to* $M_q^\lambda(\gamma, t, b)$ *then*

$$|a_2| \leq \left| \frac{d_0 c_1}{2[1]_q!(2-\lambda(t+b))} \right|$$

$$|a_3| \leq \frac{1}{3-\lambda(t^2+tb+b^2)}\left|\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}-2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$
$$|a_4| \leq \frac{1}{4-\lambda(t^3+t^2b+tb^2+b^3)}\left|\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0\right.$$
$$-\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)$$
$$-(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)$$
$$\left.\left(\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right|$$

where

$$A=\left(\frac{[2]_q!-2([1]_q!)^2}{4([1]_q!)^2[2]_q!}\right)$$
$$B=\left(\frac{4([1]_q!)^3[2]_q!-4([1]_q!)^2[3]_q!+[2]_q![3]_q!}{8([1]_q!)^3[2]_q![3]_q!}\right)$$

Proof:

Suppose $f(z) \in M_q^{\lambda}(\gamma,t,b)$, then by definition,

$$\frac{f'(z)\left(\frac{(t-b)z}{f(tz)-f(bz)}\right)^{\lambda}-1}{\varphi(z)}=\gamma_{q,m,n}(z)-1,\ (z\in\mathbb{U})$$

(10) $$f'(z)\left(\frac{(t-b)z}{f(tz)-f(bz)}\right)^{\lambda}-1=\varphi(z)(\gamma_{q,m,n}(z)-1)$$

$$f(tz)-f(bz)=(t-b)z+(t^2-b^2)a_2z^2+(t^3-b^3)a_3z^3+(t^4-b^4)a_4z^4+...$$

$$f'(z)=1+2a_2z+3a_3z^2+4a_4z^3+...$$

$$f'(z)\left(\frac{(t-b)z}{f(tz)-f(bz)}\right)^{\lambda}=$$
$$f'(z)\left[((t-b)z)\left((t-b)z+(t^2-b^2)a_2z^2+(t^3-b^3)a_3z^3+(t^4-b^4)a_4z^4+...\right)^{-1}\right]^{\lambda}$$
$$f'(z)\left[((t-b)z)((t-b)z)^{-1}\left(1+(t+b)a_2z+(t^2+tb+b^2)a_3z^2+(t^3+t^2b+tb^2+b^3)a_4z^3...\right)^{-1}\right]^{\lambda}$$
$$f'(z)\left(1+(t+b)a_2z+(t^2+tb+b^2)a_3z^2+(t^3+t^2b+tb^2+b^3)a_4z^3...\right)^{-\lambda}$$

Applying the Binomial expansion, we have,

$$f'(z)\left(\frac{(t-b)z}{f(tz)-f(bz)}\right)^{\lambda}-1=$$
$$(2-\lambda(t+b))a_2z+\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2a_2^2-\lambda(t^2+tb+b^2)a_3-2\lambda(t-b)a_2^2+3a_3\right)z^2$$
$$+\left(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)a_2a_3-\lambda(t^3+t^2b+tb^2+b^3)a_4-\frac{(\lambda)(\lambda+1)(\lambda+2)}{3!}(t+b)^3a_2^3+\right.$$
$$\left.\lambda(\lambda+1)(t+b)^2a_2^3-2\lambda(t^2+tb+b^2)a_2a_3-3\lambda(t-b)a_2a_3+4a_4\right)z^3+...$$

Furthermore,

(11) $$\gamma_{q,m,n}(\omega(z))=1+\frac{c_1}{2[1]_q!}z+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)z^2+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)z^3+...$$

Also,

$$\varphi(z) = d_0 + d_1 z + d_2 z^2 + ... \tag{12}$$

$$\varphi(z)(\gamma_{q,m,n}(\omega(z)) - 1) =$$
$$(d_0 + d_1 z + d_2 z^2 + ...)\left(\frac{c_1}{2[1]_q!}z + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)z^2 + \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3\right)z^3 + ...\right)$$

then,

$$d_0\left(\frac{c_1}{2[1]_q!}z + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)z^2 + \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3\right)z^3 + ...\right) + d_1 z\left(\frac{c_1}{2[1]_q!}z + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)z^2 + ...\right) +$$
$$d_2 z^2\left(\frac{c_1}{2[1]_q!}z + \left(\frac{c_2}{2[1]_q!} + ...\right)\frac{d_0c_1}{2[1]_q!}z + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0z^2 + \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3\right)d_0z^3\right.$$
$$+\frac{d_1c_1}{2[1]_q!}z^2 + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_1z^3 + \frac{d_2c_1}{2[1]_q!}z^3...$$
$$\varphi(z)(\gamma_{q,m,n}(\omega(z)) - 1) = \frac{d_0c_1}{2[1]_q!}z + \left(\frac{d_1c_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0\right)z^2 + \left(\frac{d_2c_1}{2[1]_q!}z^3 + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_1\right.$$
$$\left.+\left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3\right)d_0\right)z^3 + ...$$

Equating (11) and (14), we have,

$$(2-\lambda(t+b))a_2z + \left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2a_2^2 - \lambda(t^2+tb+b^2)a_3 - 2\lambda(t-b)a_2^2 + 3a_3\right)z^2$$
$$+\left(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)a_2a_3 - \lambda(t^3+t^2b+tb^2+b^3)a_4 - \frac{(\lambda)(\lambda+1)(\lambda+2)}{3!}(t+b)^3a_2^3 +\right.$$
$$\left.\lambda(\lambda+1)(t+b)^2a_2^3 - 2\lambda(t^2+tb+b^2)a_2a_3 - 3\lambda(t-b)a_2a_3 + 4a_4\right)z^3 + ...$$
$$= \frac{d_0c_1}{2[1]_q!}z + \left(\frac{d_1c_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0\right)z^2 + \left(\frac{d_2c_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_1 + \left(\frac{c_3}{2[1]_q!} + 2Ac_1c_2 + Bc_1^3\right)d_0\right)z^3 + ...$$

Comparing coefficients of z, z^2, z^3 in (15) to obtain a_2, a_3, a_4, we have

$$(2-\lambda(t+b))a_2z = \frac{d_0c_1}{2[1]_q!}z$$

$$a_2 = \frac{1}{2-\lambda(t+b)}\left(\frac{d_0c_1}{2[1]_q!}\right)$$

$$a_2 = \frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}$$

$$|a_2| \le \left|\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right| \tag{13}$$

Also,

$$\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2a_2^2 - \lambda(t^2+tb+b^2)a_3 - 2\lambda(t-b)a_2^2 + 3a_3\right)z^2 = \left(\frac{d_1c_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0\right)z^2$$
$$a_3 = \frac{1}{(3-\lambda(t^2+tb+b^2))}\left(\frac{d_1c_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0 - \left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2 - 2\lambda(t-b)\right)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)^2\right)$$
$$|a_3| \le \frac{1}{3-\lambda(t^2+tb+b^2)}\left|\frac{c_1d_1}{2[1]_q!} + \left(\frac{c_2}{2[1]_q!} + Ac_1^2\right)d_0 - \left(\frac{\lambda(\lambda+1)(t+b)^2}{2!} - 2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$

Furthermore,

$$\Big(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)a_2a_3-\lambda(t^3+t^2b+tb^2+b^3)a_4-\frac{(\lambda)(\lambda+1)(\lambda+2)}{3!}(t+b)^3a_2^3+\lambda(\lambda+1)(t+b)^2a_2^3-$$
$$2\lambda(t^2+tb+b^2)a_2a_3-3\lambda(t-b)a_2a_3+4a_4\Big)z^3+\ldots=\left(\frac{d_2c_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0\right)z^3+\ldots$$
$$a_4=\frac{1}{4-\lambda(t^3+t^2b+tb^2+b^3)}\left(\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\right.$$
$$\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))$$
$$-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\right.$$
$$\left.\left.\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right)$$
$$|a_4|\leq\frac{1}{4-\lambda(t^3+t^2b+tb^2+b^3)}\left|\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\left(\lambda(\lambda+1)(t+b)^2-\right.\right.$$
$$\left.\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)$$
$$\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\right.$$
$$\left.\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right|$$

Remark 2.1. *Setting $q=1$, $|a_2|$ and $|a_3|$ agree with Olatunji* et.al [7]

Theorem 2.2. *If $f(z)$ belongs to $M_q^{\lambda}(\gamma,t,b)$ and $\mu\in\mathbb{R}$ then*

$$|a_3-\mu a_2^2|\leq\frac{1}{3-\lambda(t^2+tb+b^2)}\left|\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}-2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)-\mu\left(\frac{d_0^2c_1^2(3-\lambda(t^2+tb+b^2))}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$

Proof:

From (16) and (17), we have,

$$a_3-\mu a_2^2=\frac{1}{3-\lambda(t^2+tb+b^2)}\left(\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}-2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)-\mu\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)$$
$$a_3-\mu a_2^2=\frac{1}{3-\lambda(t^2+tb+b^2)}\left(\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}-2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)-\mu\left(\frac{d_0^2c_1^2(3-\lambda(t^2+tb+b^2))}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)$$
$$|a_3-\mu a_2^2|\leq\frac{1}{3-\lambda(t^2+tb+b^2)}\left|\frac{c_1d_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}-2\lambda(t+b)\right)\right.$$
$$\left.\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)-\mu\left(\frac{d_0^2c_1^2(3-\lambda(t^2+tb+b^2))}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$

which completes the proof for the Fekete-Szeg$\ddot{o}$ Inequality.

Theorem 2.3. *If $f(z)$ belongs to $M_q^\lambda(\gamma,t,b)$ and $\mu\in\mathbb{R}$ then*

$$|a_2a_4-\mu a_3^2|\leq\Bigg|\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(4-\lambda(t^3+t^2b+tb^2+b^3))}\left(\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1\right.\right.$$
$$+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-$$
$$(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}\right.$$
$$\left.\left.\left.+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right)\right)-\frac{\mu}{3-\lambda(t^2+tb+b^2)^2}$$
$$\left(\frac{d_1^2c_1^2}{4([1]_q!)^2}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)^2d_0^2+\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)^2\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)^4+\left(\frac{c_1c_2d_1}{2[1]_q!}+\right.\right.$$
$$\left.\frac{Ac_1^3d_1}{[1]_q!}\right)d_0-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)\left(\frac{d_0^2c_1^3d_1}{4([1]_q!)^3(2-\lambda(t+b))^2}\right)-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)$$
$$\left.\left(\frac{d_0^2c_1^2c_2}{4([1]_q!)^3(2-\lambda(t+b))^2}+\frac{Ac_1^4d_0}{2([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$

Proof:

From (16), (17) and (18), we have,

$$a_2a_4-\mu a_3^2=\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)\left(\frac{1}{4-\lambda(t^3+t^2b+tb^2+b^3)}\left(\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1\right.\right.$$
$$+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-$$
$$(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}+\right.$$
$$\left.\left.\left.\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right)\right)-\mu\left(\frac{1}{(3-\lambda(t^2+tb+b^2))}\right.$$
$$\left.\left(\frac{d_1c_1}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)^2\right)\right)^2$$

$$a_2a_4-\mu a_3^2=\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)\left(\frac{1}{4-\lambda(t^3+t^2b+tb^2+b^3)}\left(\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1\right.\right.$$
$$+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-$$
$$(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}+\right.$$
$$\left.\left.\left.\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right)\right)-\frac{\mu}{3-\lambda(t^2+tb+b^2)^2}$$
$$\left(\frac{d_1^2c_1^2}{4([1]_q!)^2}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)^2d_0^2+\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)^2\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)^4+\left(\frac{c_1c_2d_1}{2[1]_q!}+\right.\right.$$
$$\left.\frac{Ac_1^3d_1}{[1]_q!}\right)d_0-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)\left(\frac{d_0^2c_1^3d_1}{4([1]_q!)^3(2-\lambda(t+b))^2}\right)-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)$$
$$\left.\left(\frac{d_0^2c_1^2c_2}{4([1]_q!)^3(2-\lambda(t+b))^2}+\frac{Ac_1^4d_0}{2([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)$$

$$|a_2a_4-\mu a_3^2|\leq\Bigg|\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(4-\lambda(t^3+t^2b+tb^2+b^3))}\left(\frac{c_1d_2}{2[1]_q!}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_1\right.\right.$$
$$+\left(\frac{c_3}{2[1]_q!}+2Ac_1c_2+Bc_1^3\right)d_0-\left(\lambda(\lambda+1)(t+b)^2-\frac{\lambda(\lambda+1)(\lambda+2)(t+b)^3}{3!}\right)\left(\frac{d_0^3c_1^3}{8([1]_q!)^3(2-\lambda(t+b))^3}\right)-$$
$$(\lambda(\lambda+1)(t^3+t^2b+tb^2+b^3)-2\lambda(t^2+tb+b^2))-3\lambda(t+b)\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))(3-\lambda(t^2+tb+b^2))}\right)\left(\frac{c_1d_1}{2[1]_q!}\right.$$
$$\left.\left.\left.+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)d_0-\left(\frac{\lambda(\lambda+1)(t+b)^2}{2!}+2\lambda(t+b)\right)\left(\frac{d_0^2c_1^2}{4([1]_q!)^2(2-\lambda(t+b))^2}\right)\right)\right)\right)-\frac{\mu}{3-\lambda(t^2+tb+b^2)^2}$$
$$\left(\frac{d_1^2c_1^2}{4([1]_q!)^2}+\left(\frac{c_2}{2[1]_q!}+Ac_1^2\right)^2d_0^2+\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)^2\left(\frac{d_0c_1}{2[1]_q!(2-\lambda(t+b))}\right)^4+\left(\frac{c_1c_2d_1}{2[1]_q!}+\right.\right.$$
$$\left.\frac{Ac_1^3d_1}{[1]_q!}\right)d_0-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)\left(\frac{d_0^2c_1^3d_1}{4([1]_q!)^3(2-\lambda(t+b))^2}\right)-\left(\frac{(\lambda)(\lambda+1)}{2!}(t+b)^2-2\lambda(t-b)\right)$$
$$\left.\left(\frac{d_0^2c_1^2c_2}{4([1]_q!)^3(2-\lambda(t+b))^2}+\frac{Ac_1^4d_0}{2([1]_q!)^2(2-\lambda(t+b))^2}\right)\right|$$

which completes the proof for Hankel Determinant.

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