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# STEPANOV-LIKE PSEUDO ALMOST AUTOMORPHY ON TIME SCALES: NEW DEVELOPMENTS AND APPLICATIONS

MOHSSINE ES-SAIYDY*, MOHAMMED ZARHOUNI, MOHAMED ZITANE

Moulay Ismaïl University, Faculty of Sciences, Department of Mathematics, MACS Laboratory, Meknès, Morocco

*Corresponding author: essaiydy1995@gmail.com

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Abstract. In the present paper, we introduce two new concepts called respectively Stepanov-like almost automorphic functions and Stepanov-like pseudo almost automorphic functions on time scales, and explore their essential properties including some composition theorems. Next, we establish some results about the existence and uniqueness of pseudo almost automorphic solutions to some dynamic equations involving $S^p$-pseudo almost automorphic forcing terms. Finally, we present a numerical example with simulations to illustrate the feasibility and effectiveness of our abstract results.

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## 1. Introduction

The theory of time scales was first treated and created by Stefan Hilger in 1988 ( [15]). It was introduced as a novel and effective mathematical technique for the study of certain processes in the simulation of populations of some insect species, electrical circuitry, quantum physics, DNA analysis, economics, etc. ( [1,5,9,12,18,22]). In recent years, this notion has been extended to a number of fields, especially to qualitative analysis such as periodicity, automorphy and stability of functional differential equations. For recent developments in this theory, we refer the reader to ( [6,14,16]) and the references cited therein.

In the mid 50's, in relation to certain aspects of differential geometry, S. Bochner introduced the notion of almost automorphy ( [3,4]), which generalizes the one of classical almost

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periodicity. Since then, such a notion has become of great interest to several mathematicians ( [7,8,23,27]). In ( [7]), a new generalization of the concept of almost automorphy called Stepanov-like pseudo almost automorphy was investigated. Recently, C. Lizama and J.G. Mesquita ( [19]) extended the concept of almost automorphy on time scales, and studied the existence of almost automorphic solutions to some dynamic equations. Furthermore, the notion of Stepanov-like almost automorphy and Stepanov-like pseudo almost automorphy on time scales are not defined yet.

The main objective of this paper is threefold. We first introduce the class of Stepanov-like almost automorphic functions on time scales (or $S^p-$almost automorphic functions ) and prove that a function is Stepanov-like almost automorphic if, and only if its Bochner-like transform is almost automorphic on time scales. Secondly, we extend this newly introduced notion to Stepanov-like pseudo almost automorphy on time scales. To do so, our main idea consists of enlarging the so called ergodic component utilized in ( [10,11,25]). We also prove some interesting properties of those new concepts, like the completeness and the composition theorem, which have many applications in the context of dynamic equations. Between other things, we will prove that those newly developed functions are a natural generalization of the classical almost automorphy concept and its various expansions (see Figure 1). For example, in the case of $\mathbb{T} = \mathbb{R}$, the introduced notions are the same as the traditional concepts of Stepanov-like almost automorphy ( [23]) and Stepanov-like pseudo almost automorphy ( [7]).

The third objective is to use these new functions to study the existence and uniqueness theorems of pseudo almost automorphic mild solutions to the following dynamic equations with Stepanov-like pseudo almost automorphic forcing terms

$$u^{\Delta}(t) = Au(t) + f(t), \quad t \in \mathbb{T}. \tag{1}$$

$$u^{\Delta}(t) = Au(t) + F(t, u(t)), \quad t \in \mathbb{T}. \tag{2}$$

$$u^{\Delta}(t) = Au(t) + F(t, u(t-\tau)), \quad t \in \mathbb{T} \quad and \quad \tau \in \Pi. \tag{3}$$

Where $A$ is the generator of a $C_0$-semigroup $T = \{T(t) : t \in \mathbb{T}\}$ and the coefficients $f, F$ are Stepanov-like pseudo almost automorphic and continuous on $\mathbb{T}$. Our results are completely new, even for both the case of the differential equations and the case of difference equations.

The remainder of this paper is organized as follows: In Section 2, we make some preparations for the next sections. In section 3, the concepts of Stepanov-like almost automorphy and Stepanov-like pseudo almost automorphy are introduced on time scales. Some basic properties are also investigated. In section 4, we study and obtain the existence and uniqueness of pseudo-almost automorphic solution to the dynamic equations Eq.(1), Eq.(2) and Eq.(3). In section 5, we give a numerical example to illustrate the feasibility of our abstract results.

## 2. Preliminaries and basic properties

In this section we present some basic concepts and results regarding time scales that will be necessary to prove our main results . Throughout this paper we fix $p \geq 1$ and $(X, \| \, . \, \|)$ is a Banach space. We denote by $\mathbb{N}$, $\mathbb{Z}$ and $\mathbb{R}$ the set of positive integers, the set of integers and the set of real numbers, respectively.

Now, let us recall the following definitions and basic results about time scales.

### 2.1. **Time scales.**

**Definition 2.1** ( [2]). Let $\mathbb{T}$ be a time scale, that is, a closed and nonempty subset of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \longrightarrow \mathbb{T}$ and the graininess $\mu : \mathbb{T} \longrightarrow \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} \, : \, s > t\}, \quad \rho(t) = \sup\{s \in \mathbb{T} \, : \, s < t\}, \quad \mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called right-dense continuous or rd-continuous provided that it is continuous at all right-dense points in $\mathbb{T}$ and its left-side limits exist (finite) at left-dense points in $\mathbb{T}$. A function $f : \mathbb{T} \longrightarrow \mathbb{R}$ is called continuous if and only if it is both left-dense continuous and right-dense continuous. A function $p : \mathbb{T} \longrightarrow \mathbb{R}$ is called regressive provided if $1 + \mu(t)p(t) \neq 0$ for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \longrightarrow \mathbb{R}$ will be denoted by $\mathfrak{R} = \mathfrak{R}(\mathbb{T}) = \mathfrak{R}(\mathbb{T}; \mathbb{R})$. We define the set $\mathfrak{R}^+$ of all positively regressive elements by $\mathfrak{R}^+ = \mathfrak{R}^+(\mathbb{T}) = \mathfrak{R}^+(\mathbb{T}; \mathbb{R}) = \{p \in \mathfrak{R} \, : \, 1 + \mu(t)p(t) > 0 \text{ for all } t \in \mathbb{T}\}$. Let $a, b \in \mathbb{T}$, with $a \leq b$, $[a, b]$, $[a, b)$, $(a, b]$, $(a, b)$ be the usual intervals on the real line. The intervals $[a, a)$, $(a, a]$, $(a, a)$ are understood as the empty set, and we use the following symbols:

$$[a, b]_{\mathbb{T}} = [a, b] \cap \mathbb{T} \quad [a, b)_{\mathbb{T}} = [a, b) \cap \mathbb{T} \quad (a, b]_{\mathbb{T}} = (a, b] \cap \mathbb{T} \quad (a, b)_{\mathbb{T}} = (a, b) \cap \mathbb{T}.$$

**Definition 2.2** ( [2]). For $f : \mathbb{T} \to X$ and $s \in \mathbb{T} \setminus \{\max \mathbb{T}\}$, $f^{\Delta}(t) \in X$ is the delta derivative of $f$ at $s$ if for $\varepsilon > 0$, there is a neighborhood $V$ of $s$ such that for $t \in V$,

$$\| f(\sigma(s)) - f(t) - f^{\Delta}(s)(\sigma(s) - t) \| < \varepsilon \mid \sigma(s) - t \mid .$$

Moreover, $f$ is delta differentiable on $\mathbb{T}$ provided that $f^{\Delta}(s)$ exists for $s \in \mathbb{T}$.

**Definition 2.3** ( [2]). (Exponential function). If $p \in \mathfrak{R}$, then we define the exponential function by:

$$e_p(t,s) = exp\left\{ \int_s^t \xi_{\mu(\tau)}(p(\tau))\Delta\tau \right\},$$

for $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\log(1+hz)}{h}, & if\ h \neq 0, \\ z, & if\ h = 0. \end{cases}$$

**Definition 2.4** ( [2]). If $p, q \in \mathfrak{R}$, then we define a circle plus addition by

$$(p \oplus q)(t) := p(t) + q(t) + p(t)q(t)\mu(t),$$

for all $t \in \mathbb{T} \setminus \max(\mathbb{T})$. For $p \in \mathfrak{R}$, define a circle minus p by

$$\ominus p = -\frac{p}{1 + \mu p}.$$

**lemma 2.5** ( [2]). Let $p, q \in \mathfrak{R}$, and $t, s, r \in \mathbb{T}$. Then,

1) $e_0(t,s) = 1$ and $e_p(t,t) = 1$;
2) $e_p(\sigma(t), s) = (1 + p(t)\mu(t))e_p(t,s)$;
3) $e_p(t,s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s,t)$;
4) $e_p(t,r)e_p(r,s) = e_p(t,s)$;
5) $(e_p(t,s))^{\Delta} = p(t)e_p(t,s)$;
6) If $a, b, c \in \mathbb{T}$. Then, $\int_a^b e_p(c, \sigma(t))p(t)\Delta t = e_p(c,a) - e_p(c,b)$.
7) For $t_0 \in \mathbb{T}$, $e_{\ominus\alpha}(t_0, .)$ is increasing on $(-\infty, t_0]_{\mathbb{T}}$.

**lemma 2.6** ( [2]). Assume $p \in \mathfrak{R}$, and $t_0 \in \mathbb{T}$. If $1 + \mu(t)p(t) > 0$ for all $t \in \mathbb{T}$, then, $e_p(t, t_0) > 0$ for all $t \in \mathbb{T}$.

**Definition 2.7** ( [20]). A time scale $\mathbb{T}$ is called invariant under translations if

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}; \forall t \in \mathbb{T}\} \neq \{0\}.$$

2.2. **Measure theory on time scales.**

**Definition 2.8** ( [6]). Let $F_1 = \{[t, s[_{\mathbb{T}} : \ t, s \in \mathbb{T} \text{ with } \quad t \leq s\}$. Define a countably additive measure $m_1$ on $F_1$ by assigning to every $[t, s[_{\mathbb{T}} \in F_1$ its length, i.e;

$$m_1([t, s[_{\mathbb{T}}) = s - t.$$

Using $m_1$, we can generate the outer measure $m_1^*$ on $P(\mathbb{T})$ ( the power set of $\mathbb{T}$): for $E \in P(\mathbb{T})$,

$$m_1^*(E) = \begin{cases} \inf_{\mathfrak{B}}\{\sum_{i \in I_B}(s_i - t_i)\} \in \mathbb{R}^+, & \beta \notin E, \\ +\infty, & \beta \in E, \end{cases}$$

where $\beta = \sup \mathbb{T}$, and,

$$\mathfrak{B} = \{\{[t_i, s_i[_{\mathbb{T}} \in F_1\}_{i \in I_B} \ : \ I_B \subset \mathbb{N}, \ E \subset \cup_{i \in I_B}[t_i, s_i[_{\mathbb{T}}\}.$$

A set $A \subset \mathbb{T}$ is called $\Delta-$measurable if for $E \subset \mathbb{T}$,

$$m_1^*(E) = m_1^*(E \cap A) + m_1^*(E \cap (\mathbb{T} \setminus A)).$$

Let

$$\mathcal{M}^*(m_1^*) = \{A, \ A \, \text{is} \, \Delta - \text{measurable} \ \text{subset} \, \text{in} \, \mathbb{T}\}.$$

Restricting $m_1^*$ to $\mathcal{M}^*(m_1^*)$, we get the Lebesgue $\Delta-$measure, which is denoted by $\mu_\Delta$.

**Definition 2.9** ( [21]). $f : \mathbb{T} \to X$ is a $\Delta-$measurable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that, $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$.

**Definition 2.10** ( [21]). $f : \mathbb{T} \to X$ is a $\Delta-$integrable function if there exists a simple function sequence $\{f_k : k \in \mathbb{N}\}$ such that $f_k(s) \to f(s)$ a.e. in $\mathbb{T}$ and,

$$\lim_{k \to \infty} \int_{\mathbb{T}} \| \ f_k(s) - f(s) \ \| \ \Delta s = 0.$$

Then, the integral of $f$ is defined as

$$\int_{\mathbb{T}} f(s) \Delta s = \lim_{k \to \infty} \int_{\mathbb{T}} f_k(s) \Delta s.$$

**Definition 2.11** ( [21]). For $p \geq 1$, $f : \mathbb{T} \to X$ is called locally $L^p$ $\Delta-$integrable if $f$ is $\Delta-$measurable and for any compact $\Delta-$measurable set $E \subset \mathbb{T}$, the $\Delta-$integral

$$\int_E \| \ f(s) \ \|^p \ \Delta s < \infty.$$

The set of all $L^p$ $\Delta-$integrable functions is denoted by $L^p_{loc}(\mathbb{T}; X)$.

**Theorem 2.12** ( [6]). If $a, b \in \mathbb{T}$, with $a \leq b$, then,

$\mu_\Delta([a,b)) = b - a,$

$\mu_\Delta((a,b)) = b - \sigma(a).$

**Theorem 2.13** ( [6]). If $a, b \in \mathbb{T} \setminus \{\max\mathbb{T}\}$, with $a \leq b$, then,

$\mu_\Delta((a,b]) = \sigma(b) - \sigma(a),$

$\mu_\Delta([a,b]) = \sigma(b) - a.$

2.3. **Almost autmorphy and Pseudo almost autmorphy on time scales.** This sub-section reminds us of the notion of pseudo-almost automorphic functions on time scales and also some basic properties concerning this type of functions. In addition, the following notations are presented :

- $C(\mathbb{T}; X) = \{f : \mathbb{T} \to X : f$ is continuous$\}$,
- $C(\mathbb{T} \times X; Y) = \{f : \mathbb{T} \times X \to Y : f$ is continuous$\}$,
- $BC(\mathbb{T}; X) = \{f : \mathbb{T} \to X : f$ is bounded and continuous$\}$,
- $L^p_{loc}(\mathbb{T}; X) = \{f : \mathbb{T} \to X : f$ is locally $L^p\ \Delta -$ integrable$\}$.

**Definition 2.14** ( [19]). Let $f \in BC(\mathbb{T}, X)$.

1) We say that $f : \mathbb{T} \to X$ is almost automorphic if from every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that :

$$g(t) = \lim_{n\to\infty} f(t + \tau_n)$$

is well defined for each $t \in \mathbb{T}$ and

$$\lim_{n\to\infty} g(t - \tau_n) = f(t)$$

for each $t \in \mathbb{T}$. Denote by $AA(\mathbb{T}, X)$ the set of all such functions.

2) A continuous function $f : \mathbb{T} \times X \to X$ is said to be almost automorphic if $f(t, x)$ is almost automorphic in $t \in \mathbb{T}$ uniformly in $x \in B$, where $B$ is any bounded subset of $X$. Denote by $AA(\mathbb{T} \times X, X)$ the set of all such functions.

**Definition 2.15** ( [26]). Let $t_0 \in \mathbb{T}$ and $r \in \Pi$.

1) A function $f \in BC(\mathbb{T}, X)$, is said to be ergodic if

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \mid f(s) \mid \Delta s = 0,$$

The space of all such functions is denoted by $PAA_0(\mathbb{T}, X)$.

2) Similarly, we define $PAA_0(\mathbb{T}\times X, X)$ as the collection of all functions $F:\mathbb{T}\times X\to X$ is continuous with respect to its two arguments and $F(\cdot,y)$ is bounded for each $y\in X$, and

$$\lim_{r\to+\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\mid F(s)\mid\Delta s=0,$$

uniformly in $y\in X$, where $r\in\Pi$.

**Proposition 2.16** ( [26]). $\big(PAA_0(\mathbb{T},X),\|\ .\ \|\big)$ is a Banach space.

**Proposition 2.17** ( [26]). Let $f\in PAA_0(\mathbb{T},X)$, then $f(.+\tau)\in PAA_0(\mathbb{T},X)$, for all $\tau\in\Pi$.

**Definition 2.18** ( [26]). A function $f\in BC(\mathbb{T},X)$ is called pseudo almost automorphic if $f=g+\varphi$ where $g\in AA(\mathbb{T},X)$ and $\varphi\in PAA_0(\mathbb{T},X)$.

**Definition 2.19.** A function $f\in BC(\mathbb{T}\times X,X)$ is called pseudo almost automorphic if $f=g+\varphi$ where $g\in AA(\mathbb{T}\times X,X)$ and $\varphi\in PAA_0(\mathbb{T}\times X,X)$.

**Theorem 2.20** ( [26]). The decomposition of a pseudo almost automorphic function is unique.

**Theorem 2.21** ( [26]). $(PAA(\mathbb{T},X),\|\ .\ \|)$ is a Banach space.

**lemma 2.22** ( [26]). Let $f\in PAA(\mathbb{T},X)$, then $f(.+\tau)\in PAA(\mathbb{T},X)$, for all $\tau\in\Pi$.

## 3. Stepanov-like almost automorphy and Stepanov-like Pseudo almost automorphy on time scales

3.1. **Stepanov-like almost automorphic functions on $\mathbb{T}$.** In this subsection, we shall introduce a new generalization of almost automorphic functions called Stepanov-like almost automorphic functions on time scales, and show that a function is Stepanov-like almost automorphic if and only if its Bochner-like transform is almost automorphic on time scales. Also, we investigate their basic properties.

Firstly, we recall the definition of Stepanov's bounded functions on time scales. Let

$$K=\begin{cases}\inf\big\{\mid\tau\mid;\,\tau\in\mathbb{T},\,\tau\neq0\big\}, & if\ \mathbb{T}\neq\mathbb{R},\\ 1, & if\ \mathbb{T}=\mathbb{R}.\end{cases}$$

Let $f\in L^p_{loc}(\mathbb{T},X)$. Define $\|\ .\ \|_{S^p}$: $L^p_{loc}(\mathbb{T},X)\to\mathbb{R}^+$ as

$$\|\ f\ \|_{S^p}=\sup_{t\in\mathbb{T}}\bigg(\frac{1}{K}\int_t^{t+k}\mid f(s)\mid^p\Delta s\bigg)^{\frac{1}{p}}.$$

**Definition 3.1** ( [24]). Let $f \in L^p_{loc}(\mathbb{T}, X)$. A function $f$ is said to be bounded in the sense of Stepanov on time scales if

$$\| f \|_{S^p} < \infty.$$

Denote by $BS^p(\mathbb{T}, X)$ the space of all these functions. Then, the following inclusions hold

$$BC(\mathbb{T}, X) \subset BS^p(\mathbb{T}, X) \subset L^p_{loc}(\mathbb{T}, X).$$

**Definition 3.2.** Let $f \in BS^p(\mathbb{T}, X)$. We say that $f : \mathbb{T} \to X$ is Stepanov-like almost automorphic if for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\| g(t) - f(t + \tau_n) \|_{S^p} \to 0, \quad as \quad n \to \infty,$$

is well defined for each $t \in \mathbb{T}$ and

$$\| g(t - \tau_n) - f(t) \|_{S^p} \to 0, \quad as \quad n \to \infty,$$

for each $t \in \mathbb{T}$. Denote by $S^p AA\,(\mathbb{T}, X)$ the set of all such functions.

**Definition 3.3.** Let $F \in BS^p(\mathbb{T} \times X, X)$. A function $F : \mathbb{T} \times X \to X$, $(t, x) \to F(t, x)$ is said to be Stepanov-like almost automorphic if $t \to F(t, x)$ is Stepanov almost automorphic in $t \in \mathbb{T}$ uniformly for each $x \in X$. Denote by $S^p AA\,(\mathbb{T} \times X, X)$ the collection of such functions.

*Remark* 3.4. In definition (3.1), when the convergence is uniform for $t \in \mathbb{T}$, one can show that $f$ is Stepanov-like almost periodic. Hence, one can easily see that if $X = \mathbb{R}^n$, definition (3.1) is more general than definition (3.1) in ( [7]).

Next, we recall the Bochner-like transform for general time scales.

If $\mathbb{T} \neq \mathbb{R}$, we fix a left scattered point $\omega \in \mathbb{T}$, there is a unique $n_t \in \mathbb{Z}$ such that $t - n_t K \in [\omega, \omega + k)_{\mathbb{T}}$. Let

$$N_t = \begin{cases} t, & \mathbb{T} = \mathbb{R}, \\ n_t & \mathbb{T} \neq \mathbb{R}. \end{cases}$$

**Definition 3.5** ( [25]). Let $f \in BS^p(\mathbb{T}, X)$. The Bochner-like transform of $f$ is the function $f^b : \mathbb{T} \times \mathbb{T} \to X$ defined for all $t, s \in \mathbb{T}$ by

$$f^b(t, s) = f(N_t K + s).$$

And we have

$$\| f \|_{S^p} = \| f^b \|_{\infty} .$$

In the following theorem, we propose the concept of Stepanov-like almost automorphy on time scales by using the Bochner-like transform.

**Theorem 3.6.** *The space $S^pAA(\mathbb{T}, X)$ of Stepanov-like almost automorphic functions consists of all $f \in BS^p(\mathbb{T}, X)$ such that : $f^b \in AA(\mathbb{T}, BS^p(\mathbb{T}, X))$, i.e. a function $f$ is Stepanov-like almost automorphic if and only if its Bochner-like transform $f^b$ is almost automorphic.*

*Proof.* By definitions (3.2, 3.5), for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$ we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\lim_{n\to\infty} \| g(t) - f(t+\tau_n) \|_{S^p} = 0 \Leftrightarrow \lim_{n\to\infty} \| g^b(t) - f^b(t+\tau_n) \|_{\infty} = 0,$$

and

$$\lim_{n\to\infty} \| g(t-\tau_n) - f(t) \|_{S^p} = 0 \Leftrightarrow \lim_{n\to\infty} \| g^b(t-\tau_n) - f^b(t) \|_{\infty} = 0.$$

Finally, $f \in S^pAA(\mathbb{T}, X)$ if and only if $f^b \in AA(\mathbb{T}, BS^p(\mathbb{T}, X))$. □

**Corollary 3.7.** *$f \in S^pAA(\mathbb{T} \times X, X)$ if and only if $f^b \in AA(\mathbb{T} \times X, BS^p(\mathbb{T}, X))$.*

In what follows, we will present an interesting property of Stepanov-like almost automorphic functions on time scales.

**Theorem 3.8.** *The space of all Stepanov-like almost automorphic functions $S^pAA(\mathbb{T}, X)$ is a closed linear subspace of $BS^p(\mathbb{T}, X)$.*

*Proof.* Let $f, h \in S^pAA(\mathbb{T}, X)$. Then,

$$\| f + h \|_{S^p} = \| f^b + h^b \|_{\infty} = \sup_{t\in\mathbb{T}} \left( \frac{1}{K} \int_t^{t+K} | f(s) + h(s) |^p \Delta s \right)^{\frac{1}{p}}.$$

According to Minkowski's inequality we have;

$$\begin{aligned}
\| f + h \|_{S^p} &= \sup_{t\in\mathbb{T}} \left( \frac{1}{K} \int_t^{t+K} | f(s) + h(s) |^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq \sup_{t\in\mathbb{T}} \left( \frac{1}{K} \int_t^{t+K} | f(s) |^p \Delta s \right)^{\frac{1}{p}} + \sup_{t\in\mathbb{T}} \left( \frac{1}{K} \int_t^{t+K} | h(s) |^p \Delta s \right)^{\frac{1}{p}}, \\
&= \| f^b \|_{\infty} + \| h^b \|_{\infty}, \\
&= \| f \|_{S^p} + \| h \|_{S^p} .
\end{aligned}$$

It follows that $f + h \in S^pAA(\mathbb{T}, X)$. On another side, it's clear that for any scalar $\alpha$ we get, $\alpha f \in S^pAA(\mathbb{T}, X)$.

Finally, one can prove using the Minkowski's inequality that if $(f_n)_{n\in\mathbb{N}}$ is a sequence in $S^pAA(\mathbb{T},X)$ which converges to $f$ in $S^p$ norm, then, $f$ is a $S^p$-almost automorphic functions on $\mathbb{T}$. Which ends the demonstration. □

**Proposition 3.9.** *It's easy to see that,*

1) *If $f \in AA(\mathbb{T},X)$, then $f \in S^pAA(\mathbb{T},X)$.*
2) *Let $f \in S^pAA(\mathbb{T},X)$ and $h \in AA(\mathbb{T},X)$, then $f+h \in S^pAA(\mathbb{T},X)$.*
3) *Let $1 \le p < q < \infty$ and let $f \in BS^p(\mathbb{T},X)$. If $f$ is $S^q$- almost automorphic, then $f$ is $S^p$-almost automorphic, i.e. $S^q(\mathbb{T},X) \subset S^p(\mathbb{T},X)$ and $\| f \|_{S^p} \le \| f \|_{S^q}$ .*
4) *If $f \in S^pAA(\mathbb{T},X)$, then for any $\tau \in \Pi$, we have $f(.+\tau) \in S^pAA(\mathbb{T},X)$.*
5) *If the functions $f,g : \mathbb{T} \to X$ are Stepanov-like almost automorphic, then the function $gf : \mathbb{T} \to X$ defined by $(gf)(t) = g(t)f(t)\ \forall\, t \in \mathbb{T}$, is also Stepanov-like almost automorphic.*

Now, we shall present several examples of Stepanov-like almost automorphic functions on time scales.

**Example 3.10.** Let $\mathbb{T}$ be a time scales, we consider the following functions :

1) $f : \mathbb{T} \to \mathbb{R}, \quad f(t) = \sin(2t).$
2) $g : \mathbb{T} \to \mathbb{R}, \quad g(t) = \cos\left(\frac{1}{2+\sin(t)+\sin(\sqrt{2}t)}\right).$
3 $h : \mathbb{T} \to \mathbb{R}, \quad h(t) = \sin(t) + \cos(\sqrt{2}t).$

We use the fact that $AA(\mathbb{T},X) \subset S^PAA(\mathbb{T},X)$ (proposition (3.9), 1),then we have that $f$ and $g$ are Stepanov-like almost automorphic functions on $\mathbb{T}$.

In the following, we will denote by $AAU(\mathbb{T},X)$ the closed subspace of all automorphic functions $f$ with $g \in C(\mathbb{T},X)$. That is to say, $f \in AAU(\mathbb{T},X)$ if and only if $f \in AA(\mathbb{T},X)$ and the convergences in definition (2.14) are uniform on compact intervals (in the space $C(\mathbb{T},X)$).

Consequently,

$$AP(\mathbb{T},X) \subset AAU(\mathbb{T},X) \subset AA(\mathbb{T},X) \subset BC(\mathbb{T},X).$$

**Theorem 3.11.** *The following assertions are equivalent,*

(1) $f^b \in AAU(\mathbb{T}, BS^p(\mathbb{T},X))$,
(2) *from every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that*

$$g(t) = \lim_{n\to\infty} f(t+\tau_n)$$

*exists in the space* $L^p_{Loc}(\mathbb{T}, X)$,

*and*

$$\lim_{n\to\infty} g(t-\tau_n) = f(t)$$

*in the sence of* $L^p_{Loc}(\mathbb{T}, X)$,

(3) $f \in S^pAA(\mathbb{T}, X)$.

*Proof.* By similar proof of [theorem 2.2, [23] ] and based on definition of Bochner-like transform (3.5), it's easy to show the previous theorem. □

In the sequel, for any bounded set $\Omega \subset X$ we will denote by $S^pAA_\Omega(\mathbb{T}\times X, X)$ the space af all $S^p$-almost automorphic functions on $\mathbb{T}$ and all convergences in definition (3.3) are uniform for $x \in \Omega$.

For the rest of this work, we will need the following hypothesis :

$(H_1)$ : There exists a nonnegative function $L_f(.) \in BS^p(\mathbb{T}, X)$ such that for all $u, v \in L^p_{loc}(\mathbb{T}, X)$ and $t \in \mathbb{T}$,

$$\left(\frac{1}{K}\int_t^{t+K} \| f(s,u(s)) - f(s,v(s)) \|^p \Delta s\right)^{\frac{1}{p}} \leq L_f(s)\left(\frac{1}{K}\int_t^{t+K} \| u(s)-v(s) \|^p \Delta s\right)^{\frac{1}{p}}.$$

**lemma 3.12.** *Let* $\Omega$ *be an arbitrary compact subset of* $X$*. Assume that* $(H_1)$ *holds and* $f \in S^pAA(\mathbb{T}\times X, X)$*. Then,* $f \in S^pAA_\Omega(\mathbb{T}\times X, X)$.

*Proof.* First, since $\Omega$ is a compact subset of $X$, for any $\delta > 0$, there exist $u_1, u_2, ..., u_m \in \Omega$ and $B$ an open ball with center $u_j$, $j = 1, ..., m$ and radius $\delta$ such that,

$$\Omega \subseteq \bigcup_{j=1}^{m} B(u_j, \delta).$$

Let $x \in \Omega$, there exists $u_i \in \{u_1, u_2, ..., u_m\}$ such that,

$$\| x - u_i \|_{S^p} \leq \delta.$$

We have $f \in S^pAA(\mathbb{T}, X)$, then for every sequence $\{s_n\}_{n=1}^\infty \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^\infty$ such that for each $t \in \mathbb{T}$ and $x \in X$,

$$(4) : \lim_{n\to\infty} \| f(t+\tau_n+.,x) - g(t+.,x) \|_{S^p} = \lim_{n\to\infty} \| g(t-\tau_n+.,x) - f(t+.,x) \|_{S^p} = 0.$$

So, for all $\delta > 0$ there exists integer an $N$ such that for all $n > N$ and all $j = 1, ..., m$ we get,

$$\| f(t+\tau_n+.,u_j) - g(t+.,u_j) \|_{S^p} < \frac{\delta}{2}.$$

Furthermore, for all $\delta > 0$ and $x, y \in X$ there exists $N_0 \in \mathbb{N}$ such that for all $n > N_0$ we obtain,

$$\| f(t + \tau_n + ., x) - g(t + ., x) \|_{S^p} < \frac{\delta}{4}.$$

And,

$$\| f(t + \tau_n + ., y) - g(t + ., y) \|_{S^p} < \frac{\delta}{4}.$$

Therefore, by $(H_1)$ and for all $n > N_0$ we have,

$$\begin{aligned}
\| g(t + ., x) - g(t + ., y) \|_{S^p} &\leq \| g(t + ., x) - f(t + \tau_n + ., x) \|_{S^p} \\
&+ \| f(t + \tau_n + ., x) - f(t + \tau_n + ., y) \|_{S^p} \\
&+ \| f(t + \tau_n + ., y) - g(t + ., y) \|_{S^p}, \\
&\leq \frac{\delta}{4} + L_f \| x - y \|_{S^p} + \frac{\delta}{4}, \\
&\leq \frac{\delta}{2} + L_f \| x - y \|_{S^p} .
\end{aligned}$$

Now, by using hypothesis $(H_1)$ and for all $n > N_0$ we find that,

$$\begin{aligned}
\| f(t + \tau_n + ., x) - g(t + ., x) \|_{S^p} &\leq \| f(t + \tau_n + ., x) - f(t + \tau_n + ., u_i) \|_{S^p} \\
&+ \| f(t + \tau_n + ., u_i) - g(t + ., u_i) \|_{S^p} \\
&+ \| g(t + ., u_i) - g(t + ., x) \|_{S^p}, \\
&\leq L_f \| x - u_i \|_{S^p} + \frac{\delta}{2} + \frac{\delta}{2} + L_f \| x - u_i \|_{S^p}, \\
&\leq 2L_f \| x - u_i \|_{S^p} + \delta, \\
&\leq 2\delta L_f + \delta, \\
&\leq (2L_f + 1)\delta.
\end{aligned}$$

It follows that

$$\| f(t + \tau_n + ., x) - g(t + ., x) \|_{S^p} \to 0 \quad as \quad n \to \infty$$

uniformly for $x \in \Omega$. By the same steps, we can show that

$$\| g(t - \tau_n + ., x) - f(t + ., x) \|_{S^p} \to 0 \quad as \quad n \to \infty$$

uniformly for $x \in \Omega$. Which implies that $f \in S^p AA_\Omega(\mathbb{T} \times X, X)$. □

In the following theorem, we establish a new composition theorem for Stepanov-like almost automorphic functions on time scales which is more general.

**Theorem 3.13.** *Suppose that $f \in S^pAA(\mathbb{T}\times X, X)$ and $(H_1)$ holds. If*

(1) $x \in S^pAA(\mathbb{T}, X)$.

(2) $\Omega = \overline{\{x(t); t \in \mathbb{T}\}}$ *is compact.*

*Then, $f(., x(.)) \in S^pAA(\mathbb{T}, X)$.*

*Proof.* On the one hand, by hypothesis $(H_1)$ we get,

$$\| f(., x(.)) - f(., 0) \|_{S^p} \leq L_f \| x \|_{S^p},$$

thus,

$$\| f(., x(.)) \|_{S^p} \leq \| f(., 0) \|_{S^p} + L_f \| x \|_{S^p} .$$

Yields $f \in BS^p(\mathbb{T}, X)$.

On the other hand, due to lemma (3.12) and assumptions of the theorem, for every sequence $\{s_n\}_{n=1}^{\infty} \subset \Pi$, we can extract a subsequence $\{\tau_n\}_{n=1}^{\infty}$ such that for each $t \in \mathbb{T}$ and uniformly for $x \in X$,

$$\lim_{n\to\infty} \| f(t + \tau_n + ., u) - g(t + ., u) \|_{S^p} = \lim_{n\to\infty} \| g(t - \tau_n + ., u) - f(t + ., u) \|_{S^p} = 0.$$

We have $x \in S^pAA(\mathbb{T}, X)$, then by theorem (3.11) we obtain $x^b \in AAU(BS^p(\mathbb{T}, X))$, which implies that there exists a function $y : \mathbb{T} \to X$ such that

$$\lim_{n\to\infty} \| x(t + \tau_n + .) - y(t + ., u) \|_{S^p} = \lim_{n\to\infty} \| y(t - \tau_n + .) - x(t + .) \|_{S^p} = 0$$

uniformly for $t$ on any compact interval.

Therefore, by the inequality of Minkowski, we find that

$$\| f(t + \tau_n + ., x(t + \tau_n + .)) - g(t + ., y(t + .)) \|_{S^p}$$

$$\leq \| f(t + \tau_n + ., x(t + \tau_n + .)) - f(t + \tau_n + ., y(t + .)) \|_{S^p},$$

$$+ \| f(t + \tau_n + ., y(t + .)) - g(t + ., y(t + .)) \|_{S^p} .$$

By condition $(H_1)$, we see that for all $t \in \mathbb{T}$

$$\| f(t + \tau_n + ., x(t + \tau_n + .)) - f(t + \tau_n + ., y(t + .)) \|_{S^p} \to 0 \quad as \quad n \to \infty.$$

Since $\Omega = \overline{\{x(t); t \in \mathbb{T}\}}$, we have $y(t + .) \in \Omega$. Now, from the compactness of $\Omega = \overline{\{x(t); t \in \mathbb{T}\}}$, we have

$$\| f(t + \tau_n + ., y(t + .)) - g(t + ., y(t + .)) \|_{S^p} \to 0 \quad as \quad n \to \infty,$$

for each $t \in \mathbb{T}$.

It follows that for each $t \in \mathbb{T}$,

$$\lim_{n\to\infty} \parallel f(t+\tau_n+.,x(t+\tau_n+.)) - g(t+.,y(t+.)) \parallel_{S^p} = 0.$$

Similarly, we show that

$$\lim_{n\to\infty} \parallel g(t-\tau_n+.,y(t-\tau_n+.)) - f(t+.,x(t+.)) \parallel_{S^p} = 0,$$

for each $t \in \mathbb{T}$. Finally, $f(.,x(.)) : \mathbb{T} \to X$ is Stepanov-like almost automorphic function on time scales. □

3.2. **Stepanov-like pseudo almost automorphic functions on $\mathbb{T}$.** This subsection is devoted to definitions and the important properties of Stepanov-like pseudo almost automorphic functions on $\mathbb{T}$. Namely, the completeness, translation invariance and the composition theorem.

Let $f \in BS^p(\mathbb{T}, X)$, $t_0 \in \mathbb{T}$ and $r \in \Pi$. We define the set $S^pPAA_0(\mathbb{T}, X)$ of Stepanov-like ergodic functions on time scales as follows :

$$\lim_{r\to\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_t^{t+K} \mid f(s) \mid^p \Delta s \right)^{\frac{1}{p}} \Delta t = 0.$$

Similarly, we define $S^pPAA_0(\mathbb{T}\times X, X)$ as the collection of all functions $f \in BS^p(\mathbb{T}\times X, X)$, such that $f : \mathbb{T}\times X \to X$ is continuous with respect to its two arguments and $f(.,y)$ is bounded for each $y \in X$, and

$$\lim_{r\to\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_t^{t+K} \mid f(s,y) \mid^p \Delta s \right)^{\frac{1}{p}} \Delta t = 0,$$

uniformly in $y \in X$.

We are now ready to introduce the sets $S^pPAA(\mathbb{T}, X)$ and $S^pPAA(\mathbb{T}\times X, X)$ of Stepanov-like pseudo almost automorphic functions on time scales.

**Definition 3.14.** A function $f \in BS^p(\mathbb{T}, X)$ is said to be Stepanov-like pseudo almost automorphic if $f$ is written in the following form :

$$f = g + \phi,$$

where $g \in S^pAA(\mathbb{T}, X)$ and $\phi \in S^pPAA_0(\mathbb{T}, X)$. The space of all such functions is denoted by $S^pPAA(\mathbb{T}, X)$.

As well, a function $f : \mathbb{T} \times X \to X$ such that for each $y \in X$, $f(.,y) \in BS^p(\mathbb{T} \times X, X)$ is said to be Stepanov-like pseudo almost automorphic if $f$ is written in the following form:

$$f = g + \phi,$$

where $g \in S^pAA(\mathbb{T} \times X, X)$ and $\phi \in S^pPAA_0(\mathbb{T} \times X, X)$. The space of all such functions is denoted by $S^pPAA(\mathbb{T} \times X, X)$.

By using the Bochner-like transform, we give a new definition of Stepanov-like pseudo almost automorphic functions on time scales.

**Definition 3.15.** A function $f \in BS^p(\mathbb{T}, X)$ is said to be Stepanov-like pseudo almost automorphic if its Bochner-like transform $f^b$ is pseudo almost automorphic in the sense that there exist two functions $g, \phi$ such that $f^b = g^b + \phi^b$, where $g^b \in AA(\mathbb{T}, BS^p(\mathbb{T}, X))$ and $\phi^b \in PAA_0(\mathbb{T}, BS^p(\mathbb{T}, X))$.

**Example 3.16.** Let $\mathbb{T} = \bigcup_{k=1}^{k=\infty}[2k, 2k+1]$, then $K = 2$ and $\Pi = 2\mathbb{Z}$. We consider the function $f : \mathbb{T} \to \mathbb{R}$ defined by,

$$f(t) = g(t) + h(t).$$

Where,

$$g(t) = \begin{cases} t & t \in \Pi, \\ \sin(\pi t) & t \in \mathbb{T} \backslash \Pi, \end{cases}$$

and,

$$h(t) = -\frac{1}{t\sigma(t)}.$$

It's easy to show that $g \in S^1AP(\mathbb{T}, X)$. it remains to show that $h \in S^1PAA_0(\mathbb{T}, X)$. indeed, for $t_0 = 2$ we get,

$$\begin{aligned} \frac{1}{2r}\int_{2-r}^{2+r}\left(\frac{1}{2}\int_t^{t+K} \mid h(s) \mid \Delta s\right)\Delta t &= \frac{1}{2r}\int_{2-r}^{2+r}\left(\frac{1}{2}\int_t^{t+2} \mid -\frac{1}{s\sigma(s)} \mid \Delta s\right)\Delta t, \\ &= \frac{1}{2r}\int_{2-r}^{2+r}\left(\frac{1}{2}[\frac{1}{s}]_t^{t+2}\right)\Delta t, \\ &= \frac{1}{2r}\int_{2-r}^{2+r}\left(\frac{1}{2}\left(\frac{1}{t+2} - \frac{1}{t}\right)\right)\Delta t, \end{aligned}$$

$$= \frac{1}{2r}\left[\frac{1}{2}\left(\ln\left(\frac{t+2}{t}\right)\right)\right]_{2-r}^{2+r},$$
$$= \frac{1}{2r}\left[\frac{1}{2}\left(\ln\left(\frac{2+r+2}{2+r}\right) - \ln\left(\frac{2-r+2}{2-r}\right)\right)\right],$$
$$= \frac{1}{2r}\left[\frac{1}{2}\left(\ln\left(\frac{4+r}{2+r}\right) - \ln\left(\frac{r}{r-2}\right)\right)\right].$$

So,

$$\lim_{r\to\infty}\frac{1}{2r}\int_{2-r}^{2+r}\left(\frac{1}{2}\int_{t}^{t+K} \mid h(s) \mid \Delta s\right)\Delta t = \lim_{r\to\infty}\frac{1}{2r}\left[\frac{1}{2}\left(\ln\left(\frac{4+r}{2+r}\right) - \ln\left(\frac{r}{r-2}\right)\right)\right] = 0.$$

Then, $h \in S^1PAA_0(\mathbb{T}, X)$. Finally, $f \in S^1PAA(\mathbb{T}, X)$.

**Proposition 3.17.** *If $f \in S^pPAA(\mathbb{T}, X)$, then for $\tau \in \Pi$ we have, $f(. + \tau) \in S^pPAA(\mathbb{T}, X)$.*

*Proof.* Based on proposition (2.17) and proposition ((3.9), 4), we can easily prove the previous proposition. □

**Theorem 3.18.** *Assume that $S^pPAA(\mathbb{T}, X)$ is translation invariant. Then the decomposition of a Stepanov-like pseudo almost automorphic functions according to $S^pAA(\mathbb{T}, X) \oplus S^pPAA_0(\mathbb{T}, X)$ is unique.*

*Proof.* Let $f \in S^pPAA(\mathbb{T}, X)$, we suppose there exists $g_1, g_2 \in S^pAA(\mathbb{T}, X)$ and $h_1, h_2 \in S^pPAA_0(\mathbb{T}, X)$ such taht $f = g_1 + h_1 = g_2 + h_2$. Therefore $g_1^b - g_2^b = h_2^b - h_1^b$. We know that $g_1^b - g_2^b \in AA(\mathbb{T}, BS^p(\mathbb{T}, X))$ and $g_1^b - g_2^b \in PAA_0(\mathbb{T}, BS^p(\mathbb{T}, X))$. It follows from [lemma 3.1, [26]] that $g_1^b - g_2^b = 0$, we deduce that $g_1 = g_2$ in $S^pAA(\mathbb{T}, X)$. Meanwhile, $h_1 = h_2$ in $S^pPAA(\mathbb{T}, X)$. Finally, the decomposition of a Stepanov-like pseudo almost automorphic functions is unique. □

It is easy to check the following lemmas.

**lemma 3.19.** *For $1 \le p < q < +\infty$ we have, $S^qPAA(\mathbb{T}, X) \subset S^pPAA(\mathbb{T}, X)$.*

**lemma 3.20.** *Assume that $f, h \in S^pPAA(\mathbb{T}, X)$ and $\alpha \in \mathbb{T}$. Then,*

1) $f + h \in S^pPAA(\mathbb{T}, X)$.
2) $\alpha f \in S^pPAA(\mathbb{T}, X)$.

**Theorem 3.21.** *If $f \in PAA(\mathbb{T}, X)$, then $f \in S^pPAA(\mathbb{T}, X)$, i.e. $PAA(\mathbb{T}, X) \subset S^pPAA(\mathbb{T}, X)$.*

*Proof.* Let $f \in PAA(\mathbb{T}, X)$, so, we can write $f$ in the following form $f = g + h$ such that $g \in AA(\mathbb{T}, X)$ and $h \in PAA_0(\mathbb{T}, X)$. We know that [From proposition 3.9], $AA(\mathbb{T}, X) \subset S^pAA(\mathbb{T}, X)$, which implies that $g \in S^pAA(\mathbb{T}, X)$.
Now, it remains to show that $h \in S^pPAA_0(\mathbb{T}, X)$. Indeed, let $q > 0$ such that $\frac{1}{p} + \frac{1}{q} = 1$, then by Hölder inequality, we get

$$\begin{aligned} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_t^{t+K} \| \, h(s) \, \|^p \, \Delta s \right)^{\frac{1}{p}} \Delta t &= \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_0^K \| \, h(s+t) \, \|^p \, \Delta s \right)^{\frac{1}{p}} \Delta t, \\ &\leq (2r)^{\frac{1}{q}} \left( \int_{t_0-r}^{t_0+r} \| \, h(s_0+t) \, \| \, . \, \| \, h \, \|^{p-1} \, \Delta t \right)^{\frac{1}{p}}. \end{aligned}$$

Where $h(s_0+t) = \sup_{s \in [0,K]_{\mathbb{T}}} h(s+t)$. Then,

$$\begin{aligned} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_t^{t+K} \| \, h(s) \, \|^p \, \Delta s \right)^{\frac{1}{p}} \Delta t &\leq (2r)^{\frac{1}{q}} \, . \, \| \, h \, \|^{\frac{p-1}{p}} \, . \left( \int_{t_0-r}^{t_0+r} \| \, h(s_0+t) \, \| \, \Delta t \right)^{\frac{1}{p}}, \\ &\leq 2r. \, \| \, h \, \|^{\frac{p-1}{p}} \, . \left( \frac{1}{2r} \int_{t_0-r}^{t_0+r} \| \, h(s_0+t) \, \| \, \Delta t \right)^{\frac{1}{p}}, \end{aligned}$$

Since $h \in PAA_0(\mathbb{T}, X)$ and $PAA_0(\mathbb{T}, X)$ is translation invariant we get,

$$\lim_{r \to \infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \| \, h(s_0+t) \, \| \, \Delta t = 0.$$

Which implies that

$$\lim_{r \to \infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_t^{t+K} \| \, h(s) \, \|^p \, \Delta s \right)^{\frac{1}{p}} \Delta t = 0.$$

Finally, $h \in S^pPAA_0(\mathbb{T}, X)$. □

Figure 1 : The diagram of relationship between almost automorphic functions on time scales and their extensions, with $\rightarrow$ denotes subset relation $\subset$ .

$$\begin{array}{ccccccc} AP(\mathbb{T},X) & & \rightarrow & & PAP(\mathbb{T},X) & & \\ \downarrow & \searrow & & & & \searrow & \\ \downarrow & & S^pAP(\mathbb{T},X) & & \rightarrow & \rightarrow & S^pPAP(\mathbb{T},X) \\ \downarrow & & \downarrow & & & & \downarrow \\ \downarrow & & S^pAP(\mathbb{T},X) & & \rightarrow & \rightarrow & S^pPAA(\mathbb{T},X) \\ \downarrow & \nearrow & & & & \nearrow & \\ AA(\mathbb{T},X) & & \rightarrow & & PAA(\mathbb{T},X) & & \end{array}$$

*Remark* 3.22. It is clear that the space of all Stepanov-like pseudo almost automorphic functions on time scales is more general, richer and have become more important than the classic space of Stepanov-like pseudo almost periodic, almost automorphic and pseudo almost automorphic functions on time scales.

*Remark* 3.23. When time scale $\mathbb{T} = \mathbb{R}$, our definition of the Stepanov-like pseudo almost automorphic functions is equivalent to the classical definitions of Stepanov-like pseudo almost automorphic functions ( [7]).

By combining the definition of Stepanov-like pseudo almost automorphic functions on time scales with the definition of Bochner-like transform, one can easily show the following technical lemma.

**lemma 3.24.** *If* $f = g + h \in WS^pPAA(\mathbb{T}, X, \mu)$, *where* $g \in S^pAA(\mathbb{T}, X)$ *and* $h \in WS^pPAA_0(\mathbb{T}, X, \mu)$, *then* $g(\mathbb{T}) \subset \overline{f(\mathbb{T})}$ *and* $\| g \|_{S^p} \leq \| f \|_{S^p}$ .

In the sequel, we present a result which ensures that $S^pPAA(\mathbb{T}, X)$ is a Banach space equipped with the norm $\| \, . \, \|_{S^p}$ .

**lemma 3.25.** *The space* $S^pPAA(\mathbb{T}, X)$ *equipped with the norm* $\| \, . \, \|_{S^p}$ *is a Banach space.*

*Proof.* Let $(f_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $S^pPAA(\mathbb{T}, X)$, such that $(f_n)_{n\in\mathbb{N}} \to f$ as $n \to \infty$. Let us show that $f \in S^pPAA(\mathbb{T}, X)$. In fact, there exist two sequences of functions $(g_n)_{n\in\mathbb{N}} \in S^pAA(\mathbb{T}, X)$ and $(h_n)_{n\in\mathbb{N}} \in S^pPAA_0(\mathbb{T}, X)$, such that $(f_n)_{n\in\mathbb{N}}$ is given by $f_n = g_n + h_n$, then from the statement of the above lemma (3.24) we have for all $n \in \mathbb{N}$, $\| g_n \|_{S^p} \leq \| f_n \|_{S^p}$ . Thus there exists a Stepanov-like almost automorphic function $g$ such that

$$\lim_{n\to\infty} \| g_n - g \|_{S^p} = 0.$$

Now, we need to show the closedness of $S^pPAA_0(\mathbb{T}, X)$ in $BS^p(\mathbb{T}, X)$. Indeed, from the statement of the above lemma, there exists a function $h \in BS^p(\mathbb{T}, X)$ such that

$$\lim_{n\to\infty} \| h_n - h \|_{S^p} = 0.$$

Meanwhile,

$$\begin{aligned}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}\parallel h(s)\parallel^p\Delta s\right)^{\frac{1}{p}}\Delta t\leq&\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}\parallel h_n(s)-h(s)\parallel^p\Delta s\right)^{\frac{1}{p}}\Delta t\\&+\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}\parallel h_n(s)\parallel^p\Delta s\right)^{\frac{1}{p}}\Delta t,\\&\leq\parallel h_n(s)-h(s)\parallel_{S^p}\\&+\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}\parallel h_n(s)\parallel^p\Delta s\right)^{\frac{1}{p}}\Delta t.\end{aligned}$$

It follows that

$$\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K}\parallel h(s)\parallel^p\Delta s\right)^{\frac{1}{p}}=0.$$

That is, $f\in S^pPAA(\mathbb{T},X)$, which finishes the proof. □

Next, we establish a new composition theorem for Stepanov-like pseudo almost automorphic functions on time scales which generalizes (theorem 3.5, [7]) and (theorem 3.14, [24]) .

**Theorem 3.26.** *Assume that $f=g+h\in S^pPAA(\mathbb{T},X)$ with $g\in S^pAA(\mathbb{T},X)$, $h\in S^pPAA_0(\mathbb{T},X)$ and,*

i) *For any $x,y\in X$ and $t\in\mathbb{T}$ there exist constants $L_f,L_g>0$ such that :*

$$\parallel f(t,x)-f(t,y)\parallel\leq L_f\parallel x-y\parallel\ \ \parallel g(t,x)-g(t,y)\parallel\leq L_g\parallel x-y\parallel.$$

ii) *$\varphi=u+v\in S^pPAA(\mathbb{T},X)$ with*

$$u\in S^pAA(\mathbb{T},X),v\in S^pPAA_0(\mathbb{T},X).$$

iii) *$B=\overline{\{u(t):t\in\mathbb{T}\}}$ is compact in $X$.*

*Then, the Nemytskii operator $\Gamma:\mathbb{T}\to X$ defined by $\Gamma(.)=f(.,\varphi(.))$ belongs to $S^pPAA(\mathbb{T},X)$.*

*Proof.* Let $f=g+h$ where $g\in S^pAA(\mathbb{T},X)$ and $h\in S^pPAA_0(\mathbb{T},X)$ and let $\varphi=u+v$ where $u\in S^pAA(\mathbb{T},X)$ and $v\in S^pPAA_0(\mathbb{T},X)$. Now, we decompose $f$ in the following form :

$$\begin{aligned}f(.,\varphi(.))&=g(.,u(.))+f(.,\varphi(.))-g(.,u(.)),\\&=g(.,u(.))+f(.,\varphi(.))-f(.,u(.))+h(.,u(.)).\end{aligned}$$

According to lemma (3.13), we have $g\,(.,u\,(.)) \in S^pAA(\mathbb{T},X)$. And by using the composition theorem of Stepanov-like ergodic functions on $\mathbb{T}$ (theorem 3.7, [10]) it is obvious that $h\,(.,u\,(.)) \in S^pPAA_0(\mathbb{T},X)$.

Now, it remains to show that, $F(.) = f\,(.,\varphi\,(.)) - f\,(.,u\,(.)) \in PAA_0(\mathbb{T},X)$. Indeed,

$$\begin{aligned}\frac{1}{K}\int_t^{t+K} \mid F(s) \mid^p \Delta s &= \frac{1}{K}\int_t^{t+K} \mid f(s,\varphi(s)) - f(s,u(s)) \mid^p \Delta s,\\ &\leq L_f^p \frac{1}{K}\int_t^{t+K} \mid \varphi(s) - u(s) \mid^p \Delta s.\end{aligned}$$

Thus,

$$\begin{aligned}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} \mid F(s) \mid^p \Delta s\right)^{\frac{1}{p}} \Delta t &\leq \frac{L_f}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} \mid \varphi(s) - u(s) \mid^p \Delta s\right)^{\frac{1}{p}} \Delta t,\\ &= \frac{L_f}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} \mid v(s) \mid^p \Delta s\right)^{\frac{1}{p}} \Delta t,\end{aligned}$$

we know that $v \in S^pPAA_0(\mathbb{T},X)$, consequently,

$$\lim_{r\to\infty}\frac{1}{2r}\int_{t_0-r}^{t_0+r}\left(\frac{1}{K}\int_t^{t+K} \mid F(s) \mid^p \Delta s\right)^{\frac{1}{p}} \Delta t = 0.$$

Finally, the Nemytskii operator $\Gamma : \mathbb{T} \to X$ belongs to $S^pPAA(\mathbb{T},X)$.

□

## 4. Pseudo almost automorphic solution to some dunamic equations

In this section, by using the inequality analysis techniques on time scales, the theory of calculus on time scales and Banach's fixed point theorem, we will study the existence and uniqueness of pseudo almost automorphic solution to the abstract nonautonomous semilinear dynamic equation (2) and the dynamic equation with delay (3).

First of all, we list some basic hypotheses :

$(H_2)$ The operator $A$ is the infinitesimal generator of a $C_0$-semigroup $(T(t))_{t\geq 0}$, which is asymptotically stable. Namely, there exist two constants $R,\alpha > 0$ such that

$$\parallel T(t,s) \parallel \leq Re_{\ominus\alpha}(t,s), \quad t,s \in \mathbb{T}.$$

$(H_3)$ $F = G + H \in S^pPAA(\mathbb{T}\times X, X) \cap C(\mathbb{T}\times X, X)$, where $G \in S^pAA(\mathbb{T}\times X, X)$ and $H \in S^pPAA_0(\mathbb{T}\times X, X)$.

**Definition 4.1.** [13] we say that $T = \{T(t) : t \in \mathbb{T}\} \subset \mathfrak{L}(X)$ is a $C_0$-semigroup on $\mathbb{T}$, if

i) $T(t+s) = T(t)T(s)$ for every $t, s \in \mathbb{T}$ (the semigroup property).

ii) $T(0) = I$ ($I$ is the identity operator on $X$).

iii) $\lim_{t\to 0^+} T(t)x = x$ (i.e., $T(.)x : \mathbb{T} \to X$ is continuous at 0) for each $x \in X$.

If $\lim_{t\to 0^+} \parallel T(t) - I \parallel = 0$, then $T$ is called a uniformly continuous semigroup. A linear operator $A$ is called the generator of a $C_0$-semigroup $T$ if

$$Ax = \lim_{s\to 0^+} \frac{T\left(\mu(t)\right)x - T(s)x}{\mu(t) - s}, \quad x \in D(A), \tag{4}$$

where the domain $D(A)$ of $A$ is the set of all $x \in X$ for which the above limit exists uniformly in $t$.

To investigate the existence and uniqueness of pseudo almost automorphic mild solution to Eq.(2) and Eq.(3), we first study the existence and uniqueness of pseudo almost automorphic mild solution to the linear dynamic equation (1).

**Theorem 4.2.** *Under assumption* $(H_2)$*, if* $f = g + h \in S^pPAA(\mathbb{T}, X)$*. Then, Eq.* (1) *has a unique pseudo almost automorphic mild solution* $u(t) : \mathbb{T} \to X$ *given by :*

$$u(t) = \int_{-\infty}^{t} T(t, \sigma(s))f(s)\Delta s.$$

*Proof.* Let $f = g + h \in S^pPAA(\mathbb{T}, X)$ and consider for each $n \in \mathbb{N}^*$ and $t \in \mathbb{T}$ the sequence

$$\begin{aligned} u_n(t) &= \int_{t-nK}^{t-(n-1)K} T(t, \sigma(s))f(s)\Delta s, \\ &= \int_{t-nK}^{t-(n-1)K} T(t, \sigma(s))g(s)\Delta s + \int_{t-nK}^{t-(i-n)K} T(t, \sigma(s))h(s)\Delta s, \\ &= a_n(t) + b_n(t), \end{aligned}$$

where,

$$a_n(t) = \int_{t-nK}^{t-(n-1)K} T(t, \sigma(s))g(s)\Delta s,$$

and

$$b_n(t) = \int_{t-nK}^{t-(n-1)K} T(t, \sigma(s))h(s)\Delta s.$$

For the sake of convenience, we break the proof in several steps.

**First step.** Let us show that $a_n \in AA(\mathbb{T}, X)$. From $(H_2)$ the function $s \to T(t, \sigma(s))g(s)$ is $\Delta$-integrable over $(-\infty, t)_{\mathbb{T}}$ for all $t \in \mathbb{T}$. It follows from lemma (2.5, (7)) that

$$e_{\ominus\alpha}\left(t, \sigma\left(t\right) - (n-1)K\right) \leq e_{\ominus\alpha}(t, \sigma(t)) = 1 + \alpha\mu(t) \leq 1 + \alpha\bar{\mu},$$

where $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t)$. Hence, according to hypothesis $(H_2)$ we infer that

$$\begin{aligned}
\mid a_n\left(t\right) \mid &\leq \int_{t-nK}^{t-(n-1)K} \parallel T\left(t, \sigma\left(s\right)\right) \parallel . \mid g(s) \mid \Delta s, \\
&\leq R \int_{t-nK}^{t-(n-1)K} e_{\ominus\alpha}\left(t, s\right) \mid g(s) \mid \Delta s, \\
&\leq R e_{\ominus\alpha}(t, \sigma(t-(n-1)K)) \int_{t-nK}^{t-(n-1)K} \mid g(s) \mid \Delta s, \\
&\leq R(1+\alpha\bar{\mu}) \int_{t-nK}^{t-(n-1)K} \mid g(s) \mid \Delta s.
\end{aligned}$$

Let $q > 1$ such that $\frac{1}{p} = 1 - \frac{1}{q}$. Using the Hölder's inequality, it follows that

$$\begin{aligned}
\mid a_n\left(t\right) \mid &\leq R(1+\alpha\bar{\mu}) K^{\frac{1}{q}} \left( \int_{t-nK}^{t-(n-1)K} \mid g(s) \mid^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq R(1+\alpha\bar{\mu}) K \left( \frac{1}{K} \int_{t-nK}^{t-(n-1)K} \mid g(s) \mid^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq R(1+\alpha\bar{\mu}) K \parallel g \parallel_{S^p} .
\end{aligned}$$

We deduce from the well-know Weierstrass test that the series $\sum_{i=1}^{\infty} a_n(t)$ is uniformly convergent on $\mathbb{T}$. Moreover, $a(t) = \displaystyle\int_{-\infty}^{t} T\left(t, \sigma\left(s\right)\right) g(s) \Delta s = \sum_{i=1}^{\infty} a_n\left(t\right) \in BS^p(\mathbb{T}, X)$, $a(.) \in C(\mathbb{T}, X)$ and $\parallel a(t) \parallel \leq \sum_{i=1}^{\infty} a_n\left(t\right) \leq R(1+\alpha\bar{\mu}) K \parallel g \parallel_{S^p}$ .

Next, let $\{s_n\}_n \subset \Pi$, since $g$ is Stapanov-like almost automorphic function on $\mathbb{T}$, there exists a subsequence $\{\tau_n\}_n$ of $\{s_n\}_n$ and a function $\phi \in S^p AA(\mathbb{T}, X)$ such that

$$\frac{1}{K} \left( \int_{t}^{t+k} \parallel g(\tau_n + s) - \phi(s) \parallel^p \Delta s \right)^{\frac{1}{p}} \to 0 \quad \text{as} \quad n \to \infty.$$

We set

$$W_n(t) = \int_{t-nK}^{t-(n-1)K} T(t, \sigma(s)) \phi(s) \Delta s,$$

from Hölder's inequality we obtain,

$$\begin{aligned}
\| a_n(t+\tau_n) - W_n(t) \| &= \| \int_{t-nK}^{t-(n-1)K} T(t,\sigma(s))(g(\tau_n+s)-\phi(s))\Delta s \| \\
&\leq R \int_{t-nK}^{t-(n-1)K} e_{\ominus\alpha}(t,s) \| (g(\tau_n+s)-\phi(s))\Delta s \| \\
&\leq R(1+\alpha\bar{\mu})K \left( \frac{1}{K} \int_{t-nK}^{t-(n-1)K} \| (g(\tau_n+s)-\phi(s)) \|^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq R(1+\alpha\bar{\mu})K \| g-\phi \|_{S^p} .
\end{aligned}$$

As a consequence,

$$\| a_n(t+\tau_n) - W_n(t) \| \to 0 \quad \text{as} \quad n \to \infty.$$

Equivalently, it easy to show that

$$\| W_n(t+\tau_n) - a_n(t) \| \to 0 \quad \text{as} \quad n \to \infty.$$

As a conclusion for each $n$, $a_n$ is almost automorphic on $\mathbb{T}$ and its uniform limit $a(t)$ is also almost automorphic on $\mathbb{T}$.

**Second step.** let us prove that $b_n \in PAA_0(\mathbb{T},X)$. For this, by using condition $(H_2)$ and the Hölder's inequality we have,

$$\begin{aligned}
| \phi_i(t) | &\leq \int_{t-nK}^{t-(i-n)K} \| T(t,\sigma(s)) \| . | h(s) | \Delta s, \\
&\leq R \int_{t-nK}^{t-(n-1)K} e_{\ominus\alpha}(t,s) | h(s) | \Delta s, \\
&\leq R e_{\ominus\alpha}(t,\sigma(t-(n-1)K)) \int_{t-nK}^{t-(n-1)K} | h(s) | \Delta s, \\
&\leq R(1+\alpha\bar{\mu}) \int_{t-nK}^{t-(n-1)K} | h(s) | \Delta s, \\
&\leq R(1+\alpha\bar{\mu})K^{\frac{1}{q}} \left( \int_{t-nK}^{t-(n-1)K} | h(s) |^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq R(1+\alpha\bar{\mu})K \left( \frac{1}{K} \int_{t-nK}^{t-(n-1)K} | h(s) |^p \Delta s \right)^{\frac{1}{p}}, \\
&\leq R(1+\alpha\bar{\mu})K \| h \|_{S^p} .
\end{aligned}$$

Then, we get from the well-know Weierstrass test the the series $\sum_{i=1}^{\infty} b_n(t)$ is uniformly convergent on $\mathbb{T}$. Furthermore, $b(t) = \int_{-\infty}^{t} T(t, \sigma(s))\, h(s)\Delta s = \sum_{i=1}^{\infty} b_n(t) \in BS^p(\mathbb{T}, X)$, $b(.) \in C(\mathbb{T}, X)$ and $\| b(t) \| \leq \sum_{i=1}^{\infty} b_n(t) \leq R(1+\alpha\bar{\mu})K \| g \|_{S^p}$ .

Thus, for a fixed $t_0 \in \mathbb{T}$,

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} | b_n(t) | \Delta t \leq R(1+\alpha\bar{\mu})K \lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \left( \frac{1}{K} \int_{t-nK}^{t-(n-1)K} | h(s) |^p \Delta s \right) \Delta t,$$

$$\leq R(1+\alpha\bar{\mu})K \lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \| h \|_{S^p} \Delta t,$$

we know that $h(.) \in S^pPAA_0(\mathbb{T}, X)$, for this reason we obtain,

$$\lim_{r\to+\infty} \frac{1}{2r} \int_{t_0-r}^{t_0+r} \| h \|_{S^p} \Delta t = 0,$$

consequently, $b_n(.) \in PAA_0(\mathbb{T}, X)$.

**Third step.** We study the uniqueness. Consider the homogeneous equation

$$u^{\Delta}(t) = Au(t) \quad t \in \mathbb{T}. \tag{5}$$

We suppose that $u : \mathbb{T} \to X$ is bounded and satisfies Eq. (5), it follows that

$$u(t) = T(t, \sigma(s))u(s).$$

Therefore, $\| u(t) \| \leq RNe_{\ominus\alpha}(t, s)$ . Take a sequence $\{s_n\}_n$ such that $s_n \to -\infty$ as $n \to \infty$. For any $t \in \mathbb{T}$ fixed, one can find a subsequence $\{\tau_m\}_m$ such that $\tau_m \leq t$ for all $m = 1, 2, ....$ It is clear that $u(t) = 0$ as $m \to \infty$. Now, let $v, w$ be two bounded solutions to Eq. (1), we conclude that $\breve{u} = v - w$ is a bounded solution to Eq. (5). Finally, $\breve{u} = v - w = 0$, which implies that $v = w$. □

**Theorem 4.3.** *Under previous assumptions* $(H_1) - (H_3)$. *If we assume that*

$$\begin{cases} RKL_f\left(2 + \frac{1+(\alpha\bar{\mu})^2}{\alpha\bar{\mu}}\right) < 1, & \text{If} \quad \mathbb{T} \neq \mathbb{R}, \\ \frac{RL_f}{\alpha} < 1, & \text{If} \quad \mathbb{T} = \mathbb{R}. \end{cases}$$

*Then, Eq.* (2) *has a unique pseudo almost automorphic solution* $u$ *such that*

$$u(t) = \int_{-\infty}^{t} T(t, \sigma(s))F(s, u(s))\Delta s.$$

*Proof.* Let $u = x + y \in PAA(\mathbb{T}, X)$, then $u = x + y \in S^pPAA(\mathbb{T}, X)$, where $x \in S^pAA(\mathbb{T}, X)$, $\overline{x(\mathbb{T})}$ is compact and $y \in S^pPAA_0(\mathbb{T}, X)$.

**Step 1 :** Now, let $u \in PAA(\mathbb{T}, X, \mu)$. Consider the nonlinear operator $\Lambda$ defined by

$$\Lambda(u)(t) = \int_{-\infty}^{t} T(t, \sigma(s))F(s, u(s))\Delta s, \quad t \in \mathbb{T}.$$

It is easy to get from theorem (3.26) and theorem (4.2) that

$$\Lambda(.) \in PAA(\mathbb{T}, X).$$

So, $\Lambda$ maps $PAA(\mathbb{T}, X)$ into $PAA(\mathbb{T}, X)$.

**Step 2 :** By using the Banach fixed-point theorem, we show the existence and uniqueness of solution to Eq. (2).

*Case 1:* Let $u, v \in PAA(\mathbb{T}, X)$. If $\mathbb{T} \neq \mathbb{R}$. For all $t \in \mathbb{T}$, we have $\bar{\mu} = \sup_{t\in\mathbb{T}} \mu(t) \leq K$ and there exists a right-scattered point $w_0$ such that $\bar{\mu} = \mu(w_0) \leq K$. Thus, the following interval $[\sigma(t) - nK + K, t)_{\mathbb{T}}$ contains at least $n-2$ right-scattered points, for all $t \in \mathbb{T}$ and $n \geq 3$, such that for $\beta(t) \in \mathbb{Z}$ this right-scattered point is written in the following form : $\beta(t)K + w_0$ with $\mu(\beta(t)K + w_0) = \mu(w_0) = \bar{\mu}$. Thus, for $n \in \mathbb{N}$ we get,

$$\begin{aligned}
\sum_{n=1}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (n-1)K) &\leq \left( e_{\ominus\alpha}(t, \sigma(t)) + e_{\ominus\alpha}(t, \sigma(t) - K) + \sum_{n=3}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (n-1)K) \right), \\
&\leq (e_{\ominus\alpha}(t, \sigma(t)) + e_{\ominus\alpha}(t, \sigma(t) - K) + \sum_{n=3}^{\infty} (1 + \alpha\bar{\mu})^{2-n}, \\
&\leq 1 + 1 + \alpha\bar{\mu} + \frac{1}{\alpha\bar{\mu}}, \\
&= \left( 2 + \frac{1 + (\alpha\bar{\mu})^2}{\alpha\bar{\mu}} \right).
\end{aligned}$$

Therefore, $(H_1)$, $(H_2)$ and Hölder's inequality imply that

$$\begin{aligned}
\| \Lambda(u)(t) - \Lambda(v)(t) \| &\leq \int_{-\infty}^{t} \| T(t, \sigma(s)) \| . \mid f(s, u(s)) - f(s, v(s)) \mid \Delta s, \\
&\leq R \sum_{i=1}^{\infty} \int_{t-iK}^{t-(i-1)K} e_{\ominus\alpha}(t, \sigma(s)) . \mid f(s, u(s)) - f(s, v(s)) \mid \Delta s, \\
&\leq R \sum_{i=1}^{\infty} e_{\ominus\alpha}(t, \sigma(t) - (i-1)K). \int_{t-iK}^{t-(i-1)K} \mid f(s, u(s)) - f(s, v(s)) \mid \Delta s, \\
&\leq RKL_f \left( 2 + \frac{1 + (\alpha\bar{\mu})^2}{\alpha\bar{\mu}} \right) \| u - v \|_{S^p} .
\end{aligned}$$

*Case 2:* Let $u, v \in PAA(\mathbb{T}, X)$. If $\mathbb{T} = \mathbb{R}$ i.e, $K = 1$, then $(H_1)$ and $(H_2)$ yield that

$$\begin{aligned}\| \Lambda(u)(t) - \Lambda(v)(t) \| &\leq R \int_{-\infty}^{t} e^{-\alpha(t-s)} \| F(s, u(s)) - F(s, v(s)) \| ds, \\ &\leq RL_f(s) \int_{-\infty}^{t} e^{-\alpha(t-s)} \| u - v \| ds, \\ &\leq \frac{RL_f}{\alpha} \| u - v \| .\end{aligned}$$

Namely, $\Lambda$ has a unique fixed point $u \in PAA(\mathbb{T}, X)$. □

**Corollary 4.4.** *Under assumptions $(H_1) - (H_3)$. Then, the dynamic equation with delay (3) has a unique pseudo almost automorphic solution given by*

$$u(t) = \int_{-\infty}^{t} T(t, \sigma(s)) F(s, u(s - \tau)) \Delta s.$$

*Whenever*

$$\begin{cases} RKL_f\left(2 + \frac{1+(\alpha\bar{\mu})^2}{\alpha\bar{\mu}}\right) < 1, & \text{If} \quad \mathbb{T} \neq \mathbb{R}, \\ \frac{RL_f}{\alpha} < 1, & \text{If} \quad \mathbb{T} = \mathbb{R}. \end{cases}$$

*Proof.* We can easily obtain this result by using theorem (4.3) combined with the composition theorem (3.26) and lemma (2.22). □

## 5. Examples

### 5.1. Example 1.

**Example 5.1.** Let $\mathbb{T}$ be an almost periodic time scale with $\mu(t) = \frac{5}{6}$. Let's investigate pseudo almost automorphic mild solutions on time scales to the heat equation with delay given by the system :

$$(6) \qquad \begin{cases} \dfrac{\partial_\Delta u}{\partial_\Delta t} = \dfrac{\partial_\Delta^2 u}{\partial_\Delta x^2}(t, x) + \dfrac{1}{60}.\left(\sin(2t) + h(t)\right).\sin(u(t - \theta, x)), \\ u(t - \theta, 0) = u(t - \theta, \pi) = 0. \end{cases}$$

Where, $t \in \mathbb{T}$, $\theta \in \Pi$, $x \in [0, \pi]_{\mathbb{T}}$, $\dfrac{\partial_\Delta u}{\partial_\Delta t}$ is the partial derivative and $h \in C(\mathbb{T}, \mathbb{R})$ such that $| h(t) | \leq 1$.

Take $X = L^2[0, \pi]_{\mathbb{T}}$ equipped with its natural topology and $D(A) = H_0^1[0, \pi]_{\mathbb{T}} \cap H^2[0, \pi]_{\mathbb{T}}$. Define the operator $A$ by $Au = \dfrac{\partial_\Delta^2 u}{\partial_\Delta x^2}(t, x)$ for $u \in D(A)$. A is the generator of an analytic semigroup $(T(t))_{t \geq 0}$ with $\| T(t, s) \| \leq e_{\ominus \frac{1}{2}}$ such that $R = 1$ and $\alpha = \frac{1}{2}$.

From Example (3.10) we have, $(\sin(2t)+h(t)) \in S^1PAA(\mathbb{T},\mathbb{R})$ with $\sin(2t)$ is the Stepanov-like almost automorphic component and $h(t)$ is the Stepanov-like ergodic component.
Thus,

$$F(t,u(t-\tau,x)) = \frac{1}{60}.\,(\sin(2t)+h(t))\,.\sin(u(t-\tau,x))$$

is Stepanov-like pseudo almost automorphic function on $\mathbb{T}$. It's clear that

$$\| f(t,u(t-\tau,.)) - f(t,v(t-\tau,.)) \| \leq \frac{2}{60} \| u-v \| = \frac{1}{30} \| u-v \|,$$

it follows that $L_f = \frac{1}{30}$.
*Case 1:* If $\mathbb{T} \neq \mathbb{R}$, we take $K = 1$ then,

$$RKL_f\left(2+\frac{1+(\alpha\bar{\mu})^2}{\alpha\bar{\mu}}\right) = 1\times 2\times\frac{1}{30}\left(2+\frac{1+(\frac{1}{2}\times\frac{5}{6})^2}{\frac{1}{2}\times\frac{5}{6}}\right) \simeq 0.321 < 1.$$

*Case 2:* If $\mathbb{T} = \mathbb{R}$, we take $K = 1$ then,

$$\frac{RK}{\alpha}L_f = \frac{1\times 2}{30} \simeq 0.066 < 1.$$

This means that $F$ satisfies the assumptions given in corollary (4.4). Then, Eq.(3) has a unique pseudo almost automorphic solution on $\mathbb{T}$ .

5.2. **Numerical example.** In this section, we present an example with numerical simulations to illustrate our theoretical results.

**Example 5.2.** Let $\mathbb{T}$ be an almost periodic time scale with $\mu(t) = \frac{3}{4}$, and consider the following heat equation described by:

$$(7)\qquad \begin{cases} \dfrac{\partial_\Delta u}{\partial_\Delta t}(t,x) = \dfrac{\partial^2_\Delta u}{\partial_\Delta x^2}(t,x) + \dfrac{1}{75}.\left(\sin(t)+\cos(\sqrt{2}t) - \dfrac{1}{t\sigma(t)}\right).\cos(u(t,x)), \\ u(t,0) = u(t,\pi) = 0, \\ u(t_0,x) = u_0(x). \end{cases}$$

With $t,\ t_0 \in \mathbb{T}$, $x \in [0,\pi]_{\mathbb{T}}$ and $\dfrac{\partial_\Delta u}{\partial_\Delta t}$ is the partial derivative. We pose $X = L^2\,[0,\pi]_{\mathbb{T}}$ equipped with its natural topology. According to (7), the operator $A$ is defined by $Au = \dfrac{\partial^2_\Delta u}{\partial_\Delta x^2}$ for $u \in D(A)$ with $D(A) = H^1_0\,[0,\pi]_{\mathbb{T}} \cap H^2\,[0,\pi]_{\mathbb{T}}$.
It is clear that (see subsection 3.1 in ( [17]) ) A is the generator of an analytic semigroup

$(T(t))_{t\geq 0}$ with $\parallel T(t,s) \parallel \leq e_{\ominus\frac{1}{2}}$ where $R=1$ and $\alpha = \frac{1}{2}$.

Also, from example (3.10) and example (3.16), it is easy to get that

$$\left(\sin(t)+\cos(\sqrt{2}t)-\frac{1}{t\sigma(t)}\right) \in S^1PAA(\mathbb{T},\mathbb{R}).$$

Yields,

$$F(t,u(t,x)) = \frac{1}{75}.\left(\sin(t)+\cos(\sqrt{2}t)-\frac{1}{t\sigma(t)}\right).\cos(u(t,x))$$

is Stepanov-like pseudo almost automorphic function on $\mathbb{T}$. Moreover,

$$\parallel f(t,u(t,.)) - f(t,v(t,.)) \parallel \leq \frac{3}{75} \parallel u-v \parallel = \frac{1}{25} \parallel u-v \parallel,$$

it follows that $L_f = \frac{1}{25}$.

*Case 1:* If $\mathbb{T} \neq \mathbb{R}$ we take $K=2$, therefore,

$$RKL_f\left(2+\frac{1+(\alpha\bar{\mu})^2}{\alpha\bar{\mu}}\right) = 1\times 2\times \frac{1}{25}\left(2+\frac{1+(\frac{1}{2}\times\frac{3}{4})^2}{\frac{1}{2}\times\frac{3}{4}}\right) \simeq 0.403 < 1.$$

*Case 2:* If $\mathbb{T} = \mathbb{R}$, we take $K=1$ then,

$$\frac{RK}{\alpha}L_f = \frac{1\times 2}{25} \simeq 0.08 < 1.$$

So, $F$ satisfies the assumptions of Theorem (4.3). Finally, Eq.(2) has a unique pseudo almost automorphic solution on $\mathbb{T}$.

The simulations of the unique pseudo almost automorphic solution of (7) will be given on time scales: $\mathbb{T}=[0,1.5]$, $\mathbb{T}=[0,30]$ and $\mathbb{T}=[0,200]$ with $t_0=0$ and $u(0,x)=xsin(\pi x)$ for all $x\in[0,\pi]_{\mathbb{T}}$. This is illustrated numerically in Figs. 1, and 2.

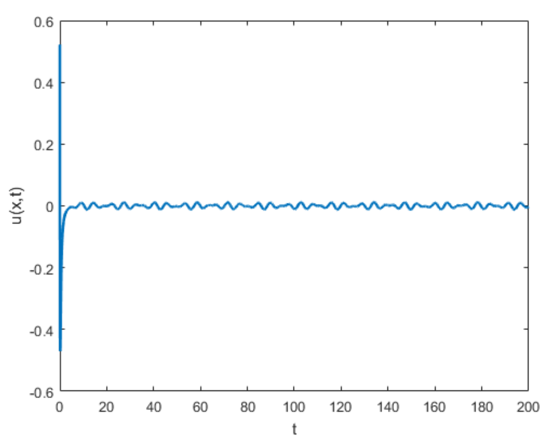


FIGURE 1. The pseudo almost automorphic solution of (7) evolution over $\mathbb{T}=[0,200]$.

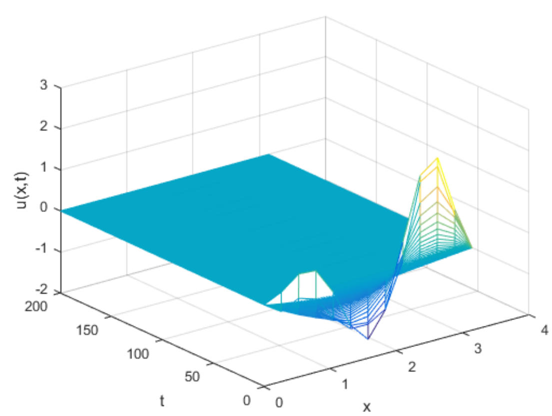


FIGURE 2. The representation of the solution of (7) over $\mathbb{T}=[0,200]$ on $[0,\pi]_{\mathbb{T}}$.

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