ESTIMATE OF SECOND HANKEL DETERMINANT FOR A SUBCLASS OF ANALYTIC FUNCTIONS WITH RESPECT TO SYMMETRIC POINTS

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ABSTRACT. In the present investigation an upper bound of second Hankel determinant $|a_2a_4 - a_3^2|$ for the functions belonging to the class $M_s(\alpha; A, B)$ is studied. The results due to various authors follow as special cases.

2010 Mathematics Subject Classification. 30C45.

Key words and phrases. Analytic functions, Subordination, Schwarz function, Second Hankel determinant.

1. Introduction

By $A$, we denote the class of functions of the form

\begin{equation}
 f(z) = z + \sum_{n=2}^{\infty} a_n z^n,
\end{equation}

analytic in the unit disc $E = \{z : |z| < 1\}$.

Let $U$ is the class of bounded analytic functions $w(z)$ in the unit disc $E$ and of the form

\[ w(z) = \sum_{n=1}^{\infty} d_n z^n, \quad z \in E, \]

which satisfy the conditions $w(0) = 0$, $|w(z)| < 1$.

Let $f(z)$ and $F(z)$ be two analytic functions in the unit disc $E$, then $f(z)$ is said to be subordinate to $F(z)$ if there exists a function $w(z) \in U$ such that $f(z) = F(w(z))$ and we write as $f(z) \prec F(z)$.
\( M_s(\alpha; A, B) \) represents the class of functions \( f(z) \) in \( A \) which satisfy the condition

\[
(1.2) \quad \frac{2zf'(z) + 2\alpha z^2 f''(z)}{(1 - \alpha)(f(z) - f(-z)) + \alpha(zf(z) - f(-z))} < \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1, \ z \in E.
\]

The following observations are obvious:

(i) \( M_s(\alpha; 1, -1) \equiv M_s(\alpha) \), the class introduced by Selvaraj and Vasanthi [12].

(ii) \( M_s(0; 1, -1) \equiv S_s^* \), the class of starlike functions with respect to symmetric points introduced by Sakaguchi [11].

(iii) \( M_s(1; 1, -1) \equiv K_s \), the class of convex functions with respect to symmetric points introduced by Das and Singh [1].

(iv) \( M_s(0; A, B) \equiv S_s^*(A, B) \), the subclass of starlike functions with respect to symmetric points introduced and studied by Goel and Mehrok [2].

(v) \( M_s(1; A, B) \equiv K_s(A, B) \), the subclass of convex functions with respect to symmetric points.

In 1976, Noonan and Thomas [9] stated the \( q \)th Hankel determinant of \( f(z) \) for \( q \geq 1 \) and \( n \geq 1 \) as

\[
H_q(n) = \begin{vmatrix}
  a_n & a_{n+1} & \ldots & a_{n+q-1} \\
  a_{n+1} & \ldots & \ldots & \ldots \\
  \ldots & \ldots & \ldots & \ldots \\
  a_{n+q-1} & \ldots & \ldots & a_{n+2q-2}
\end{vmatrix}.
\]

For our discussion in this paper, we consider the Hankel determinant in the case of \( q = 2 \) and \( n = 2 \), known as second Hankel determinant \( H_2(2) \) and obtain an upper bound to the functional \( H_2(2) \) for \( f(z) \in M_s(\alpha; A, B) \).

Easily, one can observe that the Fekete-Szegö functional is \( H_2(1) \). Earlier Janteng et al. [3, 4, 5], Mehrok and Singh [8], Singh [13, 14] obtained sharp upper bounds of \( H_2(2) \) for different classes of analytic functions.

2. Main Results

Let \( P \) be the family of all functions \( p \) analytic in \( E \) for which \( Re(p(z)) > 0 \) and

\[
p(z) = 1 + p_1z + p_2z^2 + \ldots
\]
for $z \in E$.

**Lemma 2.1.** If $p \in P$, then $|p_k| \leq 2(k = 1, 2, 3, \ldots)$.

This result is due to Pommerenke [10].

**Lemma 2.2.** If $p \in P$, then

$$2p_2 = p_1^2 + (4 - p_1^2)x,$$

$$4p_3 = p_1^3 + 2p_1(4 - p_1^2)x - p_1(4 - p_1^2)x^2 + 2(4 - p_1^2)(1 - |x|^2)z,$$

for some $x$ and $z$ satisfying $|x| \leq 1, |z| \leq 1$ and $p_1 \in [0, 2]$.

This result was proved by Libera and Zlotkiewicz [6, 7]

**Theorem 2.1.** If $f \in M_s(\alpha; A, B)$, then

(2.2) \[ |a_2a_4 - a_3^2| \leq \frac{(A - B)^2}{4(1 + 2\alpha)^2}. \]

**Proof.** As $f \in M_s(\alpha; A, B)$, so there exists a Schwarz function $w(z) \in U$ such that

(2.3) \[ \frac{2zf'(z) + 2\alpha z^2f''(z)}{(1 - \alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} = \phi(w(z)) \]

where

(2.4) \[ \phi(z) = \frac{1 + Az}{1 + Bz} = 1 + (A - B)z - B(A - B)z^2 + B^2(A - B)z^3 + \ldots = 1 + B_1z + B_2z^2 + B_3z^3 + \ldots \]

Define the function $p_1(z)$ by

(2.5) \[ p_1(z) = \frac{1 + w(z)}{1 - w(z)} = 1 + c_1z + c_2z^2 + c_3z^3 + \ldots \]

Since $w(z)$ is a Schwarz function, we see that $Re(p_1(z)) > 0$ and $p_1(0) = 1$. Define the function $h(z)$ by

(2.6) \[ h(z) = \frac{2zf'(z) + 2\alpha z^2f''(z)}{(1 - \alpha)(f(z) - f(-z)) + \alpha z(f(z) - f(-z))'} = 1 + b_1z + b_2z^2 + b_3z^3 + \ldots \]
In view of the equations (2.3), (2.5) and (2.6), we have

\[ h(z) = \phi \left( \frac{p_1(z) - 1}{p_1(z) + 1} \right) = \phi \left( \frac{c_1 z + c_2 z^2 + c_3 z^3 + \ldots}{2 + c_1 z + c_2 z^2 + c_3 z^3 + \ldots} \right) = \phi \left( \frac{1}{2} c_1 z + \frac{1}{2} \left( c_2 - \frac{c_1^2}{2} \right) z^2 + \frac{1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) z^3 \ldots \right) = 1 + \frac{B_1 c_1}{2} z + \left[ \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} \right] z^2 + \left[ \frac{B_1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{B_2 c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_3 c_1^3}{8} \right] z^3 + \ldots \]

Thus

(2.7)

\[ b_1 = \frac{B_1 c_1}{2} ; b_2 = \frac{B_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_2 c_1^2}{4} ; b_3 = \frac{B_1}{2} \left( c_3 - c_1 c_2 + \frac{c_1^3}{4} \right) + \frac{B_2 c_1}{2} \left( c_2 - \frac{c_1^2}{2} \right) + \frac{B_3 c_1^3}{8} . \]

Using (2.4) and (2.6) in (2.7), we obtain

(2.8)

\[ a_2 = \frac{(A - B) c_1}{4(1 + \alpha)} , \]

(2.9)

\[ a_3 = \frac{(A - B)}{8(1 + 2 \alpha)} [2c_2 - (B + 1)c_1^2] \]

and

(2.10)

\[ a_4 = \frac{(A - B)}{64(1 + 3 \alpha)} [8c_3 + 2(A - 5B - 4)c_1 c_2 + (B + 1)(3B - A + 2)c_1^3] . \]

Using (2.8), (2.9) and (2.10), it yields

(2.11)

\[ a_2 a_4 - a_3^2 = \frac{(A - B)^2}{C(\alpha)} [2Lc_1(4c_3) + Mc_1^2(2c_2) - Nc_1^4 - 4R(4c_2^2)] \]

where \( C(\alpha) = 256(1 + 3 \alpha)(1 + \alpha)(1 + 2 \alpha)^2 \),
\( L = (1 + 2 \alpha)^2 \),
\( M = (1 + 2 \alpha)^2 A + [8(1 + 3 \alpha)(1 + \alpha) - 5(1 + 2 \alpha)^2]B + [8(1 + 3 \alpha)(1 + \alpha) - 4(1 + 2 \alpha)^2] \),
\( N = (B + 1)(1 + 2 \alpha)^2 A + [4(1 + 3 \alpha)(1 + \alpha) - 3(1 + 2 \alpha)^2]B + [4(1 + 3 \alpha)(1 + \alpha) - 2(1 + 2 \alpha)^2] \)

and
\( R = (1 + 3 \alpha)(1 + \alpha) \).
Using Lemma 2.1 and Lemma 2.2 in (2.11), we obtain

\[ |a_2a_4 - a_3^2| = \frac{(A - B)^2}{C(\alpha)} \left| -[(1 + 2\alpha)^2AB + [4(1 + \alpha)(1 + 3\alpha) - 3(1 + 2\alpha)^2]B^2]c_1^4 \\
+[(1 + 2\alpha)^2A + [8(1 + \alpha)(1 + 3\alpha) - 5(1 + 2\alpha)^2]B]c_1^2(4 - c_1^2)x \\
-2[8(1 + \alpha)(1 + 3\alpha) - 2(1 + \alpha)(1 + 3\alpha) - (1 + 2\alpha)^2]c_1^2(4 - c_1^2)x^2 \\
+4(1 + \alpha)(1 + 3\alpha)c_1(4 - c_1^2)(1 - |x|^2)z \right| \]

Assume that \( c_1 = c \) and \( c \in [0, 2] \), using triangular inequality and \(|z| \leq 1\), we have

\[ |a_2a_4 - a_3^2| \leq \frac{(A - B)^2}{C(\alpha)} \left[ [2(4 - c^2)(8(1 + \alpha)(1 + 3\alpha) - (2(1 + \alpha)(1 + 3\alpha) - (1 + 2\alpha)^2)c^2) \\
-4(1 + \alpha)(1 + 3\alpha)c(4 - c^2)]\delta^2 \\
+[(1 + 2\alpha)^2A + [8(1 + \alpha)(1 + 3\alpha) - 5(1 + 2\alpha)^2]B](4 - c^2)c^2\delta \\
+[(1 + 2\alpha)^2AB + [4(1 + \alpha)(1 + 3\alpha) - 3(1 + 2\alpha)^2]B^2]c^4 \\
+4(1 + \alpha)(1 + 3\alpha)c(4 - c^2) \right] \]

Therefore

\[ |a_2a_4 - a_3^2| \leq \frac{(A - B)^2}{C(\alpha)} F(\delta), \]

where \( \delta = |x| \leq 1 \) and

\[ F(\delta) = [2(4 - c^2)(8(1 + \alpha)(1 + 3\alpha) - (2(1 + \alpha)(1 + 3\alpha) - (1 + 2\alpha)^2)c^2) \\
-4(1 + \alpha)(1 + 3\alpha)c(4 - c^2)]\delta^2 \\
+[(1 + 2\alpha)^2A + [8(1 + \alpha)(1 + 3\alpha) - 5(1 + 2\alpha)^2]B](4 - c^2)c^2\delta \\
+[(1 + 2\alpha)^2AB + [4(1 + \alpha)(1 + 3\alpha) - 3(1 + 2\alpha)^2]B^2]c^4 \\
+4(1 + \alpha)(1 + 3\alpha)c(4 - c^2) \]

is an increasing function.

Therefore \( \text{Max} F(\delta) = F(1) \).

Consequently

\[ (2.12) \quad |a_2a_4 - a_3^2| \leq \frac{(A - B)^2}{C(\alpha)} G(c), \]

where \( G(c) = F(1) \).

So

\[ G(c) = S(\alpha)c^4 + T(\alpha)c^2 + 64(1 + 3\alpha)(1 + \alpha) \]
where
\[ S(\alpha) = \begin{bmatrix} (1 + 2\alpha)^2AB + [4(1 + \alpha)(1 + 3\alpha) - 3(1 + 2\alpha)^2]B^2 \end{bmatrix} \]
and
\[ T(\alpha) = \begin{bmatrix} 4(1 + 2\alpha)^2A + [8(1 + \alpha)(1 + 3\alpha) - 5(1 + 2\alpha)^2]B \end{bmatrix} \]

Now
\[ G'(c) = 4S(\alpha)c^3 + 2T(\alpha)c \]
and
\[ G''(c) = 12S(\alpha)c^2 + 2T(\alpha). \]

\[ G'(c) = 0 \implies c[2S(\alpha)c^2 + T(\alpha)] = 0. \]

\[ G''(c) \text{ is negative at } c = 0. \]

So \( \text{Max}G(c) = G(1) \).

Hence from (2.12), we obtain (2.2).

The result is sharp for \( c_1 = 0, c_2 = 2 \) and \( c_3 = 0. \)

For \( A = 1, B = -1 \), Theorem 2.1 gives the following result due to Singh [13].

**Corollary 2.1.** If \( f(z) \in M_s(\alpha) \), then
\[ |a_2a_4 - a_3^2| \leq \frac{1}{(1 + 2\alpha)^2}. \]

For \( \alpha = 0, A = 1, B = -1 \), Theorem 2.1 gives the following result due to Janteng et al. [5].

**Corollary 2.2.** If \( f(z) \in S^*_s \), then
\[ |a_2a_4 - a_3^2| \leq 1. \]

For \( \alpha = 1, A = 1, B = -1 \), Theorem 2.1 gives the following result due to Janteng et al. [5].

**Corollary 2.3.** If \( f(z) \in K_s \), then
\[ |a_2a_4 - a_3^2| \leq \frac{1}{9}. \]
For $\alpha = 0$, Theorem 2.1 gives the following result.

**Corollary 2.4.** If $f(z) \in S^*_s(A, B)$, then

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(A - B)^2}{4}.$$

For $\alpha = 1$, Theorem 2.1 gives the following result.

**Corollary 2.5.** If $f(z) \in K_s(A, B)$, then

$$\left|a_2a_4 - a_3^2\right| \leq \frac{(A - B)^2}{36}.$$

**References**


