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REGION OF VARIABILITY OF SUBCLASSES OF UNIVALENT FUNCTIONS

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Abstract. Let $\mathcal{G}(\alpha)$ and $\mathcal{F}(\alpha)$ denote the subclasses of locally univalent, normalized analytic functions f in the unit disk $|z| < 1$ satisfying the condition

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < 1 + \frac{\alpha}{2}$$

and

$$Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \frac{\alpha}{2} - 1$$

for some $0 < \alpha \leq 1$ respectively. For any fixed z_0 in the unit disk and $\lambda \in [0, 1)$, we determine the region of variability for $log f'(z_0)$ when f ranges over the class $\{f \in \mathcal{G}(\alpha) : f''(0) = -\alpha\lambda\}$ and $\{f \in \mathcal{F}(\alpha) : f''(0) = (4 - \alpha)\lambda\}$.

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1. Introduction

Let $\mathcal{H}(\Delta)$ denote the class of analytic functions in the unit disk $\Delta = \{z \in \mathbb{C} : |z| < 1\}$.$\mathcal{H}(\Delta)$ may be thought as a topological vector space endowed with the topology of uniform convergence over compact subsets of Δ.Let $\mathcal{A}$ denote the set of all functions in $\mathcal{H}(\Delta)$ such that $f(0) = f'(0) - 1 = 0$ and S be the class of all functions in $\mathcal{A}$ that are univalent. Several researchers have studied the region of variability problems of f at a specified point inside the unit disk for several subclasses of S. In [2],the problem of determining the region of values of $log\left[\frac{f(z_0)}{z_0}\right]$ for a fixed $z_0 \in \Delta$ as f ranges over the class S^* of starlike function is given. Duren [3] in his paper discusses the region of variability of $f'(z_0)$ for $f \in S$ and $g(z_0)$ for $g \in S_0 = \{f \in A : f(z) \neq 0 \; in \; \Delta, f(0) = 1\}$. S.Ponnusamy *et al.* had obtained the region of variabilities for several standard subclasses of S $[6, 7]$. H.Yanagihara had discussed the region of variability for functions with bounded derivatives,convex functions and families of convex functions $[8, 9, 10]$.

In this paper we obtain the region of variability of $logf'(z_0)$ for two subclasses of analytic, univalent and normalized functions.

Let $\mathcal{G}(\alpha)$ denote the class of locally univalent , normalized analytic functions f in the unit disk Δ satisfying the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)<1+\frac{\alpha}{2} \tag{1}$$

for some $0 < \alpha \leq 1$. The class $\mathcal{G}(1)$ is included in S^* and was studied by Ozaki $[4]$. Denote $1+\frac{zf''(z)}{f'(z)}$ by $P_f(z)$, $z \in \Delta$. Then (1) is same as $ReP_f(z) < 1+\frac{\alpha}{2}$. Let $logf'$ denote the single valued branch of the logarithm of f' with $logf'(0) = 0$.Using the Herglotz representation formula for analytic functions with positive real part in the unit disk Δ, if $f \in \mathcal{G}(\alpha)$, then there exist a unique positive measure μ on $(-\pi, \pi]$ such that

$$1+\frac{zf''(z)}{f'(z)} = 1+\frac{\alpha}{2} - \left(1+\frac{\alpha}{2}-1\right)\int_{-\pi}^{\pi}\frac{1+ze^{-it}}{1-ze^{-it}}d\mu(t)$$

which gives

$$logf'(z) = \alpha\int_{-\pi}^{\pi} log(1-ze^{-it})d\mu(t)$$

Thus for each fixed $z_0 \in \Delta$, the region of variability

$$V_{\mathcal{G}}(z_0, \alpha) = \{logf'(z_0) : f \in \mathcal{G}(\alpha)\}$$

coincides with the set $\{\alpha log(1-z) : |z| \leq |z_0|\}$.Let $\mathcal{B}$ denote the class of analytic functions ω in Δ such that $|\omega(z)| \leq 1$ and $\omega(0) = 0$.If $f \in \mathcal{G}(\alpha)$, then there exists a function $\omega_f \in \mathcal{B}$ such that

$$\omega_f(z) = \frac{P_f(z)-1}{P_f(z)-(1+\alpha)} \tag{2}$$

From (2), we get $P_f'(0) = -\alpha\omega_f'(0)$. Since $|\omega_f'(0)| \leq 1$ we get
$f''(0) = -\alpha\lambda, \lambda \in \Delta$. Introduce

$$C_1(\lambda) = \{f \in \mathcal{G}(\alpha) : f''(0) = -\alpha\lambda\}$$

and

$$V_{\mathcal{G}}(z_0, \alpha, \lambda) = \{logf'(z_0) : f \in C_1(\lambda)\}.$$

Let $\mathcal{F}(\alpha)$ denote the class of locally univalent , normalized analytic functions f in the unit disk $|z| < 1$ satisfying the condition

$$Re\left(1+\frac{zf''(z)}{f'(z)}\right)>\frac{\alpha}{2}-1 \tag{3}$$

for some $0 < \alpha \leq 1$ Using the Herglotz representation formula for analytic functions with positive real part in the unit disk Δ, if $f \in \mathcal{F}(\alpha)$, then there exist a unique positive measure μ on $(-\pi, \pi]$ such that

$$1 + \tfrac{zf''(z)}{f'(z)} = (1 - \tfrac{\alpha}{2} + 1) \int_{-\pi}^{\pi} \tfrac{1+ze^{-it}}{1-ze^{-it}} d\mu(t) + \tfrac{\alpha}{2} - 1$$

which gives

$$log f'(z) = (4 - \alpha) \int_{-\pi}^{\pi} log\left(\frac{1}{(1 - ze^{-it}}\right) d\mu(t)$$

Thus, for each fixed $z_0 \in \Delta$, the region of variability

$$V_{\mathcal{F}}(z_0, \alpha) = \{log f'(z_0) : f \in \mathcal{F}(\alpha)\}$$

coincides with the set $\{-(4 - \alpha) log(1 - z) : |z| \leq |z_0|\}$. If $f \in \mathcal{F}(\alpha)$, then there exists a function $\omega_f \in \mathcal{B}$ such that

(4) $$\omega_f(z) = \frac{P_f(z) - 1}{P_f(z) + 3 - \alpha}$$

From (4), $P'_f(0) = (4 - \alpha)\omega'_f(0)$. Since $|\omega'_f(0)| \leq 1$ we get $f''(0) = (4 - \alpha)\lambda,\ \lambda \in \Delta$. Introduce

$$C_2(\lambda) = \{f \in \mathcal{F}(\alpha) : f''(0) = (4 - \alpha)\lambda\}$$

and

$$V_{\mathcal{F}}(z_0, \alpha, \lambda) = \{log f'(z_0) : f \in C_2(\lambda)\}.$$

We now determine the sets $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ and $V_{\mathcal{F}}(z_0, \alpha, \lambda)$.To do this we prove some basic properties of the sets $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ and $V_{\mathcal{F}}(z_0, \alpha, \lambda)$.

2. Basic Properties of the sets $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ and $V_{\mathcal{F}}(z_0, \alpha, \lambda)$

For a positive integer p, let $(S^*)^p = \{f = f_0^p : f_0 \in S^*\}$.We first recall a result proved in $[4, 8]$.

Lemma 2.1. *Let f be an analytic function in Δ with $f(z) = z^p + ...$ If $Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, z \in \Delta$, then $f \in (S^*)^p$*

We now discuss some basic properties of the sets $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ and$V_{\mathcal{F}}(z_0, \alpha, \lambda)$.

(1) $V_{\mathcal{G}}(z_0, \alpha, \lambda) \subseteq V_{\mathcal{G}}(z_0, \alpha)$ and $V_{\mathcal{F}}(z_0, \alpha, \lambda) \subseteq V_{\mathcal{F}}(z_0, \alpha)$

(2) $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ and $V_{\mathcal{F}}(z_0, \alpha, \lambda)$ are compact subsets of $\mathbb{C}$, since both $C_1(\lambda)$ and $C_2(\lambda)$ are compact.

(3) $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ are convex sets, for if $f_0, f_1 \in C_j(\lambda), j=1,2$ and $t \in [0,1]$, then the function

$$f_t(z) = \int_0^z exp\{(1-t)log f_0'(\zeta) + t log f_1'(\zeta)\}d\zeta$$

is in $C_j(\lambda), i = 1,2$. Convexity of the sets $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ follows from $log f_t'(z) = (1-t)log f_0'(z) + t log f_1'(z)$.

(4) If $z_0 = 0$ or $|\lambda| = 1$, then $V_{\mathcal{G}}(z_0,\alpha,\lambda) = \{\alpha log(1-\lambda z_0)\}$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda) = \{-(4-\alpha)log(1-\lambda z_0)\}$. For $|\lambda| < 1$ and $z_0 \neq 0$, $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ has $\alpha log(1-\lambda z_0)$ and $-(4-\alpha)log(1-\lambda z_0)$ as interior points respectively. If $f \in \mathcal{G}(\alpha)$ and $|\lambda| = |\omega_f'(0)| = 1$, then by Schwarz lemma, $\omega_f(z) = \lambda z$.This implies $P_f(z) = \frac{1-(1+\alpha)\lambda z}{1-\lambda z}$ which gives $log f'(z) = \alpha log(1-\lambda z)$ and hence $V_{\mathcal{G}}(z_0,\alpha,\lambda) = \{\alpha log(1-\lambda z_0)\}$. Similarly if $f \in \mathcal{F}(\alpha)$ and $|\lambda| = |\omega_f'(0)| = 1$, then by Schwarz lemma, $\omega_f(z) = \lambda z$. Therefore $P_f(z) = \frac{(3-\alpha)\lambda z+1}{1-\lambda z}$ which gives $log f'(z) = -(4-\alpha)log(1-\lambda z)$ and hence $V_{\mathcal{F}}(z_0,\alpha,\lambda) = \{-(4-\alpha)log(1-\lambda z_0)\}$. If $z_0 = 0$, then $V_{\mathcal{G}}(z_0,\alpha,\lambda) = V_{\mathcal{F}}(z_0,\alpha,\lambda) = \{0\}$.
For $\lambda \in \Delta$, $a \in \bar{\Delta}$, we introduce

(5) $$\delta(z,\lambda) = \frac{z+\lambda}{1+\bar{\lambda}z}$$

(6) $$F_{a,\lambda}(z) = \int_0^z exp\int_0^{\zeta_2} \frac{\alpha\delta(a\zeta_1,\lambda)d\zeta_1}{\zeta_1\delta(a\zeta_1,\lambda)-1}d\zeta_2$$

(7) $$H_{a,\lambda}(z) = \int_0^z exp\int_0^{\zeta_2} \frac{(4-\alpha)\delta(a\zeta_1,\lambda)d\zeta_1}{1-\zeta_1\delta(a\zeta_1,\lambda)}d\zeta_2$$

Then

$$1 + \frac{zF_{a,\lambda}^{''}(z)}{F_{a,\lambda}^{'}(z)} = \frac{1-(1+\alpha)z\delta(az,\lambda)}{1-z\delta(az,\lambda)}$$

which implies $F_{a,\lambda} \in C_1(\lambda)$ and

(8) $$\omega_{F_{a,\lambda}}(z) = z\delta(az,\lambda)$$

Similarly

$$1 + \frac{zH_{a,\lambda}^{''}(z)}{H_{a,\lambda}^{'}(z)} = \frac{1+(3-\alpha)z\delta(az,\lambda)}{1-z\delta(az,\lambda)}$$

implies $H_{a,\lambda} \in C_2(\lambda)$ and

(9) $$\omega_{H_{a,\lambda}}(z) = z\delta(az,\lambda)$$

Fix $\lambda \in \Delta$ and $z_0 \in \Delta - \{0\}$.Then the functions

$$\Delta \ni a \mapsto log F_{a,\lambda}^{'}(z_0) = \int_0^{z_0} \frac{\alpha\delta(a\zeta,\lambda)}{\zeta\delta(a\zeta,\lambda)-1}d\zeta$$

and

$$\Delta \ni a \mapsto logH'_{a,\lambda}(z_0) = \int_0^{z_0} \frac{(4-\alpha)\delta(a\zeta,\lambda)}{1-\zeta\delta(a\zeta,\lambda)} d\zeta$$

are non-constant analytic functions of $a \in \Delta$ and hence are open maps. This implies that $logF'_{0,\lambda}(z_0) = \alpha log(1-\lambda z_0)$ is an interior point of
$\{logF'_{a,\lambda}(z_0) : a \in \Delta\} \subset V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $logH'_{0,\lambda}(z_0) = -(4-\alpha)log(1-\lambda z_0)$ is an interior point of $\{logH'_{a,\lambda}(z_0) : a \in \Delta\} \subset V_{\mathcal{F}}(z_0,\alpha,\lambda)$

(5) For each $\theta \in \mathbb{R}$, $V_{\mathcal{G}}(e^{i\theta}z_0,\alpha,\lambda) = V_{\mathcal{G}}(z_0,\alpha,e^{i\theta}\lambda)$ and $V_{\mathcal{F}}(e^{i\theta}z_0,\alpha,\lambda) = V_{\mathcal{F}}(z_0,\alpha,e^{i\theta}\lambda)$ because $e^{-i\theta}f(e^{i\theta}z) \in C_j(e^{i\theta}\lambda)$ if and only if $f \in C_j(\lambda), j = 1,2.$ Hence it is sufficient to determine $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ for $0 \le \lambda < 1$ and $z_0 \in \Delta - \{0\}$. Since $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ are compact convex subsets of $\mathbb{C}$ and has nonempty interior, the boundaries $\partial V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $\partial V_{\mathcal{F}}(z_0,\alpha,\lambda)$ are Jordan curves and each $V_{\mathcal{G}}(z_0,\alpha,\lambda)$ and $V_{\mathcal{F}}(z_0,\alpha,\lambda)$ are the union of their respective boundaries and interior domains. From now on it is enough to consider the class

$$C_1(\lambda) = \{f \in \mathcal{G} : f''(0) = -\alpha\lambda\},\ 0 \le \lambda < 1$$

and

$$C_2(\lambda) = \{f \in \mathcal{F} : f''(0) = (4-\alpha)\lambda\},\ 0 \le \lambda < 1$$

We now state the main results.

Theorem 2.1. *For $0 \le \lambda < 1$ and $z_0 \in \Delta - \{0\}$, the boundary $\partial V_{\mathcal{G}}(z_0,\alpha,\lambda)$ is the Jordan curve given by*

$$(10) \qquad (-\pi,\pi] \ni \theta \mapsto logF'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{\alpha\delta(e^{i\theta}z,\lambda)}{z\delta(e^{i\theta}z,\lambda)-1} dz,\ z \in \Delta$$

If $logf'(z_0) = logF'_{e^{i\theta},\lambda}(z_0)$ for some $f \in C_1(\lambda)$ and $\theta \in (-\pi,\pi]$, then
$f(z) = F_{e^{i\theta},\lambda}(z)$, where $F_{e^{i\theta},\lambda}$ is given by (6)

Theorem 2.2. *For $0 \le \lambda < 1$ and $z_0 \in \Delta - \{0\}$, the boundary $\partial V_{\mathcal{F}}(z_0,\alpha,\lambda)$ is the Jordan curve given by*

$$(11) \qquad (-\pi,\pi] \ni \theta \mapsto logH'_{e^{i\theta},\lambda}(z_0) = \int_0^{z_0} \frac{(4-\alpha)\delta(e^{i\theta}z,\lambda)}{1-z\delta(e^{i\theta}z,\lambda)} dz,\ z \in \Delta$$

If $logf'(z_0) = logH'_{e^{i\theta},\lambda}(z_0)$ for some $f \in C_2(\lambda)$ and $\theta \in (-\pi,\pi]$, then
$f(z) = H_{e^{i\theta},\lambda}(z)$, where $H_{e^{i\theta},\lambda}$ is given by (7)

3. Region of variability of $V_{\mathcal{G}}(z_0,\alpha,\lambda)$

To prove theorem 2.1, we first state few results without proof.

Theorem 3.1. *For* $f \in C_1(\lambda)$

$$\left|\frac{f''(z)}{f'(z)} - c(z,\lambda)\right| \le r(z,\lambda),\ z \in \Delta \tag{12}$$

where

$$c(z,\lambda) = -\alpha\left[\frac{\lambda(1-|z|^2) + \bar{z}(|z|^2 - \lambda^2)}{(1-|z|^2)(1-2Re(\lambda z)+|z|^2)}\right]$$

$$r(z,\lambda) = |\alpha|\left[\frac{(1-|\lambda|^2)|z|}{(1-|z|^2)(1-2Re(\lambda z)+|z|^2)}\right]$$

For each $z \in \Delta - \{0\}$, *equality holds if and only if* $f = F_{e^{i\theta},\lambda}$ *for some* $\theta \in \mathbb{R}$.

Proof. Let $f \in C_1(\lambda)$. Then there exist $\omega_f \in \mathcal{B}$ such that $\omega_f = \frac{P_f - 1}{P_f - (1+\alpha)}$, $z \in \Delta$ and $\omega_f'(z) = -f''(0) = -\alpha\lambda$. By Schwarz lemma,

$$\left|\frac{\frac{\omega_f(z)}{z} - \lambda}{1 - \bar{\lambda}\frac{\omega_f(z)}{z}}\right| \le |z| \tag{13}$$

Using the definition of P_f, (13) is equivalent to

$$\left|\frac{\frac{f''(z)}{f'(z)} - A(z,\lambda)}{\frac{f''(z)}{f'(z)} + B(z,\lambda)}\right| \le |z||\tau(z,\lambda)| \tag{14}$$

where

$$A(z,\lambda) = -\frac{\alpha\lambda}{1-\lambda z},\ B(z,\lambda) = -\frac{\alpha}{z-\bar{\lambda}},\ \tau(z,\lambda) = \frac{z-\bar{\lambda}}{1-\lambda z} \tag{15}$$

The inequality (14) is equivalent to

$$\left|\frac{f''(z)}{f'(z)} - \frac{A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2}\right| \le \frac{|z||\tau(z,\lambda)||A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2} \tag{16}$$

But

$$1 - |z|^2|\tau(z,\lambda)|^2 = \frac{(1-|z|^2)(1-2Re(\lambda z)+|z|^2)}{|1-\lambda z|^2}$$

$$A(z,\lambda) + B(z,\lambda) = -\alpha\left[\frac{1-|\lambda|^2}{(1-\lambda z)(z-\bar{\lambda})}\right]$$

$$A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda) = -\alpha\left[\frac{\lambda(1-|z|^2) + \bar{z}(|z|^2-\lambda^2)}{|1-\lambda z|^2}\right]$$

Using these we have

$$\frac{A(z,\lambda) + |z|^2|\tau(z,\lambda)|^2 B(z,\lambda)}{1-|z|^2|\tau(z,\lambda)|^2} = c(z,\lambda)$$

$$\frac{|z||\tau(z,\lambda)||A(z,\lambda)+B(z,\lambda)|}{1-|z|^2|\tau(z,\lambda)|^2} = r(z,\lambda)$$

The inequality (12) follows from last two equalities and (16). The equality occurs for any $z \in \Delta$ in (12) when $f = F_{e^{i\theta},\lambda}$ for some $\theta \in \mathbb{R}$. Conversely if equality occurs for some $z \in \Delta - \{0\}$ in (12) then the equality must hold in (13). Thus by Schwarz lemma there exists $\theta \in \mathbb{R}$ such that $\omega_f(z) = z\delta(e^{i\theta}z, \lambda)$ for all $z \in \Delta$. This implies $f = F_{e^{i\theta},\lambda}$. □

When $\lambda = 0$, we get an interesting corollary.

Corollary 3.1. *If $f \in C_1(0)$ then*

$$\left|\frac{f''(z)}{f'(z)} + \alpha\frac{\bar{z}|z|^2}{1-|z|^4}\right| \le \frac{|\alpha||z|}{1-|z|^4}$$

In particular,

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \le |\alpha||z|$$

Corollary 3.2. *Let $\gamma : z(t)$ $(0 \le t \le 1)$ be a C^1-curve in Δ with $z(0) = 0$ and $z(1) = z_0$. Then*

$$V_{\mathcal{G}}(z_0, \alpha, \lambda) \subset \bar{\Delta}(C(\lambda, \gamma), R(\lambda, \gamma))$$

where

$$\bar{\Delta}(C(\lambda,\gamma), R(\lambda,\gamma)) = \{w \in \mathbb{C} : |w - C(\lambda,\gamma)| \le R(\lambda,\gamma)\}$$
$$C(\lambda,\gamma) = \int_0^1 c(z(t),\lambda)z'(t)dt$$
$$R(\lambda,\gamma) = \int_0^1 r(z(t),\lambda)|z'(t)|dt.$$

Proof. Let $f \in C_1(\lambda)$. Then

$$\int_0^1 \frac{f''(z(t)}{f'(z(t))}z'(t)dt = logf'(z(1)) - logf'(z(0)) = logf'(z_0)$$

It follows from Theorem (3.1) that

$$\begin{aligned}\left|logf'(z_0) - C(\lambda,\gamma)\right| &= \left|logf'(z_0) - \int_0^1 c(z(t),\lambda)z'(t)dt\right| \\ &= \left|\int_0^1\{\frac{f''(z(t))}{f'(z(t))} - c(z(t),\lambda)\}z'(t)dt\right| \\ &\le \int_0^1 r(z(t),\lambda)|z'(t)|dt = R(\lambda,\gamma)\end{aligned}$$

and hence the conclusion follows. □

We now state a lemma proved in [7]

Lemma 3.1. *For $\theta \in \mathbb{R}$ and $\lambda \in \Delta$ the function*

$$G(z) = \int_0^z \frac{e^{i\theta}\zeta}{\{1 + (\bar{\lambda}e^{i\theta} - \lambda)\zeta - e^{i\theta}\zeta^2\}^2} d\zeta$$

has a double zero at the origin and no zeros elsewhere in Δ. Furthermore there exists a starlike univalent function G_0 in Δ such that $G = e^{i\theta}G_0^2$ and $G_0(0) = G_0'(0) - 1 = 0$.

Theorem 3.2. *Let $z_0 \in \Delta - \{0\}$.Then, for $\theta \in (-\pi, \pi]$, we have $logF'_{e^{i\theta},\lambda}(z_0) \in \partial V_{\mathcal{G}}(z_0, \alpha, \lambda)$.Furthermore if $logf'(z_0) = logF'_{e^{i\theta},\lambda}(z_0)$ for some $f \in C_1(\lambda)$ and $\theta \in (-\pi, \pi]$, then $f = F_{e^{i\theta},\lambda}$*

Proof. From (6)

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} = \frac{\alpha(az + \lambda)}{az^2 + (\lambda - a\bar{\lambda})z - 1}$$

From (15)

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - A(z, \lambda) = \frac{\alpha(1 - |\lambda|^2)az}{(1 - \lambda z)(az^2 + (\lambda - a\bar{\lambda})z - 1)}$$

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} + B(z, \lambda) = \frac{\alpha(1 - |\lambda|^2)}{(z - \bar{\lambda})(az^2 + (\lambda - a\bar{\lambda})z - 1)}$$

Hence

$$\begin{aligned}\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - c(z, \lambda) &= \frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - \frac{A(z, \lambda) + |z|^2|\tau(z, \lambda)|^2 B(z, \lambda)}{1 - |z|^2|\tau(z, \lambda)|^2} \\ &= \frac{1}{1 - |z|^2|\tau(z, \lambda)|^2}\left\{\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - A(z, \lambda) - |z|^2|\tau(z, \lambda)|^2\left(\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} + B(z, \lambda)\right)\right\} \\ &= \alpha\frac{(1 - |\lambda|^2)[a(1 - \bar{\lambda}\bar{z})z - |z|^2(\bar{z} - \lambda)]}{(1 - |z|^2)(1 - 2Re(\lambda z) + |z|^2)(az^2 + (\lambda - a\bar{\lambda})z - 1)}\end{aligned}$$

Putting $a = e^{i\theta}$ we get

$$\begin{aligned}\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - c(z, \lambda) &= \frac{-\alpha(1 - |\lambda|^2)e^{i\theta}z\overline{(1 + (\bar{\lambda}e^{-i\theta} - \lambda)z - e^{-i\theta}z^2)}}{(1 - |z|^2)(1 - 2Re(\lambda z) + |z|^2)(1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2)} \\ &= \frac{-\alpha(1 - |\lambda|^2)e^{i\theta}z|1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2|^2}{(1 - |z|^2)(1 - 2Re(\lambda z) + |z|^2)(1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2)^2} \\ &= r(z, \lambda)\frac{|1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2|^2 e^{i\theta}z}{(1 + (\bar{\lambda}e^{i\theta} - \lambda)z - e^{i\theta}z^2)^2|z|}\end{aligned}$$

Using lemma (3.1) the above equation can be written as

$$\frac{F''_{a,\lambda}(z)}{F'_{a,\lambda}(z)} - c(z, \lambda) = r(z, \lambda)\frac{G'(z)}{|G'(z)|} \tag{17}$$

where

$$\frac{G^{'}(z)}{|G^{'}(z)|}=\frac{|1+(\bar{\lambda}e^{i\theta}-\lambda)z-e^{i\theta}z^2|^2e^{i\theta}z}{(1+(\bar{\lambda}e^{i\theta}-\lambda)z-e^{i\theta}z^2)^2|z|}$$

Since the curve G_0 is starlike, for any $z_0\in\Delta-\{0\}$, the linear segment joining 0 and $G_0(z_0)$ lies entirely in $G_0(\Delta)$. Define the curve γ_0 by

(18) $$\gamma_0 : z(t)=G_0^{-1}(tG_0(z_0)),\ 0\leq t\leq 1$$

Siince $G(z(t))=2^{-1}e^{i\theta}G_0(z(t))^2=2^{-1}e^{i\theta}(tG_0(z_0))^2=t^2G(z_0)$ we have

(19) $$G^{'}(z(t))z^{'}(t)=2tG(z_0),\ t\in[0,1]$$

This along with (17) gives

$$\begin{aligned}
logF^{'}_{e^{i\theta},\lambda}(z_0)-C(\lambda,\gamma_0)&=\int_0^1\left\{\frac{F^{''}_{e^{i\theta},\lambda}(z(t))}{F^{'}_{e^{i\theta},\lambda}(z(t))}-c(z(t),\lambda)\right\}z'(t)dt\\
&=\int_0^1 r(z(t),\lambda)\frac{G^{'}(z(t))z^{'}(t)}{|G^{'}(z(t))z^{'}(t)|}|z^{'}(t)|dt\\
&=\frac{G(z_0)}{|G(z_0)|}\int_0^1 r(z(t),\lambda)|z^{'}(t)|dt\\
&=\frac{G(z_0)}{|G(z_0)|}R(\lambda,\gamma_0)
\end{aligned}$$

Thus we have $logF^{'}_{e^{i\theta},\lambda}(z_0)\in\partial\bar{\Delta}(C(\lambda,\gamma_0),R(\lambda,\gamma_0))$.Corollary 3.2 gives $logF^{'}_{e^{i\theta},\lambda}(z_0)\in V_{\mathcal{G}}(z_0,\alpha,\lambda)\subset\bar{\Delta}(C(\lambda,\gamma_0),R(\lambda,\gamma_0))$. Hence we have $logF^{'}_{e^{i\theta},\lambda}(z_0)\in\partial V_{\mathcal{G}}(z_0,\alpha,\lambda)$.
We now prove the uniqueness.Suppose that $logf^{'}(z_0)=logF^{'}_{e^{i\theta},\lambda}(z_0)$ for some $f\in C_1(\lambda)$ and $\theta\in(-\pi,\pi]$. Let

$$h(t)=\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f''(z(t))}{f'(z(t))}-c(z(t),\lambda)\right\}z^{'}(t)$$

where $\gamma_0 : z(t), 0\leq t\leq 1$ as in (18). Then h is continuous function of $t\in[0,1]$ and satisfies $|h(t)|\leq r(z(t),\lambda)|z'(t)|$. Also

$$\begin{aligned}
\int_0^1 Reh(t)dt&=\int_0^1 Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}\left\{\frac{f^{''}(z(t))}{f^{'}(z(t))}-c(z(t),\lambda)\right\}z^{'}(t)\right]dt\\
&=Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}(logf^{'}(z_0)-C(\gamma_0,\lambda))\right]\\
&=Re\left[\frac{\overline{G(z_0)}}{|G(z_0)|}(logF^{'}_{e^{i\theta},\lambda}(z_0)-C(\gamma_0,\lambda))\right]\\
&=\int_0^1 r(z(t),\lambda)|z^{'}(t)|dt.
\end{aligned}$$

Thus $h(t) = r(z(t), \lambda)|z'(t)|$ for all $t \in [0,1]$. From (17) and (19) this implies $\frac{f''}{f'} = \frac{F''_{e^{i\theta},\lambda}}{F'_{e^{i\theta},\lambda}}$ on γ_0. By identity theorem for analytic functions we get $\frac{f''}{f'} = \frac{F''_{e^{i\theta},\lambda}}{F'_{e^{i\theta},\lambda}}$ in Δ. By normalization we get $f = F_{e^{i\theta},\lambda}$.

□

Proof of Theorem 2.1 To prove that the closed curve $(-\pi, \pi] \ni \theta \mapsto logF'_{e^{i\theta},\lambda}(z_0)$ is simple. Suppose not. Then there are $\theta_1, \theta_2 \in (\pi, \pi]$ with $\theta_1 \neq \theta_2$ such that

$$logF'_{e^{i\theta_1},\lambda}(z_0) = logF'_{e^{i\theta_2},\lambda}(z_0)$$

By theorem (3.2), $F'_{e^{i\theta_1},\lambda} = F'_{e^{i\theta_2},\lambda}$. From (8),

$$e^{i\theta_1}z = \tau\left(\frac{\omega_{F_{e^{i\theta_1},\lambda}}}{z}, \lambda\right) = \tau\left(\frac{\omega_{F_{e^{i\theta_2},\lambda}}}{z}, \lambda\right) = e^{i\theta_2}z$$

which is a contradiction.Thus the curve is simple.Since $V_{\mathcal{G}}(z_0, \alpha, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, the boundary $\partial V_{\mathcal{G}}(z_0, \alpha, \lambda)$ is a simple closed curve.It follows from Theorem (3.2) that the curve $(-\pi, \pi] \ni \theta \mapsto logF'_{e^{i\theta},\lambda}(z_0)$ is a subcurve of $\partial V_{\mathcal{G}}(z_0, \alpha, \lambda)$. Since a simple closed curve cannot contain any simple closed curve other than itself, $\partial V_{\mathcal{G}}(z_0, \alpha, \lambda)$ is given by $(-\pi, \pi] \ni \theta \mapsto logF'_{e^{i\theta},\lambda}(z_0)$.

4. Region of variability of $V_{\mathcal{F}}(z_0, \alpha, \lambda)$

Theorem 4.1. *For* $f \in C_2(\lambda)$

$$\left|\frac{f''(z)}{f'(z)} - c(z,\lambda)\right| \leq r(z,\lambda), z \in \Delta \tag{20}$$

where

$$c(z,\lambda) = (4-\alpha)\left[\frac{\lambda(1-|z|^2) + \bar{z}(|z|^2 - \lambda^2)}{(1-|z|^2)(1-2Re(\lambda z) + |z|^2)}\right]$$

$$r(z,\lambda) = (4-\alpha)\left[\frac{(1-|\lambda|^2)|z|}{(1-|z|^2)(1-2Re(\lambda z) + |z|^2)}\right]$$

For each $z \in \Delta - \{0\}$*, equality holds if and only if* $f = H_{e^{i\theta},\lambda}$ *for some* $\theta \in \mathbb{R}$.

Corollary 4.1. *If* $f \in C_2(0)$ *then*

$$\left|\frac{f''(z)}{f'(z)} - (4-\alpha)\frac{\bar{z}|z|^2}{1-|z|^4}\right| \leq \frac{(4-\alpha)|z|}{1-|z|^4}$$

In particular,

$$(1-|z|^2)\left|\frac{f''(z)}{f'(z)}\right| \leq (4-\alpha)|z|$$

Corollary 4.2. *Let* $\gamma : z(t)(0 \leq t \leq 1)$ *be a* C^1*-curve in* Δ *with* $z(0) = 0$ *and* $z(1) = z_0$. *Then*

$$V_{\mathcal{F}}(z_0, \alpha, \lambda) \subset \bar{\Delta}(C(\lambda, \gamma), R(\lambda, \gamma))$$

where

$$\bar{\Delta}(C(\lambda, \gamma), R(\lambda, \gamma)) = \{w \in \mathbb{C} : |w - C(\lambda, \gamma)| \leq R(\lambda, \gamma)\}$$
$$C(\lambda, \gamma) = \int_0^1 c(z(t), \lambda) z'(t) dt$$
$$R(\lambda, \gamma) = \int_0^1 r(z(t), \lambda) |z'(t)| dt.$$

Theorem 4.2. *Let* $z_0 \in \Delta - \{0\}$*.Then, for* $\theta \in (-\pi, \pi]$*, we have* $logH'_{e^{i\theta},\lambda}(z_0) \in \partial V_{\mathcal{F}}(z_0, \alpha, \lambda)$*.Furthermore if* $logf'(z_0) = logH'_{e^{i\theta},\lambda}(z_0)$ *for some* $f \in C_2(\lambda)$*,* $\theta \in (-\pi, \pi]$*, then* $f = H_{e^{i\theta},\lambda}$

Proofs of the above results follows in line with the results proved in section 3. Hence we omit them.

Proof of Theorem 2.2 To prove that the closed curve $(-\pi, \pi] \ni \theta \mapsto logH'_{e^{i\theta},\lambda}(z_0)$ is simple. Suppose not. Then there are $\theta_1, \theta_2 \in (\pi, \pi]$ with $\theta_1 \neq \theta_2$ such that

$$logH'_{e^{i\theta_1},\lambda}(z_0) = logH'_{e^{i\theta_2},\lambda}(z_0)$$

By theorem (3.2), $H'_{e^{i\theta_1},\lambda} = H'_{e^{i\theta_2},\lambda}$. From (9),

$$e^{i\theta_1} z = \tau\left(\frac{\omega_{H_{e^{i\theta_1},\lambda}}}{z}, \lambda\right) = \tau\left(\frac{\omega_{H_{e^{i\theta_2},\lambda}}}{z}, \lambda\right) = e^{i\theta_2} z$$

Thus the curve is simple.Since $V_{\mathcal{F}}(z_0, \alpha, \lambda)$ is a compact convex subset of $\mathbb{C}$ and has nonempty interior, the boundary $\partial V_{\mathcal{F}}(z_0, \alpha, \lambda)$ is a simple closed curve.It follows from Theorem (4.2) that the curve $(-\pi, \pi] \ni \theta \mapsto logH'_{e^{i\theta},\lambda}(z_0)$ is a subcurve of $\partial V_{\mathcal{F}}(z_0, \alpha, \lambda)$. Since a simple closed curve cannot contain any simple closed curve other than itself, $\partial V_{\mathcal{F}}(z_0, \alpha, \lambda)$ is given by $(-\pi, \pi] \ni \theta \mapsto logH'_{e^{i\theta},\lambda}(z_0)$.

Remark 4.1. *When* $\alpha = 1$ *the results obtained here corresponds to those obtained in* [6].

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