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ON THE STUDY OF SINGULAR FRACTIONAL EVOLUTION PROBLEMS VIA A GENERALIZED FIXED POINT THEOREM IN COLOMBEAU ALGEBRA

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Abstract. In this paper, we give the existence and uniqueness of solutions for an abstract Caputo-type fractional evolution problem with generalized real numbers in the initial conditions. The mild solutions of our proposed model is constructed by using Laplace transform and a density function. By applying some fixed point theorems on Colombeau algebra, we prove our main results. As application, an illustrative example is given to show the applicability of our theoretical results.

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1. Introduction

In this manuscript, we are concerned with the existence, uniqueness, regularity of solutions for the following generalized fractional evolution problem

$$\begin{cases} \mathcal{D}^{\alpha}u(t) = Au(t) + f(t, u(t)), \quad 0 < \alpha < 1, \\ u(0) = x_0 \end{cases} \tag{1.1}$$

where $x_0 \in (\tilde{\mathbb{R}})^n$ is generalized real number, A be the infinitesimal generator of C_0-generalized semigroup $(S(t))_{t\geq 0}$ of uniformly bounded linear operators on a class of Colombeau algebras $\mathcal{G}(\mathbb{R}^n)$. Nets of smooth functions $(x_{0\varepsilon})_{\varepsilon\in(0,1]}$ are used to approximate the initial data. The purpose of this study is to prove the existence of a net of smooth solutions $(x_\varepsilon)_{\varepsilon\in[0,1)}$ up to a

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$\mathcal{O}(\varepsilon^{\infty})$ asymptotic error term. The paper is written in the Colombeau generalized functions framework. By using the Laplace transform and a density function, we present a new existence result of (1.1) with nonlinearity f, for initial data belong to the non-Archimedian ring $\widetilde{R}^n$. Precisely, this sort of problem can be found, for example, in the study of singularly perturbed partial differential equations, semi-classical analysis, and the regularization of partial differential operators with non-smooth coefficients or pseudo-differential operators with irregular symbols. We approach the problem from the perspective of asymptotic analysis: the right-hand side and solution regularity, as well as the operator's mapping qualities, will be explained using asymptotic estimates in terms of the parameter $(\varepsilon \to 0)$. In [2, 3] the author constructed an algebra, commutative, associative, differential, where $\mathcal{D}'$ is injected so that the product of indefinitely differentiable functions and the normal derivative are both respected. When it comes to algebra, non-linear operations $\mathcal{G}$ are more general than multiplication. Therefore, this algebra is very convenient for finding and studying the solutions to nonlinear differential equations with singular data and coefficients. This type of algebra is essential for calculating the multiplication of distributions [5] and [16]. As a nonlinear extension of distribution theory to deal with nonlinearities and singularities of data and coefficients in PDE theory [16], these algebras contain the space of distributions $\mathcal{D}'$ as a subspace with an embedding realized through convolution with a suitable mollifier. The elements of $\mathcal{G}^s$ are classes of smooth functions called moderate functions with respect to a set of negligible functions.

In the classical literature, over the past few years, many researcher's of focus within the theory of fractional calculus as an interesting and popular tool in modelling many phenomena in various fields of engineering, physics and economics. It often appears in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. (see [4,6]). The fractional evolution problem studied in [1] by E. Bazhlekova. In its work, Emilia provides necessary and sufficient conditions for an unbounded closed operator A in a Banach space X in order to solve the abstract Cauchy problem for the fractional evolution equation $\mathcal{D}^{\alpha}u = Au$, $0 < \alpha < 1$. The authors in [21], to reduce the Cauchy problem for a linear in-homogeneous partial differential equation to the Cauchy problem for the corresponding homogeneous equation, the authors use Duhamel's principle. In [23], the non-local Cauchy problem for fractional evolution equations in an arbitrary Banach space is studied by Y. Zhou et al, and several criteria for the existence and uniqueness of mild solutions are found. The author in [13] offer a method for dealing with fractional derivatives, including singularities based on Colombeau's theory of algebras of generalized functions. It is interested to solving fractional nonlinear ODEs and PDEs with singularities in the space of classical functions or distributions that have no solutions. There are relevant techniques such as regularization using delta sequences and multiplication with distinct cutoff functions to embed different kinds of fractional derivatives into the space of Colombeau special algebra of generalized functions for these goals. S. Mirjana in [20] gives an extension of Colombeau algebra of generalized functions to fractional derivatives. We apply it in solving ODEs and PDEs with entire and fractional derivatives with respect to temporal and spatial variables. In [14] the authors introduce the notion of C_0-Semigroup with polynomial growth in $\varepsilon \to 0$. Moreover, they show some properties concerning such notion, so that they use them to solve the heat equation. Fixed point concept is very essential for proving the existence and uniqueness of diverse mathematical models (partial differential equations, variational inequalities, etc). This theory has been studied by many researchers, but it is rare to find a paper that presented the fixed point theory in Colombeau algebra. We will rely on the work of J.Martin in [11], and we will use the topology of locally convex spaces to make sense of the concept of a fixed point in a class of Colombeau algebra compatible with our study of the evolution problem.

Motivated by the previous works, we write the mild solution of a representative of the evolution problem (**1.1**) by using the Laplace transform and density function. The class of such solution constituted an element of $\mathcal{G}^s(\mathbb{R}^n)$, and we demonstrate that is the unique solution in the spacial algebra $\mathcal{G}^s(\mathbb{R}^n)$.

The organization of the paper is as follows. In section 2, we recall some basic properties of the generalized functions theory. In section 3, we deal with the fixed point concept in a locally convex space of generalized function. The new notion of generalized semigroup take place in section 4. Section 5 is consecrated for the proof of the existence and uniqueness in Colombeau algebra to the problem given in (**1.1**). In Section 6 we have introduced an example to illustrate our work.

2. Preliminaries

In this section we introduce preliminary facts which are used throughout this paper. For this, we recall a few basics from the theory of generalized functions. The regularization methods of Colombeau type is to model non-smooth objects by approximating nets of any smooth functions. The elements of Colombeau algebras $\mathcal{G}^s$ are equivalence classes of regularization's nets, i.e., sequences of smooth functions satisfying asymptotic conditions in the regularization parameter ε. Let $n \in \mathbb{N}^*$, as in [5], we define the set

$$\mathcal{E}(\mathbb{R}^n) = \left(\mathcal{C}^\infty\left(\mathbb{R}^n\right)\right)^{(0,1)}.$$

The set of moderate functions is given as follows

$$\mathcal{E}_M^s(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n)/\forall K \subset\subset \mathbb{R}^n, \forall\alpha \in \mathbb{N}_0^n, \\ \exists N \in \mathbb{N} \ / \sup_{y\in K} |\partial^\alpha u_\varepsilon(y)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-N})\Big\}.$$

The ideal of negligible functions is defined by

$$\mathcal{N}(\mathbb{R}^n) = \Big\{(u_\varepsilon)_{\varepsilon>0} \subset \mathcal{E}(\mathbb{R}^n)/\forall K \subset\subset \mathbb{R}^n, \forall\alpha \in \mathbb{N}_0^n, \\ \forall p \in \mathbb{N} \ / \sup_{y\in K} |\partial^\alpha u_\varepsilon(y)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{p})\Big\}.$$

The Colombeau algebra is defined as a factor set

$$\mathcal{G}^s(\mathbb{R}^n) = \mathcal{E}_M^s(\mathbb{R}^n)/\mathcal{N}(\mathbb{R}^n).$$

For each x in $\mathcal{G}^s(\mathbb{R}^n)$, we can write $x = \overline{[(x_\epsilon)_\epsilon]}$.

The ring of all generalized real numbers is given by the following set

$$\widetilde{\mathbb{R}} = \mathcal{E}\left(\mathbb{R}\right)/\mathcal{I}\left(\mathbb{R}\right),$$

where

$$\mathcal{E}(\mathbb{R}) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)}/\exists m \in \mathbb{N}, |x_\varepsilon| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-m})\Big\},$$

and

$$\mathcal{I}(\mathbb{R}) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathbb{R})^{(0,1)}/\forall m \in \mathbb{N}, |x_\varepsilon| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

Where $|.|$ is the absolute value, and note that $\widetilde{\mathbb{R}}$ is a ring obtained by factoring moderate families of real numbers with respect to negligible families.

We need basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. [13]

I) The Caputo derivative of order $0 < \alpha < 1$ for a function f is given by

$${}^c\mathcal{D}^\alpha f(t) \;=\; \frac{1}{\Gamma(1-\alpha)}\int_0^t \frac{f'(\tau)}{(t-\tau)^\alpha}d\tau.$$

II) The fractional integral of order $0 < \alpha < 1$ for a function f can be written as

$$I^{\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\tau)^{\alpha-1} f(\tau) d\tau.$$

Definition 2.2. The fractional derivative of order in the sense of Caputo in the Colombeau algebra is defined by

$${}^{c}\mathcal{D}^{\alpha} F = [({}^{c}\mathcal{D}^{\alpha} f_{\varepsilon} * \rho_{\varepsilon})_{\varepsilon}], \qquad 0 < \alpha < 1$$

and the fractional integral in the algebra of Colombeau is defined by

$$I^{\alpha} F = [(I^{\alpha} f_{\varepsilon} * \rho_{\varepsilon})_{\varepsilon}] \qquad 0 < \alpha < 1.$$

Where $(f_{\varepsilon})_{\varepsilon>0}$ is a representative of $F \in \mathcal{G}^{s}(\mathbb{R})$ and $\rho_{\varepsilon}(x) = \frac{1}{\varepsilon}\rho(\frac{x}{\varepsilon})$, ρ is a test function such that $\rho \in \mathcal{C}^{\infty}(\mathbb{R}), \ \int_{\mathbb{R}} \rho(x) dx = 1, \quad \rho \geq 0.$

3. Generalized Fixed Points

3.1. **Contraction operator in $\tilde{X}$.** The contraction map in Colombeau type algebra is discussed in this subsection, this contraction type is inspired by that in the classic case in locally convex spaces X. We need some basic definitions and properties of the contraction in locally convex space. Firstly, we present the notion of locally convex spaces.

Definition 3.1. Let X be a vector space indowed with a familly $(N_i)_{i\in I}$ of seminorms. For all $i \in I$, we denote τ_i the topology induced by the seminorm N_i, and τ the topology generated by the classe of the all union sets τ_i . The pair $\big(X, \tau\big)$ is said to be locally convex space.

The set of all balls of the form is called a basis of 0-neighbourhood

$$B(i,r) = \Big\{x \in X/ \quad N_i(x) < r\Big\}, \quad \forall i \in I \text{ and } r > 0.$$

where $(N_i)_i$ is a family of seminorms.

Definition 3.2. We recall that a map $T_{\varepsilon} : X \longrightarrow X$ is called contraction if for all $i \in I$ there exits $k_i < 1$ such that

$$\forall (x_{\varepsilon}, y_{\varepsilon}) \in X \times X, \ N_i(T_{\varepsilon}x_{\varepsilon} - T_{\varepsilon}y_{\varepsilon}) \leq k_i \, N_i(x_{\varepsilon} - y_{\varepsilon}).$$

We will give a notion of contraction map in type Colombeau algebra.

Definition 3.3. Consider a locally convex space X endowed with a familly of seminorms $(N_i)_{i\in I}$. A class of moderate functions compatible with properties of the space X is defined by

$$\mathcal{E}_M^s(X) = \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} / \exists m \in \mathbb{N}, \forall i \in I, N_i(x_\varepsilon) = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-m})\Big\}.$$

The corresponding class of negligible functions is given as follows

$$\mathcal{N}(X) = \Big\{(x_\varepsilon)_\varepsilon \in (X)^{(0,1)} / \forall m \in \mathbb{N}, \forall i \in I, \quad N_i(x_\varepsilon) = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

The Colombeau algebra type is given in this case by

$$\tilde{X} = \mathcal{E}_M^s(X)/\mathcal{N}^s(X).$$

First, we will see if it's possible to give a definition of a map $T : \tilde{X} \to \tilde{X}$ by the data of a family $(T_\varepsilon)_{\varepsilon\in(0,1)}$ of maps $T_\varepsilon : X \to X$ where T_ε is a linear and continuous operator. The general idea is given in the following result

Lemma 3.4. [12, 19] *Let $(T_\varepsilon)_{\varepsilon\in(0,1)}$ be a given family of maps $T_\varepsilon : X \to X$. For each $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M(X)$ and $(y_\varepsilon)_\varepsilon \in \mathcal{N}(X)$, suppose that*

(1) $\Big(T_\varepsilon x_\varepsilon\Big)_\varepsilon \in \mathcal{E}_M(X)$,
(2) $\Big(T_\varepsilon(x_\varepsilon + y_\varepsilon)\Big)_\varepsilon - \Big(T_\varepsilon x_\varepsilon\Big)_\varepsilon \in \mathcal{N}(X)$.

Then

$$T : \begin{cases} \tilde{X} \longrightarrow \tilde{X} \\ x = \overline{[(x_\varepsilon)_\varepsilon]} \mapsto Tx = \overline{[(T_\varepsilon x_\varepsilon)_\varepsilon]}, \end{cases}$$

is well defined.

Definition 3.5. [11] A map $T : \tilde{X} \to \tilde{X}$ is called a contraction if only if

a) $(x_\varepsilon)_\varepsilon \in \mathcal{E}_M^s(X)$, implies $(T_\varepsilon x_\varepsilon)_\varepsilon \in \mathcal{E}_M^s(X)$ for all $\varepsilon \in (0,1)$.

b) T_ε is a contraction in $\big(X, \tau_\varepsilon\big)$ endowed with the family $M_\varepsilon = (M_{\varepsilon,i})_{i\in I}$ and the corresponding contraction constants are denoted by $l_{\varepsilon,i} < 1$.

c) For every $i \in I$ and $\varepsilon \in (0,1]$, $\exists \alpha_{\varepsilon,i} > 0$ and $\beta_{\varepsilon,i} > 0$, such that

$$\alpha_{\varepsilon,i}\, N_i \le M_{\varepsilon,i} \le \beta_{\varepsilon,i}\, N_i.$$

d) For each $i \in I$ and $\forall \varepsilon \in (0,1]$, we have $(\frac{\beta_{\varepsilon,i}}{\alpha_{\varepsilon,i}})_\varepsilon$, $(\frac{1}{1-l_{\varepsilon,i}})_\varepsilon \in |\mathcal{E}_M^s(\mathbb{R})|$.

The essential result given in the next theorem which has been proven in [11]

Theorem 3.6. *With the same previous notations, any contraction $T : \tilde{X} \to \tilde{X}$ has a fixed point in $\tilde{X}$.*

4. Generalized Semigroup

The generalized semigroup is introduced by M. Nedeljkov et Al [14]. Here, we liste its definition and fundamental properties.

Definition 4.1. [14]

• $\mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(S_\varepsilon)_\varepsilon$ of strongly continuous mappings

$S_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X),\ \ \varepsilon \in (0,1)$ with the property that for every $T > 0$ there exists $a \in \mathbb{R}$ such that

$$\sup_{t\in[0,T)} \|S_\varepsilon(t)\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{4.1}$$

• $\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}_c(X))$ is the space of nets $(N_\varepsilon)_\varepsilon$ of strongly continuous mappings

$N_\varepsilon : \mathbb{R}_+ \longrightarrow \mathcal{L}_c(X),\ \ \varepsilon \in (0,1)$. with the properties

For every $b \in \mathbb{R}$ and $T > 0$

$$\sup_{t\in[0,T)} \|N_\varepsilon(t)\| = O_{\varepsilon\to 0}(\varepsilon^b). \tag{4.2}$$

There exist $t_0 > 0$ and $a \in \mathbb{R}$ such that

$$\sup_{t<t_0} \|\frac{N_\varepsilon(t)}{t}\| = O_{\varepsilon\to 0}(\varepsilon^a). \tag{4.3}$$

There exists a net $(H_\varepsilon)_\varepsilon$ in $\mathcal{L}_c(X)$ and $\varepsilon_0 \in (0,1)$ such that

$$\lim_{t\longrightarrow 0} \frac{N_\varepsilon(t)}{t} = H_\varepsilon x, \quad x \in X, \quad \varepsilon < \varepsilon_0. \tag{4.4}$$

For every $b > 0$,

$$\|H_\varepsilon\| = O_{\varepsilon\to 0}(\varepsilon^b). \tag{4.5}$$

M. Nedeljkov et Al introduce [14] a Colombeau type algebra as the factor algebra:

$$\mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X)) = \mathcal{SE}_M(\mathbb{R}_+ : \mathcal{L}(X))/\mathcal{SN}(\mathbb{R}_+ : \mathcal{L}(X)).$$

Definition 4.2. [14]

$S \in \mathcal{SG}(\mathbb{R}_+ : \mathcal{L}(X))$ is called a Colombeau C_0-Semigroup if it has a representative $(S_\varepsilon)_\varepsilon$ such that, for some $\varepsilon_0 > 0$, S_ε is a C_0-Semigroup, for every $\varepsilon < \varepsilon_0$.

M. Nedeljkov et Al introduce the infinitesimal generator of a Colombeau C_0-semigroup S and it's denote by $\mathcal{A}$ the set of pairs $((A_\varepsilon)_\varepsilon, (D(A_\varepsilon))_\varepsilon)$ where A_ε is a closed linear operator on X with the dense domain $D(A_\varepsilon) \subset X$, for every $\varepsilon \in (0,1)$.

5. Main Result

In this section, the evolution problem for the fractional equations with initial data are distributions is discussed, and various criteria on the existence and uniqueness of Colombeau generalized solutions, one introduces the algebra of generalized functions suitable to this context.

We define the simplified algebra of global generalized functions, which must be compatible with the study of the fractional evolution equations, denoted $\mathcal{G}^s([0,\infty))$, by the quotient algebra

$$\mathcal{G}^s([0,\infty)) = \mathcal{E}^s_M([0,\infty))/\mathcal{N}^s([0,\infty)),$$

where

$$\mathcal{E}^s_M([0,\infty)) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(\mathbb{R}^+))^{(0,1)}/\exists m \in \mathbb{N}, \forall T > 0, \\ \sup_{t\in[0,T]} |x_\epsilon(t)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{-m})\Big\},$$

and

$$\mathcal{N}^s([0,\infty)) = \Big\{(x_\varepsilon)_\varepsilon \in (\mathcal{C}^\infty(\mathbb{R}^+))^{(0,1)}/\forall m \in \mathbb{N}, \forall T > 0, \\ \sup_{t\in[0,T]} |x_\varepsilon(t)| = \mathcal{O}_{\varepsilon\to 0}(\varepsilon^{m})\Big\}.$$

Now, we recall the problem (1.1) of the following form

$$\begin{cases} {}^c\mathcal{D}^\alpha u(t) = Au(t) + f(t,u(t)), \quad u \in \mathcal{G}^s(\mathbb{R}^+),\ t \in \mathbb{R}^+ \\ u(0) = x_0 \in \tilde{\mathbb{R}}. \end{cases} \tag{5.1}$$

where ${}^c\mathcal{D}^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, A is the infinitesimal generator of a Colombeau C0-semigroup $S = \overline{[(S_\varepsilon)_\varepsilon]}$, $x_0 \in \tilde{\mathbb{R}}$ and $f : [0,\infty] \times \mathcal{G}^s([0,\infty)) \to \tilde{\mathbb{R}}$.

In the rest of the paper we will need some definitions and basic properties.

Definition 5.1. [23] The one-sided stable probability density is given by

$$\Psi_\alpha(\tau) = \frac{1}{\pi}\sum_{n=1}^{\infty}(-1)^{n-1}(-\tau)^{-\alpha n-1}\frac{\Gamma(1+\alpha n)}{n!}\sin(n\pi\alpha), \quad \tau \in (0,\infty). \tag{5.2}$$

and

$$h_\alpha(\tau) = \frac{1}{\alpha}\tau^{-(1+\frac{1}{\alpha})}\Psi_\alpha(\tau^{-\frac{1}{\alpha}})$$

is the probability density function defined on $(0,\infty)$.

According to [23], we have

$$\int_0^\infty \tau h_\alpha(\tau)d\tau = \int_0^\infty \frac{1}{\tau^\alpha}\Psi_\alpha(\tau)d\tau = \frac{1}{\Gamma(1+\alpha)}. \tag{5.3}$$

The following Lemma plays a key role in the proof of our main results.

Lemma 5.2. (***Holder Inequality***) [23]. *Assume that $q, p \geq 1$, and $\frac{1}{q} + \frac{1}{q} = 1$. If $g \in L^q(J, \mathbb{R}),\ h \in {}^p(J, \mathbb{R})$, then for $1 \leq p \leq \infty$, $gh \in L^1(J, \mathbb{R})$ and*

$$\| gh \| \leq \| g \|_{l^q(J)} \| h \|_{l^p(J)} .$$

Lemma 5.3. [23] *A function u is a solution of problem (**5.1**), if only if u satisfies the following integral equation*

$$\begin{aligned} u(t) &= \int_0^\infty h_\alpha(\tau)S(t^\alpha\tau)u_0 d\tau \\ &+ \alpha \int_0^t \int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S((t-s)^\alpha\tau)f(s, u(s))d\tau ds. \end{aligned}$$

We make the following hypotheses will be used in the sequel.

(H_1) There exists a constant $M_\epsilon \geq 0$, such that

$$\sup_{t\in[0,\infty)} \|S_\varepsilon(t)\| \leq M_\epsilon,\ \forall \epsilon \in (0,1).$$

(H_2) For each $x_\varepsilon \in \mathcal{C}([0,T], \mathbb{R})$, the function $f_\varepsilon(\cdot, x_\varepsilon) : [0,T] \to \mathbb{R}$ is strongly measurable and for each $t \in [0,T]$, the function $f_\varepsilon(t, \cdot) : \mathbb{R} \to \mathbb{R}$ is continuous.

(H_3) There exist $\beta \in [0, \alpha)$, $m_\epsilon \in L^{\frac{1}{\beta}}([0,T], \mathbb{R}^+)$ such that

$$\sup_{t\in[0,T]} |m_\epsilon^\beta(s)| = M_{0\epsilon} \quad \forall \epsilon \in (0,1),$$

and

$$|f_\varepsilon(t, x_\varepsilon(t)) - f_\varepsilon(t, y_\varepsilon(t))| \leq m_\epsilon(t)|x_\varepsilon - y_\varepsilon|,$$

for $x_\varepsilon, y_\varepsilon \in \mathcal{C}([0,T], \mathbb{R})$ and $t \in [0,T]$.

Theorem 5.4. *Assume that the hypotheses (H_1)-(H_3) hold. then the problem (**5.1**) has a unique solution.*

Proof. Consider the operator $Q : \mathcal{G}^s([0,\infty)) \to \mathcal{G}^s([0,\infty))$ such that

$$Q : \begin{cases} \mathcal{G}^s(\mathbb{R}^+) & \longrightarrow \mathcal{G}^s(\mathbb{R}^+) \\ u & \mapsto Qu, \end{cases}$$

where

$$Qu(t) = \int_0^\infty h_\alpha(\tau)S(t^\alpha\tau)u_0 d\tau + \alpha\int_0^t\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S((t-s)^\alpha\tau)f(s,u(s))d\tau ds.$$

In the following, we will prove that Q has a unique fixed point on $\mathcal{G}^s(\mathbb{R}^+)$, and for this we have to prove that the conditions of the definition3.5 is holds.

a) Firstly, we will prove that Q is will defined. For $(u_\varepsilon)_\varepsilon \in \mathcal{E}^s_M(\mathbb{R}^+)$, we have

$$Q_\varepsilon u_\varepsilon(t) = \int_0^\infty h_\alpha(\tau)S_\varepsilon(t^\alpha\tau)u_{0\varepsilon} d\tau + \alpha\int_0^t\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S_\varepsilon((t-s)^\alpha\tau)f_\varepsilon(s,u_\varepsilon(s))d\tau ds.$$

Then, Q_ε is defined from $\mathcal{C}^\infty(\mathbb{R}^+)$ into $\mathcal{C}^\infty(\mathbb{R}^+)$. Endowed with the topology τ given by the family of norms $(N_T)_{T\in\mathbb{R}^+}$, $\mathcal{C}^\infty(\mathbb{R}^+)$ is a topological space, such that $N_T(u_\varepsilon) = \sup_{t\in[0,T]} |u_\varepsilon(t)|$.

Let $(u_\varepsilon)_\varepsilon \in \mathcal{E}^s_M(\mathbb{R}^+)$. By (**5.3**), we can obtain

$$\left|\int_0^\infty h_\alpha(\tau)S_\varepsilon(t^\alpha\tau)u_{0\varepsilon} d\tau\right| \leq M_\epsilon|u_{0\varepsilon}|$$

aand

$$\begin{aligned}
&\left|\int_0^t\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S_\varepsilon((t-s)^\alpha\tau)f_\varepsilon(s,u_\varepsilon(s))d\tau ds\right| \\
&\leq \int_0^t \left|\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S_\varepsilon((t-s)^\alpha\tau)f_\varepsilon(s,u_\varepsilon(s))d\tau\right| ds \\
&\leq M_\epsilon \int_0^t\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)|f_\varepsilon(s,u_\varepsilon(s))|d\tau ds \\
&\leq \frac{M_\epsilon}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}|f_\varepsilon(s,u_\varepsilon(s))|ds.
\end{aligned}$$

Then

$$|Q_\varepsilon u_\varepsilon(t)| \leq M_\epsilon|u_{0\varepsilon}| + \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}|f_\varepsilon(s,u_\varepsilon(s))|,$$

which implies

$$|Q_\varepsilon u_\varepsilon| \leq M|u_{0\varepsilon}| + \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}N_T(f_\varepsilon)\int_0^t (t-s)^{\alpha-1}ds.$$

On the other hand, direct calculation gives that

$$(t-s)^{\alpha-1} \in L^{\frac{1}{1-q}}[0,T], \qquad \text{for } q \in [0,\alpha).$$

Let

$$b = \frac{\alpha - 1}{1 - q} \in]-1, 0[, \qquad M_{1\epsilon} = \| m_\epsilon \|_{L^{\frac{1}{q}}} . \tag{5.4}$$

Using Lemma 5.2, we get

$$\begin{aligned} |Q_\varepsilon u_\varepsilon| &\leq M_\epsilon |u_{0\varepsilon}| + \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} N_T(f_\varepsilon) \bigg(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}} ds \bigg)^{1-q} T \\ &\leq M_\epsilon |u_{0\varepsilon}| + \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} N_T(f_\varepsilon) \frac{T^{1+(1+b)(1-q)}}{(1+b)^{1-q}} \\ &\leq M_\epsilon |u_{0\varepsilon}| + \frac{\alpha M_\epsilon \, T^{1+(1+b)(1-q)}}{\Gamma(1+\alpha)(1+b)^{1-q}} N_T(f_\varepsilon). \end{aligned}$$

It follows that

$$\big(Q_\varepsilon u_\varepsilon\big)_\varepsilon \in \mathcal{E}_M^s\big(\mathbb{R}^+\big).$$

- Let $(v_\varepsilon)_\varepsilon \in \mathcal{N}^s\big(\mathbb{R}^+\big)$. We have

$$\begin{aligned} Q_\varepsilon\big(u_\varepsilon(t) + v_\varepsilon(t)\big) - Q_\varepsilon u_\varepsilon(t) = \alpha \int_0^t \int_0^\infty \tau (t-s)^{\alpha-1} h_\alpha(\tau) S_\varepsilon((t-s)^\alpha \tau) \\ \Big[f_\varepsilon(s, u_\varepsilon(s) + v_\varepsilon(s)) - f_\varepsilon(s, u_\varepsilon(s)) \Big] d\tau ds. \end{aligned}$$

Therefore,

$$\begin{aligned} |Q_\varepsilon\big(u_\varepsilon(t) + v_\varepsilon(t)\big) - Q_\varepsilon u_\varepsilon(t)| \leq \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} \\ |f_\varepsilon(s, u_\varepsilon(s) + v_\varepsilon(s)) - f_\varepsilon(s, u_\varepsilon(s))| ds, \end{aligned}$$

so,

$$\begin{aligned} |Q_\varepsilon\big(u_\varepsilon(t) + v_\varepsilon(t)\big) - Q_\varepsilon u_\varepsilon(t)| &\leq \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} \int_0^t (t-s)^{\alpha-1} m(s) |v_\varepsilon| ds \\ &\leq \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} N_T(v_\varepsilon) \int_0^t (t-s)^{\alpha-1} m_\epsilon(s) ds. \end{aligned}$$

From Lemma 5.2 and (5.4), we obtain

$$\begin{aligned} &|Q_\varepsilon\big(u_\varepsilon(t) + v_\varepsilon(t)\big) - Q_\varepsilon u_\varepsilon(t)| \\ &\qquad \leq \frac{\alpha M_\epsilon T}{\Gamma(1+\alpha)} N_T(v_\varepsilon) \bigg(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}} ds \bigg)^{1-q} \| m_\epsilon \|_{L^{\frac{1}{q}}} ds \\ &\qquad \leq \frac{\alpha M_\epsilon \, M_{1\epsilon} \, T^{(1+b)(1-q)}}{\Gamma(1+\alpha)(1+b)^{1-q}} N_T(v_\varepsilon) \end{aligned}$$

Since

$$(v_\varepsilon)_\varepsilon \in \mathcal{N}^s\big(\mathbb{R}^+\big).$$

So,

$$\big(Q_\varepsilon(u_\varepsilon+v_\varepsilon)-Q_\varepsilon u_\varepsilon\big)_\varepsilon \in \mathcal{N}^s\big(\mathbb{R}^+\big).$$

According to the definition 3.5, we get the map Q is well defined.

b) Second step, we will show that Q_ε is a contration. The regularization of problem (5.1), defined as follows

$$\begin{cases} {}^c\mathcal{D}^\alpha u_\varepsilon(t) = A_\varepsilon u_\varepsilon(t) + f_\varepsilon(t,u_\varepsilon(t)), & t\in\mathbb{R}^+ \\ u_\varepsilon(0) = x_{0\varepsilon}\in\mathbb{R}. \end{cases} \tag{5.5}$$

Endowed with the topology τ_ε given by the family of norms $(M_{T,\varepsilon})_{T\in\mathbb{R}^+}$, $\mathcal{C}^\infty(\mathbb{R}^+)$ is a topological space, such that for all $u_\varepsilon\in\mathcal{C}^\infty(\mathbb{R}^+)$.

$$M_{T,\varepsilon}(y_\varepsilon) = \sup_{t\in[0,T]}\{e^{-t\,H_{T,\epsilon}}|u_\varepsilon(t)|\},$$

where

$$H_{T,\epsilon} = \frac{\alpha\ M_{0\epsilon}\ M_\epsilon\ T^{(1+b)(1-q}}{\Gamma(1+\alpha)\ (1+b)^{(1-q)}}.$$

Let $(u_\varepsilon)_\varepsilon, (v_\varepsilon)_\varepsilon\in\mathcal{E}^s_M(\mathbb{R}^+)$, we have

$$\begin{aligned} Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t) &= \alpha\int_0^t\int_0^\infty \tau(t-s)^{\alpha-1}h_\alpha(\tau)S_\varepsilon((t-s)^\alpha\tau) \\ &\qquad \Big[f_\varepsilon(s,u_\varepsilon(s)) - f_\varepsilon(s,v_\varepsilon(s))\Big]d\tau ds. \end{aligned}$$

From (5.3), then

$$\begin{aligned} |Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)| \le{}& \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)} \\ & \int_0^t (t-s)^{\alpha-1}|f_\varepsilon(s,u_\varepsilon(s)) - f_\varepsilon(s,v_\varepsilon(s))|ds. \end{aligned}$$

Hence,

$$|Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)| \le \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\int_0^t (t-s)^{\alpha-1}m_\epsilon(s)|u_\varepsilon(s)-v_\varepsilon(s)|ds.$$

By (5.4) and Lemma 5.2, we get

$$\begin{aligned} |Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)| &\le \frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\left(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds\right)^{1-q} \\ &\qquad \left(\int_0^t \big[m_\epsilon(s)|u_\varepsilon(s)-v_\varepsilon(s)|\big]^{\frac{1}{q}}ds\right)^q \\ &\le \frac{\alpha\ M_\epsilon\ T^{(1+b)(1-q}}{\Gamma(1+\alpha)\ (1+b)^{(1-q)}}M_{0\epsilon}\left(\int_0^t |u_\varepsilon(s)-v_\varepsilon(s)|^{\frac{1}{q}}ds\right)^q, \end{aligned}$$

which implies

$$|Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)| \le H_{T,\epsilon}\bigg(\int_0^t |u_\varepsilon(s) - v_\varepsilon(s)|^{\frac{1}{q}} ds \bigg)^q.$$

Multipling the both sides of the inequality by $\exp\{-t\ H_{T,\epsilon}\}$, we get

$$\exp\{-t\ H_{T,\epsilon}\}|Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)| \le \exp\{-t\ H_{T,\epsilon}\}H_{T,\epsilon} \bigg(\int_0^t |u_\varepsilon(s) - v_\varepsilon(s)|^{\frac{1}{q}} ds \bigg)^q.$$

Scince

$$\begin{aligned} e^{-t\ H_{T,\epsilon}} H_{T,\epsilon} \int_0^t |u_\varepsilon(s) - v_\varepsilon(s)| ds &= e^{-t\ H_{T,\epsilon}} H_{T,\epsilon}\bigg(\int_0^t |u_\varepsilon(s) - v_\varepsilon(s)|^{\frac{1}{q}} ds \bigg)^q \\ &= e^{-t\ H_{T,\epsilon}} H_{T,\epsilon} \\ &\bigg(\int_0^t \left[e^{s\ H_{T,\epsilon}} e^{-s\ H_{T,\epsilon}} |u_\varepsilon(s) - v_\varepsilon(s)|\right]^{\frac{1}{q}} ds \bigg)^q \\ &= e^{-t\ H_{T,\epsilon}} H_{T,\epsilon} \\ &\bigg(\int_0^t (e^{s\ H_{T,\epsilon}})^{\frac{1}{q}} \left[e^{-s\ H_{T,\epsilon}} |u_\varepsilon(s) - v_\varepsilon(s)|\right]^{\frac{1}{q}} ds \bigg)^q, \end{aligned}$$

which gives

$$\begin{aligned} e^{-t\ H_{T,\epsilon}} H_{T,\epsilon} \int_0^t |u_\varepsilon(s) - v_\varepsilon(s)| ds &\le \sup_{t\in(0,T]} \{e^{-t\ H_{T,\epsilon}} |u_\varepsilon(s) - v_\varepsilon(s)|\}\ e^{-t\ H_{T,\epsilon}} H_{T,\epsilon}\bigg(\int_0^t e^{s\ H_{T,\epsilon}})^{\frac{1}{q}} ds \bigg)^q \\ &\le \sup_{t\in(0,T]} \{e^{-t\ H_{T,\epsilon}} |u_\varepsilon(s) - v_\varepsilon(s)|\}\ e^{-t\ H_{T,\epsilon}} H_{T,\epsilon}\bigg(\int_0^t e^{s\ \frac{H_{T,\epsilon}}{q}} ds \bigg)^q \\ &\le M_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)\ e^{-t\ H_{T,\epsilon}} H_{T,\epsilon}\bigg(\left[e^{s\ \frac{H_{T,\epsilon}}{q}}\right]_0^t \bigg)^q \\ &\le M_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)\ \frac{q^q}{H_{T,\epsilon}^{q-1}} \bigg(1 - e^{-t\ \frac{H_{T,\epsilon}}{q}} \bigg)^q. \end{aligned}$$

Hence,

$$\sup_{t\in(0,T]} \{e^{-t\ H_{T,\epsilon}} |Q_\varepsilon u_\varepsilon(t) - Q_\varepsilon v_\varepsilon(t)|\} \le M_{T,\varepsilon}(u_\varepsilon - v_\varepsilon)\ \frac{q^q}{H_{T,\epsilon}^{q-1}} \bigg(1 - e^{-T\ \frac{H_{T,\epsilon}}{q}} \bigg)^q,$$

that is

$$M_{T,\varepsilon}(Q_\varepsilon u_\varepsilon - Q_\varepsilon v_\varepsilon) \le \ \frac{q^q}{H_{T,\epsilon}^{q-1}} \bigg(1 - e^{-T\ \frac{H_{T,\epsilon}}{q}} \bigg)^q\ M_{T,\varepsilon}(u_\varepsilon - v_\varepsilon),$$

but $\frac{q^q}{H_{T,\epsilon}^{q-1}}\left(1-e^{-T\,\frac{H_{T,\epsilon}}{q}}\right)^q<1$, we can conclude that Q_ε is a contraction in $\left(\mathcal{C}^\infty(\mathbb{R}^+),\tau_\varepsilon\right)$.

c) For all $T\in\mathbb{R}^+$ and $u_\varepsilon\in C^\infty(\mathbb{R}^+)$, we have

$$\begin{aligned} e^{-T\,H_{T,\epsilon}}\sup_{t\in[0,T]}\{|u_\varepsilon(t)|\} &\leq \sup_{t\in[0,T]}\{|u_\varepsilon(t)|e^{-t\,H_{T,\epsilon}}\} \\ &\leq \sup_{t\in[0,T]}|u_\varepsilon(t)|, \end{aligned}$$

then

$$e^{-T\,H_{T,\epsilon}}N_T\leq M_{T,\varepsilon}\leq N_T.$$

d) For each $T\in\mathbb{R}^+$, we have

$$e^{T\,H_{T,\epsilon}}\in|\mathcal{E}_M^s| \text{ and } 1/(1-e^{-T\,H_{T,\epsilon}})\in|\mathcal{E}_M^s|.$$

Finally, according to the definition 3.5 the map

$$Q:\begin{cases}\mathcal{G}^s(\mathbb{R}^+)\longrightarrow\mathcal{G}^s(\mathbb{R}^+)\\ u(t)=\overline{[(u_\varepsilon(t))_\varepsilon]}\longmapsto Qu(t)=\overline{[(Q_\varepsilon u_\varepsilon(t))_\varepsilon]},\end{cases}$$

is a contraction. By using Theorem 3.6, the map Q has a fixed point u on $\mathcal{G}^s(\mathbb{R}^+)$. Hence u is a solution of (5.1).

Uniqueness: Suppose that $v=[v_\varepsilon]$ is another solution of the problem (5.1), we set

$$v_\varepsilon=Q_\varepsilon(v_\varepsilon)+n_\varepsilon,$$

where $n_\varepsilon\in\mathcal{N}^s(\mathbb{R}^+)$. We have

$$\begin{aligned} u_\varepsilon(t)-v_\varepsilon(t)=\alpha\int_0^t\int_0^\infty &\tau(t-s)^{\alpha-1}h_\alpha(\tau)S_\varepsilon((t-s)^\alpha\tau) \\ &\Big[f_\varepsilon(s,u_\varepsilon(s))-f_\varepsilon(s,v_\varepsilon(s))\Big]d\tau ds+n_\varepsilon(t). \end{aligned}$$

By (5.3), we get

$$|u_\varepsilon(t)-v_\varepsilon(t)|\leq\frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}|f_\varepsilon(s,u_\varepsilon(s))-f_\varepsilon(s,v_\varepsilon(s))|ds+|n_\varepsilon(t)|.$$

and thus,

$$|u_\varepsilon(t)-v_\varepsilon(t)|\leq\frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\int_0^t(t-s)^{\alpha-1}m_\epsilon(s)|u_\varepsilon(s)-v_\varepsilon(s)|ds+|n_\varepsilon(t)|.$$

From (5.4) and Lemma 5.2, we get

$$\begin{aligned}|u_\varepsilon(t)-v_\varepsilon(t)| &\leq |n_\varepsilon(t)|+\frac{\alpha M_\epsilon}{\Gamma(1+\alpha)}\bigg(\int_0^t (t-s)^{\frac{\alpha-1}{1-q}}ds\bigg)^{1-q}\\ &\qquad \bigg(\int_0^t \big[m_\epsilon(s)|u_\varepsilon(s)-v_\varepsilon(s)|\big]^{\frac{1}{q}}ds\bigg)^q\\ &\leq |n_\varepsilon(t)|+\frac{\alpha\ M_\epsilon\ T^{(1+b)(1-q}}{\Gamma(1+\alpha)\ (1+b)^{(1-q)}}M_{0\epsilon}\\ &\qquad \bigg(\int_0^t |u_\varepsilon(s)-v_\varepsilon(s)|^{\frac{1}{q}}ds\bigg)^q,\end{aligned}$$

which implies

$$|u_\varepsilon(t)-v_\varepsilon(t)| \ \leq\ |n_\varepsilon(t)|+H_{T,\epsilon}\bigg(\int_0^t |u_\varepsilon(s)-v_\varepsilon(s)|^{\frac{1}{q}}ds\bigg)^q.$$

Then,

$$\sup_{t\in[0,T]}\{|u_\varepsilon(t)-v_\varepsilon(t)|\} \ \leq\ \sup_{t\in[0,T]}|n_\varepsilon(t)|\ \frac{1}{(1-H_{T,\epsilon}\ T^q)},$$

and thus

$$N_T(u_\varepsilon-v_\varepsilon)\leq \frac{1}{(1-H_{T,\epsilon}\ T^q)}N_T(n_\varepsilon).$$

Since $(n_\varepsilon)_\varepsilon\in\mathcal{N}^s(\mathbb{R}^+)$, then $(u_\varepsilon-v_\varepsilon)_\varepsilon\in\mathcal{N}^s(\mathbb{R}^+)$, it follows that

$$u=v,$$

which prove the uniqueness of solution.

□

6. Example

Let us consider the following fractional partial differential equations.

$$\text{(6.1)}\qquad \begin{cases}\partial_t^{1/2}v(t,y)=\partial_y^2 v(t,y)+t^{1/3}\sin(v(t,y)), & (t,y)\in\mathbb{R}^+\times[0,\pi],\\ v(t,o)=v(t,\pi)=0,\\ v(0,y)=\delta(y), \qquad y\in[0,\pi].\end{cases}$$

where $\delta=[(\delta_\varepsilon)_\varepsilon]$ is the embedding of the Dirac measure in $\mathcal{G}(\mathbb{R})$ and

$$\delta_\varepsilon(y)=\delta*\psi_\varepsilon(y)=\psi_\varepsilon(y)=\frac{1}{\varepsilon}\psi(\frac{y}{\varepsilon}),\ \ y\in\mathbb{R},\quad \text{for all } \varepsilon\in(0,1),$$

where ψ is a test function such that $\psi\in\mathcal{C}^\infty(\mathbb{R})$, $\int_{\mathbb{R}}\psi(x)dx=1,\quad \psi(x)\geq 0.$

We define an operator A by

$$Av = \overline{[(A_\varepsilon v_\varepsilon)_\varepsilon]},\ \text{where } A_\varepsilon v_\varepsilon = v_\varepsilon'',$$

with the domain

$$\begin{aligned}\mathcal{D}(A_\varepsilon) = \{v_\varepsilon \in \mathcal{C}(\mathbb{R})\ /\ v_\varepsilon, \qquad & \partial v_\varepsilon \text{ absolutely continuous},\\ & \partial^2 v_\varepsilon \in \mathcal{C}(\mathbb{R}),\ v_\varepsilon(0) = v_\varepsilon(\pi) = 0\}.\end{aligned}$$

Then A is the infinitesimal generator of a Colombeau C0-semigroup $S = \overline{[(S_\varepsilon)_\varepsilon]}$. Moreover, (H_1) is satisfied.

The equations (6.1) can be reformulation as the following Cauchy problem in $\mathcal{G}^s$

$$\begin{cases} {}^c\mathcal{D}^\alpha u(t) = Au(t) + f(t, u(t)), \quad u \in \mathcal{G}^s(\mathbb{R}^+),\ t \in \mathbb{R}^+, \\ u(0) = x_0 \in \tilde{\mathbb{R}}; \end{cases} \tag{6.2}$$

where $u(t) = v(t, .)$, that is $u(t)y = v(t, y), \quad t \in [0, T], y \in [0, \pi]$, $\alpha = 1/2$ and the function $f : \mathbb{R}^+ \times \mathcal{G}^s(\mathbb{R}^+) \longrightarrow \tilde{R}$, such that

$$f : \begin{cases} \mathbb{R}^+ \times \mathcal{G}^s(\mathbb{R}^+) & \longrightarrow \tilde{\mathbb{R}} \\ \qquad (t, u) & \mapsto f(t, u(t)) = \overline{[(f_\varepsilon(t, u_\varepsilon(t)))_\varepsilon]}, \end{cases}$$

where

$$f_\varepsilon(t, u_\varepsilon(t) = t^{1/3}\sin(u_\varepsilon(t)).$$

Moreover, we have

$$\begin{aligned}|f_\varepsilon(t, u_\varepsilon(t)) - f_\varepsilon(t, w_\varepsilon(t))| &= |t^{1/3}\sin(u_\varepsilon(t)) - t^{1/3}\sin(w_\varepsilon(t))|\\ &= t^{1/3}|\sin(u_\varepsilon(t)) - t^{1/3}\sin(w_\varepsilon(t))|\\ &\le t^{1/3}|u_\varepsilon(t) - w_\varepsilon(t)|.\end{aligned}$$

We can take $m_\epsilon(t) = t^{1/3}$. It is clear

$$m_\epsilon \in L^{\frac{1}{1/3}},\ \ \beta = 1/3 \in [0, 1/2),$$

and $\sup\limits_{t\in[0,T]} |m_\epsilon(t)| = T^{1/3}$.

Then, (H_2) and (H_3) are satisfied.

Finally, according to Theorem 5.4, equations (6.1) has a unique solution in $\mathcal{G}^s$.

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