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INTUITIONISTIC $\mathscr{N}$-FUZZY STRUCTURES OVER HILBERT ALGEBRAS

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Abstract. The notions of intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras are introduced, and several properties are investigated. Conditions for intuitionistic $\mathscr{N}$-fuzzy structures to be intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras are provided. It is also explored how intuitionistic $\mathscr{N}$-fuzzy subalgebras (intuitionistic $\mathscr{N}$-fuzzy ideals) relate to their t-level subsets. Hilbert algebras are also investigated in terms of the homomorphic pre-images of intuitionistic $\mathscr{N}$-fuzzy subalgebras (intuitionistic $\mathscr{N}$-fuzzy ideals) and other related properties.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [23]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted

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on the generalizations of fuzzy sets, one of which is the intuitionistic fuzzy set defined by Atanassov [2]. The integration between fuzzy sets and some uncertainty approaches such as soft sets and rough sets has been discussed in [1,3,6,19]. The idea of intuitionistic fuzzy sets suggested by Atanassov [2] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [12–14]. The concept of Hilbert algebras was introduced in early 50-ties by Henkin [15] for some investigations of implication in intuitionistic and other non-classical logics. In 60-ties, these algebras were studied especially by Diego [8] from algebraic point of view. Diego [8] proved that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by Busneag [4,5] and Jun [16] and some of their filters forming deductive systems were recognized. Dudek [9–11] considered the fuzzification of subalgebras/ideals and deductive systems in Hilbert algebras.

The study of $\mathscr{N}$-fuzzy structures has continued, for example, in 2017, Smarandache et al. [18] introduced neutrosophic $\mathscr{N}$-structures over semigroups. In 2018, Songsaeng and Iampan [22] studied $\mathscr{N}$-fuzzy UP-subalgebras, $\mathscr{N}$-fuzzy UP-filters, $\mathscr{N}$-fuzzy UP-ideals, and $\mathscr{N}$-fuzzy strong UP-ideals of UP-algebras. Rangsuk et al. [20] studied neutrosophic $\mathscr{N}$-structures over UP-algebras in 2019. In 2022, Simuen et al. [21] studied picture N-structures over semigroups.

We presented the concepts of intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras in this work and looked into a variety of characteristics. Criteria are given for intuitionistic $\mathscr{N}$-fuzzy structures to be intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras. It is also explored how intuitionistic $\mathscr{N}$-fuzzy subalgebras (intuitionistic $\mathscr{N}$-fuzzy ideals) relate to their t-level subsets. Moreover, the homomorphic pre-images of intuitionistic $\mathscr{N}$-fuzzy subalgebras (intuitionistic $\mathscr{N}$-fuzzy ideals) are studied, along with other related features, for Hilbert algebras.

2. Preliminaries

Before we begin our study, we will give the definition of a Hilbert algebra.

Definition 2.1. [8] A *Hilbert algebra* is a triplet with the formula $X = (X, \cdot, 1)$, where X is a nonempty set, $\cdot$ is a binary operation, and 1 is a fixed member of X that is true according to the axioms stated below:

(1) $(\forall x, y \in X)(x \cdot (y \cdot x) = 1)$,

(2) $(\forall x, y, z \in X)((x \cdot (y \cdot z)) \cdot ((x \cdot y) \cdot (x \cdot z)) = 1)$,

(3) $(\forall x, y \in X)(x \cdot y = 1, y \cdot x = 1 \Rightarrow x = y)$.

In [9], the following conclusion was established.

Lemma 2.2. *Let $X = (X, \cdot, 1)$ be a Hilbert algebra. Then*

(1) $(\forall x \in X)(x \cdot x = 1)$,

(2) $(\forall x \in X)(1 \cdot x = x)$,

(3) $(\forall x \in X)(x \cdot 1 = 1)$,

(4) $(\forall x, y, z \in X)(x \cdot (y \cdot z) = y \cdot (x \cdot z))$.

In a Hilbert algebra $X = (X, \cdot, 1)$, the binary relation $\leq$ is defined by

$$(\forall x, y \in X)(x \leq y \Leftrightarrow x \cdot y = 1),$$

which is a partial order on X with 1 as the largest element.

Definition 2.3. [24] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called a *subalgebra* of X if $x \cdot y \in D$ for all $x, y \in D$.

Definition 2.4. [7] A nonempty subset D of a Hilbert algebra $X = (X, \cdot, 1)$ is called an *ideal* of X if the following conditions hold:

(1) $1 \in D$,

(2) $(\forall x, y \in X)(y \in D \Rightarrow x \cdot y \in D)$,

(3) $(\forall x, y_1, y_2 \in X)(y_1, y_2 \in D \Rightarrow (y_1 \cdot (y_2 \cdot x)) \cdot x \in D)$.

A *fuzzy set* [23] in a nonempty set X is defined to be a function $\mu : X \to [0, 1]$, where $[0, 1]$ is the unit closed interval of real numbers.

Definition 2.5. [19] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X)(\mu(x \cdot y) \geq \min\{\mu(x), \mu(y)\}).$$

Definition 2.6. [11] A fuzzy set μ in a Hilbert algebra $X = (X, \cdot, 1)$ is said to be a *fuzzy ideal* of X if the following conditions hold:

(1) $(\forall x \in X)(\mu(1) \geq \mu(x))$,

(2) $(\forall x, y \in X)(\mu(x \cdot y) \geq \mu(y))$,

(3) $(\forall x, y_1, y_2 \in X)(\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\mu(y_1), \mu(y_2)\})$.

Definition 2.7. [2] An *intuitionistic fuzzy set* on a nonempty set X is defined to be a structure

$$A := \{(x, \mu(x), \gamma(x)) \mid x \in X\}, \tag{2.1}$$

where $\mu : X \to [0,1]$ is a membership function and $\gamma : X \to [0,1]$ is a non-membership membership function. The intuitionistic fuzzy set in (2.1) is simply denoted by $A = (\mu, \gamma)$.

Definition 2.8. [18] We denote the family of all functions from a nonempty set X to the closed interval $[-1, 0]$ of the real line by $\mathscr{F}(X, [-1, 0])$. An element of $\mathscr{F}(X, [-1, 0])$ is called a *negative-valued function* from X to $[-1, 0]$ (briefly, $\mathscr{N}$-function on X). An ordered pair of a nonempty set X and an $\mathscr{N}$-function on X is called an *$\mathscr{N}$-fuzzy structure*. An intuitionistic $\mathscr{N}$-fuzzy structure over a nonempty set X is defined to be the structure (X, μ, γ), where μ and γ are $\mathscr{N}$-functions on X which are called the negative membership function and the negative non-membership function on X, respectively.

For the sake of simplicity, we will use the notation X_n instead of the intuitionistic $\mathscr{N}$-fuzzy structure (X, μ, γ) [17].

Definition 2.9. [20] Let X_n be an intuitionistic $\mathscr{N}$-fuzzy structure over a nonempty set X. The intuitionistic $\mathscr{N}$-fuzzy structure $\overline{X_n} = (X, \overline{\gamma}, \overline{\mu})$ defined by

$$(\forall x, \in X) \begin{pmatrix} \overline{\gamma}(x) = -1 - \gamma(x) \\ \overline{\mu}(x) = -1 - \mu(x) \end{pmatrix} \tag{2.2}$$

is called the *complement* of X_n in X.

3. Intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals

In this section, we introduce the notions of intuitionistic $\mathscr{N}$-fuzzy subalgebras and intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras and provide some interesting properties.

In what follows, let X denote a Hilbert algebra $(X, \cdot, 1)$ unless otherwise specified.

Definition 3.1. An intuitionistic $\mathscr{N}$-fuzzy structure X_n over X is called an *intuitionistic $\mathscr{N}$-fuzzy subalgebra* of X if the following condition holds:

$$(\forall x, y \in X) \begin{pmatrix} \mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\} \\ \gamma(x \cdot y) \geq \min\{\gamma(x), \gamma(y)\} \end{pmatrix} \tag{3.1}$$

Example 3.2. Let $X = \{1, x, y, z, 0\}$ with the following Cayley table:

$\cdot$	1	x	y	z	0
1	1	x	y	z	0
x	1	1	y	z	0
y	1	x	1	z	z
z	1	1	y	1	y
0	1	1	1	1	1

Then X is a Hilbert algebra. We define an intuitionistic $\mathcal{N}$-fuzzy structure X_n over X as follows:

X	1	x	y	z	0
μ	-1	-0.8	-0.8	-0.7	-0.4
γ	-0.3	-0.5	-0.7	-0.3	-0.6

Hence, X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.

Proposition 3.3. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X, then*

$$(\forall x \in X)\begin{pmatrix} \mu(1) \leq \mu(x) \\ \gamma(1) \geq \gamma(x) \end{pmatrix}. \tag{3.2}$$

Proof. For any $x \in X$, we have

$$\mu(1) = \mu(x \cdot x) \leq \max\{\mu(x), \mu(x)\} = \mu(x),$$

$$\gamma(1) = \gamma(x \cdot x) \geq \min\{\gamma(x), \gamma(x)\} = \gamma(x).$$

□

Definition 3.4. An intuitionistic $\mathcal{N}$-fuzzy structure X_n over X is called an *intuitionistic $\mathcal{N}$-fuzzy ideal* of X if (3.2) and the following conditions hold:

$$(\forall x, y \in X)\begin{pmatrix} \mu(x \cdot y) \leq \mu(y) \\ \gamma(x \cdot y) \geq \gamma(y) \end{pmatrix} \tag{3.3}$$

$$(\forall x, y_1, y_2 \in X)\begin{pmatrix} \mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\mu(y_1), \mu(y_2)\} \\ \gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\gamma(y_1), \gamma(y_2)\} \end{pmatrix} \tag{3.4}$$

Example 3.5. From Example 3.2, we define an intuitionistic $\mathcal{N}$-fuzzy structure X_n over X as follows:

X	1	x	y	z	0
μ	-1	-0.8	-0.8	-0.7	-0.4
γ	-0.3	-0.5	-0.7	-0.3	-0.6

Hence, X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X.

Proposition 3.6. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X, then*

$$(\forall x, y \in X)\left(\begin{array}{l}\mu((y\cdot x)\cdot x) \leq \mu(y)\\ \gamma((y\cdot x)\cdot x) \geq \gamma(y)\end{array}\right). \tag{3.5}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (3.4), we have

$$\mu((y\cdot x)\cdot x) \leq \max\{\mu(y), \mu(1)\} = \mu(y),$$

$$\gamma((y\cdot x)\cdot x) \geq \min\{\gamma(y), \gamma(1)\} = \gamma(y).$$

□

Lemma 3.7. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X, then*

$$(\forall x, y \in X)\left(x \leq y \Rightarrow \left\{\begin{array}{l}\mu(x) \geq \mu(y)\\ \gamma(x) \leq \gamma(y)\end{array}\right.\right). \tag{3.6}$$

Proof. Let $x, y \in X$ be such that $x \leq y$. Then $x \cdot y = 1$ and so

$$\begin{aligned}\mu(y) &= \mu(1\cdot y)\\ &= \mu(((x\cdot y)\cdot(x\cdot y))\cdot y)\\ &\leq \max\{\mu(x\cdot y), \mu(x)\}\\ &= \max\{\mu(1), \mu(x)\}\\ &= \mu(x),\end{aligned}$$

$$\begin{aligned}\gamma(y) &= \gamma(1\cdot y)\\ &= \gamma(((x\cdot y)\cdot(x\cdot y))\cdot y)\\ &\geq \min\{\gamma(x\cdot y), \gamma(x)\}\\ &= \min\{\gamma(1), \gamma(x)\}\\ &= \gamma(x).\end{aligned}$$

□

Theorem 3.8. *Every intuitionistic $\mathcal{N}$-fuzzy ideal of X is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.*

Proof. Let X_n be an intuitionistic $\mathcal{N}$-fuzzy ideal of X. Let $x, y \in X$. It follows from (3.3) that

$$\mu(x \cdot y) \leq \mu(y) \leq \max\{\mu(x), \mu(y)\},$$

$$\gamma(x \cdot y) \geq \gamma(y) \geq \min\{\gamma(x), \gamma(y)\}.$$

Hence, X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. □

Definition 3.9. Let $\{X_n^i \mid i \in \Delta\}$ be a family of intuitionistic $\mathcal{N}$-fuzzy structures over a nonempty set X. We define the intuitionistic $\mathcal{N}$-fuzzy structure $\bigwedge_{i\in\Delta} H_n^i = (X, \bigvee_{i\in\Delta} \mu_i, \bigwedge_{i\in\Delta} \gamma_i)$ by $(\bigvee_{i\in\Delta} \mu_i)(x) = \sup_{i\in\Delta}\{\mu_i(x)\}$ and $(\bigwedge_{i\in\Delta} \gamma_i)(x) = \inf_{i\in\Delta}\{\gamma_i(x)\}$ for all $x \in X$.

Proposition 3.10. *If $\{X_n^i \mid i \in \Delta\}$ is a family of intuitionistic $\mathcal{N}$-fuzzy ideals of X, then $\bigwedge_{i\in\Delta} X_n^i$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X.*

Proof. Let $\{X_n^i \mid i \in \Delta\}$ be a family of intuitionistic $\mathcal{N}$-fuzzy ideals of X. Let $x \in X$. Then

$$(\bigvee_{i\in\Delta} \mu_i)(1) = \sup_{i\in\Delta}\{\mu_i(1)\} \leq \sup_{i\in\Delta}\{\mu_i(x)\} = (\bigvee_{i\in\Delta} \mu_i)(x),$$

$$(\bigwedge_{i\in\Delta} \gamma_i)(1) = \inf_{i\in\Delta}\{\gamma_i(1)\} \geq \inf_{i\in\Delta}\{\gamma_i(x)\} = (\bigwedge_{i\in\Delta} \gamma_i)(x).$$

Let $x, y \in X$. Then

$$(\bigvee_{i\in\Delta} \mu_i)(x \cdot y) = \sup_{i\in\Delta}\{\mu_i(x \cdot y)\} \leq \sup_{i\in\Delta}\{\mu_i(y)\} = (\bigvee_{i\in\Delta} \mu_i)(y),$$

$$(\bigwedge_{i\in\Delta} \gamma_i)(x \cdot y) = \inf_{i\in\Delta}\{\gamma_i(x \cdot y)\} \geq \inf_{i\in\Delta}\{\gamma_i(y)\} = (\bigwedge_{i\in\Delta} \gamma_i)(y).$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}
(\bigvee_{i\in\Delta} \mu_i)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \sup_{i\in\Delta}\{\mu_i((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \\
&\leq \sup_{i\in\Delta}\{\min\{\mu_i(y_1), \mu_i(y_2)\}\} \\
&\leq \max\{\sup_{i\in\Delta} \mu_i(y_1), \sup_{i\in\Delta} \mu_i(y_2)\} \\
&= \max\{(\bigvee_{i\in\Delta} \mu_i)(y_1), (\bigvee_{i\in\Delta} \mu_i)(y_2)\},
\end{aligned}$$

$$\begin{aligned}
(\bigwedge_{i\in\Delta} \gamma_i)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \inf_{i\in\Delta}\{\gamma_i((y_1 \cdot (y_2 \cdot x)) \cdot x)\} \\
&\geq \inf_{i\in\Delta}\{\max\{\gamma_i(y_1), \gamma_i(y_2)\}\} \\
&\geq \min\{\inf_{i\in\Delta} \gamma_i(y_1), \inf_{i\in\Delta} \gamma_i(y_2)\} \\
&= \min\{(\bigwedge_{i\in\Delta} \gamma_i)(y_1), (\bigwedge_{i\in\Delta} \gamma_i)(y_2)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} H_n^i$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. □

Proposition 3.11. *If $\{X_n^i \mid i \in \Delta\}$ is a family of intuitionistic $\mathcal{N}$-fuzzy subalgebras of X, then $\bigwedge_{i\in\Delta} X_n^i$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.*

Proof. Let $\{X_n^i \mid i \in \Delta\}$ be a family of intuitionistic $\mathcal{N}$-fuzzy subalgebras of X. Let $x, y \in X$. Then

$$\begin{aligned}
(\bigvee_{i\in\Delta} \mu_i)(x \cdot y) &= \sup_{i\in\Delta}\{\mu_i(x \cdot y)\} \\
&\leq \sup_{i\in\Delta}\{\min\{\mu_i(x), \mu_i(y)\}\} \\
&\leq \max\{\sup_{i\in\Delta} \mu_i(x), \sup_{i\in\Delta} \mu_i(y)\} \\
&= \max\{(\bigvee_{i\in\Delta} \mu_i)(x), (\bigvee_{i\in\Delta} \mu_i)(y)\},
\end{aligned}$$

$$\begin{aligned}
(\bigwedge_{i\in\Delta} \gamma_i)(x \cdot y) &= \inf_{i\in\Delta}\{\gamma_i(x \cdot y)\} \\
&\geq \inf_{i\in\Delta}\{\max\{\gamma_i(x), \gamma_i(y)\}\} \\
&\geq \min\{\inf_{i\in\Delta} \gamma_i(x), \inf_{i\in\Delta} \gamma_i(y)\} \\
&= \min\{(\bigwedge_{i\in\Delta} \gamma_i)(x), (\bigwedge_{i\in\Delta} \gamma_i)(y)\}.
\end{aligned}$$

Hence, $\bigwedge_{i\in\Delta} H_n^i$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. □

Definition 3.12. Let X_n be an intuitionistic $\mathcal{N}$-fuzzy structure over a nonempty set X. The intuitionistic $\mathcal{N}$-fuzzy structures $\oplus X_n$ and $\otimes X_n$ are defined as $\oplus X_n = (X, \mu, \overline{\mu})$ and $\otimes X_n = (X, \overline{\gamma}, \gamma)$.

Theorem 3.13. *An intuitionistic $\mathcal{N}$-fuzzy structure X_n over X is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X if and only if $\oplus X_n$ and $\otimes X_n$ are intuitionistic intuitionistic $\mathcal{N}$-fuzzy subalgebras of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. Let $x, y \in X$. Then

$$\begin{aligned}
\overline{\mu}(x \cdot y) &= -1 - \mu(x \cdot y) \\
&\geq -1 - \max\{\mu(x), \mu(y)\} \\
&= \min\{-1 - \mu(x), -1 - \mu(y)\} \\
&= \min\{\overline{\mu}(x), \overline{\mu}(y)\}.
\end{aligned}$$

Hence, $\oplus X_n$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.

Let $x, y \in X$. Then

$$\begin{aligned}\overline{\gamma}(x \cdot y) &= -1 - \gamma(x \cdot y)\\ &\leq -1 - \min\{\gamma(x), \gamma(y)\}\\ &= \max\{-1 - \gamma(x), -1 - \gamma(y)\}\\ &= \max\{\overline{\gamma}(x), \overline{\gamma}(y)\}.\end{aligned}$$

Hence, $\otimes X_n$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.

The converse of the theorem is true immediately in the order of μ and γ in $\oplus X_n$ and $\otimes X_n$, respectively. □

Theorem 3.14. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X, then the sets $X_\mu := \{x \in X \mid \mu(x) = \mu(1)\}$ and $X_\gamma := \{x \in X \mid \gamma(x) = \gamma(1)\}$ are subalgebras of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. Let $x, y \in X_\mu$. Then $\mu(x) = \mu(1) = \mu(y)$, so $\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\} = \mu(1)$. By (3.2), we have $\mu(x \cdot y) = \mu(1)$, that is, $x \cdot y \in X_\mu$. Hence, X_μ is a subalgebra of X. Again, let $x, y \in X_\gamma$. Then $\gamma(x) = \gamma(1) = \gamma(y)$, so $\gamma(x \cdot y) \geq \min\{\gamma(x), \gamma(y)\} = \gamma(1)$. Again, by (3.2), we have $\gamma(x \cdot y) = \gamma(1)$, that is, $x \cdot y \in X_\gamma$. Hence, X_γ is a subalgebra of X. □

By proving Theorem 3.13, we get the following corollary.

Corollary 3.15. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X, then $\overline{X_n}$ is also an intuitionistic $\mathcal{N}$-fuzzy ideal of X.*

Theorem 3.16. *An intuitionistic $\mathcal{N}$-fuzzy structure X_n over X is an intuitionistic $\mathcal{N}$-fuzzy ideal of X if and only if $\oplus X_n$ and $\otimes X_n$ are intuitionistic intuitionistic $\mathcal{N}$-fuzzy ideals of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. Let $x \in X$. Then $\overline{\mu}(1) = -1 - \mu(1) \geq -1 - \mu(x) \geq \overline{\mu}(x)$. Let $x, y \in X$. Then $\overline{\mu}(x \cdot y) = -1 - \mu(x \cdot y) \geq -1 - \mu(y) \geq \overline{\mu}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}\overline{\mu}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= -1 - \mu((y_1 \cdot (y_2 \cdot x)) \cdot x)\\ &\geq -1 - \max\{\mu(y_1), \mu(y_2)\}\\ &= \min\{-1 - \mu(y_1), -1 - \mu(y_2)\}\\ &= \min\{\overline{\mu}(y_1), \overline{\mu}(y_2)\}.\end{aligned}$$

Hence, $\oplus X_n$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X.

Let $x \in X$. Then $\overline{\gamma}(1) = -1 - \gamma(1) \leq -1 - \gamma(x) \leq \overline{\gamma}(x)$. Let $x, y \in X$. Then $\overline{\gamma}(x \cdot y) = -1 - \gamma(x \cdot y) \leq -1 - \gamma(y) \leq \overline{\gamma}(y)$. Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned} \overline{\gamma}((y_1 \cdot (y_2 \cdot x)) \cdot x) &= -1 - \gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) \\ &\leq -1 - \min\{\gamma(y_1), \gamma(y_2)\} \\ &= \max\{-1 - \gamma(y_1), -1 - \gamma(y_2)\} \\ &= \max\{\overline{\gamma}(y_1), \overline{\gamma}(y_2)\}. \end{aligned}$$

Hence, $\otimes X_n$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X.

The converse of the theorem is true immediately in the order of μ and γ in $\oplus X_n$ and $\otimes X_n$, respectively. □

Theorem 3.17. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X, then the sets X_μ and X_γ are ideals of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. Clearly, $1 \in X_\mu \cap X_\gamma$. Let $x, y \in X$ be such that $y \in X_\mu$. Then $\mu(y) = \mu(1)$. By (3.3), we have $\mu(x \cdot y) \leq \mu(y) = \mu(1)$, whence $\mu(x \cdot y) = \mu(1)$, by (3.2). This means that $x \cdot y \in X_\mu$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in X_\mu$. Then $\mu(y_1) = \mu(1)$ and $\mu(y_2) = \mu(1)$. By (3.4), we have $\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\mu(y_1), \mu(y_2)\} = \mu(1)$, whence $\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) = \mu(1)$, by (3.2). This means that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in X_\mu$. Hence, X_μ is an ideal of X.

Let $x, y \in X$ be such that $y \in X_\gamma$. Then $\gamma(y) = \gamma(1)$. By (3.3), we have $\gamma(x \cdot y) \geq \gamma(y) = \gamma(1)$, whence $\gamma(x \cdot y) = \gamma(1)$, by (3.2). This means that $x \cdot y \in X_\gamma$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in X_\gamma$. Then $\gamma(y_1) = \gamma(1)$ and $\gamma(y_2) = \gamma(1)$. By (3.4), we have $\gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\gamma(y_1), \gamma(y_2)\} = \gamma(1)$, whence $\gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) = \gamma(1)$, by (3.2). This means that $(y_1 \cdot (y_2 \cdot x)) \cdot x \in X_\gamma$. Hence, X_γ is an ideal of X. □

By proving Theorem 3.16, we get the following corollary.

Corollary 3.18. *If X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X, then $\overline{X_n}$ is also an intuitionistic $\mathcal{N}$-fuzzy ideal of X.*

Definition 3.19. Let $f \in \mathscr{F}(X, [-1, 0])$. For any $t \in [-1, 0]$, the sets $U(f : t) = \{x \in X \mid f(x) \geq t\}$ is called an *upper t-level subset* of f, $L(f : t) = \{x \in X \mid f(x) \leq t\}$ is called a *lower t-level subset* of f, and $E(f : t) = \{x \in X \mid f(x) = t\}$ is called an *equal t-level subset* of f.

Theorem 3.20. *An intuitionistic $\mathcal{N}$-fuzzy structure X_n over X is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X if and only if for all $a, b \in [-1, 0]$, the sets $L(\mu : a)$ and $U(\gamma : b)$ are either empty or subalgebras of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. Let $a, b \in [-1, 0]$ be such that $L(\mu : a)$ and $U(\gamma : b)$ are nonempty. Let $x, y \in L(\mu : a)$. Then $\mu(x) \leq a$ and $\mu(y) \leq a$, so a is an upper bound of $\{\mu(x), \mu(y)\}$. By (3.1), we have $\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\} \leq a$. Thus $x \cdot y \in L(\mu : a)$. Let $x, y \in U(\gamma : b)$. Then $\gamma(x) \geq b$ and $\gamma(y) \geq b$, so b is a lower bound of $\{\gamma(x), \gamma(y)\}$. By (3.1), we have $\gamma(x \cdot y) \geq \min\{\gamma(x), \gamma(y)\} \geq b$. Thus $x \cdot y \in U(\gamma : b)$. Hence, $L(\mu : a)$ and $U(\gamma : b)$ are subalgebras of X.

Conversely, assume that for all $a, b \in [-1, 0]$, the sets $L(\mu : a)$ and $U(\gamma : b)$ are either empty or subalgebras of X. Let $x, y \in X$. Then $\mu(x) \leq \max\{\mu(x), \mu(y)\}$ and $\mu(y) \leq \max\{\mu(x), \mu(y)\}$. Thus $x, y \in L(\mu : \max\{\mu(x), \mu(y)\}) \neq \emptyset$. By assumption, we have $L(\mu : \max\{\mu(x), \mu(y)\})$ is a subalgebra of X. Then $x \cdot y \in L(\mu : \max\{\mu(x), \mu(y)\})$. Thus $\mu(x \cdot y) \leq \max\{\mu(x), \mu(y)\}$. Let $x, y \in X$. Then $\gamma(x) \geq \min\{\gamma(x), \gamma(y)\}$ and $\gamma(y) \geq \min\{\gamma(x), \gamma(y)\}$. Thus $x, y \in U(\gamma : \min\{\gamma(x), \gamma(y)\}) \neq \emptyset$. By assumption, we have $U(\gamma : \min\{\gamma(x), \gamma(y)\})$ is a subalgebra of X. Then $x \cdot y \in U(\gamma : \min\{\gamma(x), \gamma(y)\}$. Thus $\gamma(x \cdot y) \geq \max\{\gamma(x), \gamma(y)\}$. Hence, X_n is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. □

Theorem 3.21. *An intuitionistic $\mathcal{N}$-fuzzy structure X_n over X is an intuitionistic $\mathcal{N}$-fuzzy ideal of X if and only if for all $a, b \in [-1, 0]$, the sets $L(\mu : a)$ and $U(\gamma : b)$ are either empty or ideals of X.*

Proof. Assume that X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. Let $a, b \in [-1, 0]$ be such that $L(\mu : a)$ and $U(\gamma : b)$ are nonempty. Let $x \in L(\mu : a)$ and $y \in U(\gamma : b)$. By (3.2), we have $\mu(1) \leq \mu(x) \leq a$ and $\gamma(1) \geq \gamma(x) \geq b$. Thus $1 \in L(\mu : a) \cap U(\gamma : b)$. Let $x, y \in X$ be such that $y \in L(\mu : a)$. Then $\mu(y) \leq a$. By (3.3), we have $\mu(x \cdot y) \leq \mu(y) \leq a$. Thus $x \cdot y \in L(\mu : a)$. Let $x, y \in X$ be such that $y \in U(\gamma : b)$. Then $\gamma(y) \geq b$. By (3.3), we have $\gamma(x \cdot y) \geq \gamma(y) \geq b$. Thus $x \cdot y \in U(\gamma : b)$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in L(\mu : a)$. Then $\mu(y_1) \leq a$ and $\mu(y_2) \leq a$, so a is an upper bound of $\{\mu(y_1), \mu(y_2)\}$. By (3.4), we have $\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq \max\{\mu(y_1), \mu(y_2)\} \leq a$. Thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\mu : a)$. Let $x, y_1, y_2 \in X$ be such that $y_1, y_2 \in U(\gamma : b)$. Then $\gamma(y_1) \geq b$ and $\gamma(y_2) \geq b$, so b is a lower bound of $\{\gamma(y_1), \gamma(y_2)\}$. By (3.4), we have $\gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq \min\{\gamma(y_1), \gamma(y_2)\} \geq b$. Thus $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\gamma : b)$. Hence, $L(\mu : a)$ and $U(\gamma : b)$ are ideals of X.

Conversely, assume that for all $a, b \in [-1, 0]$, the sets $L(\mu : a)$ and $U(\gamma : b)$ are either empty or ideals of X. Let $x \in X$. Then $\mu(x) \in [-1, 0]$. Choose $a = \mu(x)$. Then $\mu(x) \leq a$, so $x \in L(\mu : a) \neq \emptyset$. By assumption, we have $L(\mu : a)$ is an ideal of X and so $1 \in L(\mu : a)$. Thus $\mu(1) \leq a = \mu(x)$. Let $x \in X$. Then $\gamma(x) \in [-1, 0]$. Choose $b = \gamma(x)$. Then $\gamma(x) \geq b$, so $x \in U(\gamma : b) \neq \emptyset$. By assumption, we have $U(\gamma : b)$ is an ideal of X and so $1 \in U(\gamma : b)$.

Thus $\gamma(1) \geq b = \gamma(x)$. Let $x, y \in X$. Then $\mu(y) \in [-1, 0]$. Choose $a = \mu(y)$. Then $\mu(y) \leq a$, so $y \in L(\mu : a) \neq \emptyset$. By assumption, we have $L(\mu : a)$ is an ideal of X and so $x \cdot y \in L(\mu : a)$. Thus $\mu(x \cdot y) \leq a = \mu(y)$. Let $x, y \in X$. Then $\gamma(y) \in [-1, 0]$. Choose $b = \gamma(y)$. Then $\gamma(y) \geq b$, so $y \in U(\gamma : b) \neq \emptyset$. By assumption, we have $U(\gamma : b)$ is an ideal of X and so $x \cdot y \in U(\gamma : b)$. Thus $\gamma(x \cdot y) \geq b = \gamma(y)$. Let $x, y_1, y_2 \in X$. Then $\mu(y_1), \mu(y_2) \in [-1, 0]$. Choose $a = \max\{\mu(y_1), \mu(y_2)\}$. Thus $\mu(y_1) \leq a$ and $\mu(y_2) \leq a$, so $y_1, y_2 \in L(\mu : a) \neq \emptyset$. By assumption, we have $L(\mu : a)$ is an ideal of X and so $(y_1 \cdot (y_2 \cdot x)) \cdot x \in L(\mu : a)$. Thus $\mu((y_1 \cdot (y_2 \cdot x)) \cdot x) \leq a = \max\{\mu(y_1), \mu(y_2)\}$. Let $x, y_1, y_2 \in X$. Then $\gamma(y_1), \gamma(y_2) \in [-1, 0]$. Choose $b = \min\{\gamma(y_1), \gamma(y_2)\}$. Thus $\gamma(y_1) \geq b$ and $\gamma(y_2) \geq b$, so $y_1, y_2 \in U(\gamma : b) \neq \emptyset$. By assumption, we have $U(\gamma : b)$ is an ideal of X and so $(y_1 \cdot (y_2 \cdot x)) \cdot x \in U(\gamma : b)$. Thus $\gamma((y_1 \cdot (y_2 \cdot x)) \cdot x) \geq b = \min\{\gamma(y_1), \gamma(y_2)\}$. Hence, X_n is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. □

Definition 3.22. Let $X_n = (X, \mu_X, \gamma_X)$ and $Y_n = (Y, \mu_Y, \gamma_Y)$ be intuitionistic $\mathcal{N}$-fuzzy structures over nonempty sets X and Y, respectively. The Cartesian product $X_n \times Y_n = (X \times Y, \Phi, \Upsilon)$ defined by $\Phi(x, y) = \max\{\mu_X(x), \mu_Y(y)\}$ and $\Upsilon(x, y) = \min\{\gamma_X(x), \gamma_Y(y)\}$, where $\Phi : X \times Y \to [-1, 0]$ and $\Upsilon : X \times Y \to [-1, 0]$ for all $x \in X$ and $y \in Y$.

Remark 3.23. Let $(X, \cdot, 1_X)$ and $(Y, \star, 1_Y)$ be Hilbert algebras. Then $(X \times Y, \diamond, (1_X, 1_Y))$ is a Hilbert algebra defined by $(x, y) \diamond (u, v) = (x \cdot u, y \star v)$ for every $x, u \in X$ and $y, v \in Y$.

Proposition 3.24. *If $X_n = (X, \mu_X, \gamma_X)$ and $Y_n = (Y, \mu_Y, \gamma_Y)$ are intuitionistic $\mathcal{N}$-fuzzy subalgebras of Hilbert algebras X and Y, respectively, then the Cartesian product $X_n \times Y_n$ is also an intuitionistic $\mathcal{N}$-fuzzy subalgebra of $X \times Y$.*

Proof. Assume that $X_n = (X, \mu_X, \gamma_X)$ and $Y_n = (Y, \mu_Y, \gamma_Y)$ are intuitionistic $\mathcal{N}$-fuzzy subalgebras of Hilbert algebras X and Y, respectively. Let $(x_1, y_1), (x_2, y_2) \in X \times Y$. Then

$$\begin{aligned}
& \Phi((x_1, y_1) \diamond (x_2, y_2)) \\
= \; & \Phi((x_1 \cdot x_2), (y_1 \star y_2)) \\
= \; & \max\{\mu_X(x_1 \cdot x_2), \mu_Y(y_1 \star y_2)\} \\
\leq \; & \max\{\max\{\mu_X(x_1), \mu_X(x_2)\}, \max\{\mu_Y(y_1), \mu_Y(y_2)\}\} \\
= \; & \max\{\max\{\mu_X(x_1), \mu_Y(y_1)\}, \max\{\mu_X(x_2), \mu_Y(y_2)\}\} \\
= \; & \max\{\Phi(x_1, y_1), \Phi(x_2, y_2)\},
\end{aligned}$$

$$\begin{aligned} & \Upsilon((x_1,y_1)\diamond(x_2,y_2)) \\ = & \Upsilon((x_1\cdot x_2),(y_1\star y_2)) \\ = & \min\{\gamma_X(x_1\cdot x_2),\gamma_Y(y_1\star y_2)\} \\ \geq & \min\{\min\{\gamma_X(x_1),\gamma_X(x_2)\},\min\{\gamma_Y(y_1),\gamma_Y(y_2)\}\} \\ = & \min\{\min\{\gamma_X(x_1),\gamma_Y(y_1)\},\min\{\gamma_X(x_2),\gamma_Y(y_2)\}\} \\ = & \min\{\Upsilon(x_1,y_1),\Upsilon(x_2,y_2)\}. \end{aligned}$$

Hence, $X_n \times Y_n$ is an intuitionistic $\mathscr{N}$-fuzzy subalgebra of $X \times Y$. □

Theorem 3.25. *If $X_n = (X,\mu_X,\gamma_X)$ and $Y_n = (Y,\mu_Y,\gamma_Y)$ are intuitionistic $\mathscr{N}$-fuzzy subalgebras of Hilbert algebras X and Y, respectively, then $\oplus(X_n \times Y_n)$ is an intuitionistic $\mathscr{N}$-fuzzy subalgebra of $X \times Y$.*

Proof. It follows from Theorem 3.13 and Proposition 3.24. □

Proposition 3.26. *If $X_n = (X,\mu_X,\gamma_X)$ and $Y_n = (Y,\mu_Y,\gamma_Y)$ are intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras X and Y, respectively, then the Cartesian product $X_n \times Y_n$ is also an intuitionistic $\mathscr{N}$-fuzzy ideal of $X \times Y$.*

Proof. Assume that $X_n = (X,\mu_X,\gamma_X)$ and $Y_n = (Y,\mu_Y,\gamma_Y)$ are intuitionistic $\mathscr{N}$-fuzzy ideals of Hilbert algebras X and Y, respectively. Let $(x,y) \in X \times Y$. Then

$$\begin{aligned} \Phi(1_X,1_Y) &= \max\{\mu_X(1_X),\mu_Y(1_Y)\} \\ &\leq \max\{\mu_X(x),\mu_Y(y)\} \\ &= \Phi(x,y), \end{aligned}$$

$$\begin{aligned} \Upsilon(1_X,1_Y) &= \min\{\gamma_X(1_X),\gamma_Y(1_Y)\} \\ &\leq \min\{\gamma_X(x),\gamma_Y(y)\}\} \\ &= \Upsilon(x,y). \end{aligned}$$

Let $(x_1,x_2),(y_1,y_2) \in X \times Y$. Then

$$\begin{aligned} \Phi((x_1,x_2)\diamond(y_1,y_2)) &= \Phi((x_1\cdot y_1),(x_2\star y_2)) \\ &= \max\{\mu_X(x_1\cdot y_1),\mu_Y(x_2\star y_2)\} \\ &\leq \max\{\mu_X(y_1),\mu_Y(y_2)\} \\ &= \Phi(y_1,y_2), \end{aligned}$$

$$
\begin{aligned}
\Upsilon((x_1,x_2)\diamond(y_1,y_2)) &= \Upsilon((x_1\cdot y_1),(x_2\star y_2))\\
&= \min\{\gamma_X(x_1\cdot y_1),\gamma_Y(x_2\star y_2)\}\\
&\geq \min\{\gamma_X(y_1),\gamma_Y(y_2)\}\\
&= \Upsilon(y_1,y_2).
\end{aligned}
$$

Let $(x_1,y_1),(x_2,y_2),(x_3,y_3)\in X\times Y$. Then

$$
\begin{aligned}
&\Phi(((x_2,y_2)\diamond((x_3,y_3)\diamond(x_1,y_1)))\diamond(x_1,y_1))\\
= &\Phi((x_2\cdot(x_3\cdot x_1))\cdot x_1,(y_2\star(y_3\star y_1))\star y_1)\\
= &\max\{\mu_X((x_2\cdot(x_3\cdot x_1))\cdot x_1),\mu_Y((y_2\star(y_3\star y_1))\star y_1)\}\\
\leq &\max\{\max\{\mu_X(x_2),\mu_X(x_3)\},\max\{\mu_Y(y_2),\mu_Y(y_3)\}\}\\
= &\max\{\max\{\mu_X(x_2),\mu_Y(y_2)\},\max\{\mu_X(x_3),\mu_Y(y_3)\}\}\\
= &\max\{\Phi(x_2,y_2),\Phi(x_3,y_3)\},
\end{aligned}
$$

$$
\begin{aligned}
&\Upsilon(((x_2,y_2)\diamond((x_3,y_3)\diamond(x_1,y_1)))\diamond(x_1,y_1))\\
= &\Upsilon((x_2\cdot(x_3\cdot x_1))\cdot x_1,(y_2\star(y_3\star y_1))\star y_1)\\
= &\min\{\gamma_X((x_2\cdot(x_3\cdot x_1))\cdot x_1),\gamma_Y((y_2\cdot(y_3\cdot y_1))\cdot y_1)\}\\
\geq &\min\{\min\{\gamma_X(x_2),\gamma_X(x_3)\},\min\{\gamma_Y(y_2),\gamma_Y(y_3)\}\}\\
= &\min\{\min\{\gamma_X(x_2),\gamma_Y(y_2)\},\min\{\gamma_X(x_3),\gamma_Y(y_3)\}\}\\
= &\min\{\Upsilon(x_2,y_2),\Upsilon(x_3,y_3)\}.
\end{aligned}
$$

Hence, $X_n\times Y_n$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of $X\times Y$. □

Theorem 3.27. *If $X_n=(X,\mu_X,\gamma_X)$ and $Y_n=(Y,\mu_Y,\gamma_Y)$ are intuitionistic $\mathcal{N}$-fuzzy ideals of Hilbert algebras X and Y, respectively, then $\oplus(X_n\times Y_n)$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of $X\times Y$.*

Proof. It follows from Theorem 3.16 and Proposition 3.26. □

A mapping $f:(X,\cdot,1_X)\to(Y,\star,1_Y)$ of Hilbert algebras is called a *homomorphism* if $f(x\cdot y)=f(x)\star f(y)$ for all $x,y\in X$. Note that if $f:X\to Y$ is a homomorphism of Hilbert algebras, then $f(1_X)=1_Y$.

Definition 3.28. Let f be a function from a nonempty set X to a nonempty set Y. If $Y_n=(Y,\mu,\gamma)$ is an intuitionistic $\mathcal{N}$-fuzzy structure over Y, then the intuitionistic $\mathcal{N}$-fuzzy structure $f^{-1}(Y_n)=(\mu\circ f,\gamma\circ f)$ over X is called the *pre-image of Y_n under f*.

Theorem 3.29. *Let $f:(X,\cdot,1_X)\to(Y,\star,1_Y)$ be a homomorphism of Hilbert algebras. If $Y_n=(Y,\mu,\gamma)$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of Y, then $f^{-1}(Y_n)$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X.*

Proof. Assume that $Y_n = (Y, \mu, \gamma)$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of Y. Let $x, y \in X$. Then

$$\begin{aligned}(\mu \circ f)(x \cdot y) &= \mu(f(x \cdot y)) \\ &= \mu(f(x) \star f(y)) \\ &\leq \max\{\mu(f(x)), \mu(f(y))\} \\ &= \max\{(\mu \circ f)(x), (\mu \circ f)(y)\},\end{aligned}$$

$$\begin{aligned}(\gamma \circ f)(x \cdot y) &= \gamma(f(x \cdot y)) \\ &= \gamma(f(x) \star f(y)) \\ &\geq \min\{\gamma(f(x)), \gamma(f(y))\} \\ &= \min\{(\gamma \circ f)(x), (\gamma \circ f)(y)\}.\end{aligned}$$

Hence, $f^{-1}(Y_n)$ is an intuitionistic $\mathcal{N}$-fuzzy subalgebra of X. □

Theorem 3.30. *Let $f : (X, \cdot, 1_X) \to (Y, \star, 1_Y)$ be a homomorphism of Hilbert algebras. If $Y_n = (Y, \mu, \gamma)$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of Y, then $f^{-1}(Y_n)$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X.*

Proof. Assume that $Y_n = (Y, \mu, \gamma)$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of Y. Since f is a homomorphism of X into Y, we have $f(1_X) = 1_Y$. Thus $(\mu \circ f)(1_X) = \mu(f(1_X)) = \mu(1_Y) \leq \mu(f(x)) = (\mu \circ f)(x)$ for $x \in X$. Also, $(\gamma \circ f)(1_X) = \gamma(f(1_X)) = \gamma(1_Y) \geq \gamma(f(x)) = (\gamma \circ f)(x)$ for every $x \in X$. Let $x, y \in X$. Then

$$(\mu \circ f)(x \cdot y) = \mu(f(x \cdot y)) = \mu(f(x) \star f(y)) \leq \mu(f(y)) = (\mu \circ f)(y),$$

$$(\gamma \circ f)(x \cdot y) = \gamma(f(x \cdot y)) = \gamma(f(x) \star f(y)) \geq \gamma(f(y)) = (\gamma \circ f)(y).$$

Let $x, y_1, y_2 \in X$. Then

$$\begin{aligned}(\mu \circ f)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \mu(f((y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &= \mu((f(y_1) \star (f(y_2) \star f(x))) \star f(x)) \\ &\leq \max\{\mu(f(y_1)), \mu(f(y_2))\} \\ &= \max\{(\mu \circ f)(y_1), (\mu \circ f)(y_2)\},\end{aligned}$$

$$\begin{aligned}(\gamma \circ f)((y_1 \cdot (y_2 \cdot x)) \cdot x) &= \gamma(f((y_1 \cdot (y_2 \cdot x)) \cdot x)) \\ &= \gamma((f(y_1) \star (f(y_2) \star f(x))) \star f(x)) \\ &\geq \min\{\gamma(f(y_1)), \gamma(f(y_2))\} \\ &= \min\{(\gamma \circ f)(y_1), (\gamma \circ f)(y_2)\}.\end{aligned}$$

Hence, $f^{-1}(Y_n)$ is an intuitionistic $\mathcal{N}$-fuzzy ideal of X. □

4. Conclusion

In this paper, we have introduced the notions of intuitionistic $\mathcal{N}$-fuzzy subalgebras and intuitionistic $\mathcal{N}$-fuzzy ideals of Hilbert algebras and investigated some of their important properties. We have given certain requirements for intuitionistic $\mathcal{N}$-fuzzy structures to be intuitionistic $\mathcal{N}$-fuzzy subalgebras and intuitionistic $\mathcal{N}$-fuzzy ideals of Hilbert algebras. The relationship between $\mathcal{N}$-fuzzy subalgebras (intuitionistic $\mathcal{N}$-fuzzy ideals) and their t-level subsets is also examined. The homomorphic pre-images of intuitionistic $\mathcal{N}$-fuzzy subalgebras (intuitionistic $\mathcal{N}$-fuzzy ideals) and other associated features are also examined in relation to Hilbert algebras.

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