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HARMONIC CLASSES ASSOCIATED WITH AN OPERATOR DEFINED BY POISSON DISTRIBUTION SERIES

SAJEDA HUSAM AL-KHALDI, BASEM AREF FRASIN*

Faculty of Science, Department of Mathematics, Al al-Bayt University, Mafraq, Jordan

*Corresponding author: bafrasin@yahoo.com

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Abstract. In the present paper, we obtain the inclusion relations of the harmonic class $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$ with the classes $\mathcal{K}_{\mathbb{H}}$ and $\mathcal{S}^{*}_{\mathbb{H}}$, $\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$ and $\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$ associated with the operator Ξ defined by by applying certain convolution operator involving Poisson distribution series. Several corollaries and consequences of the main results are also obtained.

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1. Introduction and definitions

Let $\mathbb{H}$ be the family of all harmonic functions of the form $f=\varpi+\overline{\varkappa}$, where

$$\varpi(\zeta)=\zeta+\sum_{j=2}^{\infty}c_j\zeta^j, \qquad \varkappa(\zeta)=\sum_{j=1}^{\infty}d_j\zeta^j, \quad |d_1|<1. \tag{1}$$

are analytic in the open unit disk $E=\{\zeta:|\zeta|<1\}$ for which $f(0)=f_{\zeta}(0)-1=0$.Furthermore, let $\mathcal{S}_{\mathbb{H}}$ denote the family of functions $f=\varpi+\overline{\varkappa}$ that are harmonic univalent and sense preserving in E.

Note that the family $\mathcal{S}_{\mathbb{H}}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions if the co-analytic part of its member is zero.

In 1984 Clunie and Sheil-Small [9] investigated the class $\mathcal{S}_{\mathbb{H}}$ as well as its geometric subclasses and obtained some coefficient bounds. For more results on harmonic functions one may refer to [4,11,12,18,19,24,25,32,33].

We also let the subclass $\mathcal{S}^0_{\mathbb{H}}$ of $\mathcal{S}_{\mathbb{H}}$ as

$$\mathcal{S}^0_{\mathbb{H}}=\left\{f=\varpi+\overline{\varkappa}\in\mathcal{S}_{\mathbb{H}}:\varkappa'(0)=d_1=0\right\}.$$

The classes $\mathcal{S}^0_{\mathbb{H}}$ and $\mathcal{S}_{\mathbb{H}}$ were first studied in [9].

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A sense-preserving harmonic mapping $f \in \mathcal{S}_{\mathbb{H}}^{0}$ is in the class $\mathcal{S}^{*}$ if the range $f(E)$ is starlike with respect to the origin. A function $f \in \mathcal{S}_{\mathbb{H}}^{*}$ is called a harmonic starlike mapping in E. Also a function f defined in E belongs to the class $\mathcal{K}_{\mathbb{H}}$ if $f \in \mathcal{S}_{\mathbb{H}}^{0}$ and if $f(E)$ is a convex domain. A function $f \in \mathcal{K}_{\mathbb{H}}$ is called harmonic convex in E. Analytically, we have

$$f \in \mathcal{S}_{\mathbb{H}}^{*} \text{ iff } arg\left(\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right) \geq 0,$$

and

$$f \in \mathcal{K}_{\mathbb{H}} \text{ iff } \frac{\partial}{\partial\theta}\left\{arg\left(arg\left(\frac{\partial}{\partial\theta} f\left(re^{i\theta}\right)\right)\right)\right\} \geq 0,$$
$$\zeta = re^{i\theta} \in E,\ 0 \leq \theta \leq 2\pi,\ 0 \leq r \leq 1.$$

For definitions and properties of these classes, one may refer to [4].

Let $\mathcal{T}_{\mathbb{H}}$ be the class of functions in $\mathcal{S}_{\mathbb{H}}$ that may be expressed as $f = \varpi + \overline{\varkappa}$, where

$$\varpi(\zeta) = \zeta - \sum_{j=2}^{\infty} |c_j|\,\zeta^j,\ \ \varkappa(\zeta) = \sum_{j=1}^{\infty} |d_j|\,\zeta^j, \quad |d_1| < 1. \tag{2}$$

For $0 \leq \sigma < 1$, let

$$\Psi_{\mathbb{H}}(\sigma) = \left\{f \in \mathbb{H} : \operatorname{Re}\left(\frac{f'(\zeta)}{\zeta'}\right) \geq \sigma,\ \zeta = re^{i\theta} \in E\right\},$$

and

$$\Phi_{\mathbb{H}}(\sigma) = \left\{f \in \mathbb{H} : \operatorname{Re}\left(\frac{f''(\zeta)}{\zeta''}\right) \geq \sigma,\ \zeta = re^{i\theta} \in E\right\}$$

where

$$\zeta' = \frac{\partial}{\partial\theta}\left(\zeta = re^{i\theta}\right), \zeta'' = \frac{\partial}{\partial\theta}\left(\zeta'\right), f'(\zeta) = \frac{\partial}{\partial\theta} f\left(re^{i\theta}\right), f'' = \frac{\partial}{\partial\theta}\left(f'(\zeta)\right).$$

Define

$$\mathcal{T}\Psi_{\mathbb{H}}(\sigma) = \Psi_{\mathbb{H}}(\sigma) \cap \mathcal{T}_{\mathbb{H}} \quad \text{and} \quad \mathcal{T}\Phi_{\mathbb{H}}(\sigma) = \Phi_{\mathbb{H}}(\sigma) \cap \mathcal{T}_{\mathbb{H}}.$$

The classes $\mathcal{T}_{\mathbb{H}}$, $\Psi_{\mathbb{H}}(\sigma)$, $\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$, $\Phi_{\mathbb{H}}(\sigma)$ and $\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$ were introduced and studied in [3] and [32].

Aini and Suzeini [5] introduced and studied the class $\mathcal{T}_{\mathbb{H}}(\varepsilon, \xi)$ of functions of the form (2) that satisfy the condition

$$\operatorname{Re}\left\{f'(\zeta) + \varepsilon\zeta f''(\zeta)\right\} > 1 - |\xi|,$$

for some $\varepsilon \geq 0$ and $\xi \in \mathbb{C}$. In particular for $\varepsilon = 0$, we get the class $\mathcal{T}_{\mathbb{H}}(\xi)$ which satisfy the condition

$$\operatorname{Re} f'(\zeta) > 1 - |\xi|.$$

It is well known that the special functions play an important role in geometric function theory, especially in the solution by de Branges of the famous Bieberbach conjecture. Recently several researchers have studied the geometric properties of analytic functions associating with hypergeometric functions (see [7,15,20,34]), Bessel functions (see [8,15]), Struve functions (see [16,35]), Poisson distribution series (see [6,21,24]) and Pascal distribution series (see [10,14]).

Very recently, Porwal [27](see also, [17,23]) introduced a Poisson distribution series as

$$K\left(t,\zeta\right)=\zeta+\sum_{j=2}^{\infty}\frac{t^{j-1}}{(j-1)!}e^{-t}\zeta^{j}.$$

Now, for $t_1, t_2 > 0$, Porwal and Srivastava [26] introduced the operator $\Xi(t_1,t_2)$ for $f(\zeta)\in\mathcal{S}_{\mathbb{H}}$ as

$$\Xi(f)=\Xi(t_1,t_2)f(\zeta)=K(t_1,\zeta)*\varpi(\zeta)+\overline{K(t_2,\zeta)*\varkappa(\zeta)}=\varpi(\zeta)+\overline{\varkappa(\zeta)}, \tag{3}$$

where

$$\varpi(\zeta)=\zeta+\sum_{j=2}^{\infty}\frac{t_1^{j-1}}{(j-1)!}e^{-t_1}c_j\zeta^j,\ \varkappa(\zeta)=d_1\zeta+\sum_{j=2}^{\infty}\frac{t_2^{j-1}}{(j-1)!}e^{-t_2}d_j\zeta^j. \tag{4}$$

for any function $f=\varpi+\overline{\varkappa}$ in $\mathbb{H}$.

Motivated the work of Porwal and Srivastava [26] (see also, [1], [13], [28]- [31], [36]), and by applying the convolution operator Ξ, we establish a number of connections between the classes $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$, $\mathcal{K}_{\mathbb{H}}$, $\Psi_{\mathbb{H}}(\sigma)$ and $\Phi_{\mathbb{H}}(\sigma)$.

2. Preliminary lemmas

To establish our main results, we need the following Lemmas.

Lemma 1. [5] *Let $f=\varpi+\overline{\varkappa}$ where ϖ and $\varkappa$ are given by* (2) *and suppose that $\varepsilon\geq 0$ and $\xi\in\mathbb{C}$. Then $f\in\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$ if*

$$\sum_{j=2}^{\infty}j\left(1-\varepsilon+j\varepsilon\right)|c_j|+\sum_{j=1}^{\infty}j\left(1-\varepsilon+j\varepsilon\right)|d_j|\leq|\xi|. \tag{5}$$

Moreover, if $f\in\mathbb{H}(\varepsilon,\xi)$, then

$$|c_j|\leq\frac{|\xi|}{j\left(1-\varepsilon+j\varepsilon\right)},\ j\geq 2, \tag{6}$$

and

$$|d_j|\leq\frac{|\xi|}{j\left(1-\varepsilon+j\varepsilon\right)},\ j\geq 1. \tag{7}$$

Lemma 2. [4] *Let $f=\varpi+\overline{\varkappa}$ where ϖ and $\varkappa$ are given by* (2) *and suppose that $0\leq\sigma<1$. Then $f\in\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$ if and only if*

$$\sum_{j=2}^{\infty}j|c_j|+\sum_{j=1}^{\infty}j|d_j|\leq 1-\sigma. \tag{8}$$

Moreover, if $f\in\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$, then

$$|c_j|\ \leq\frac{1-\sigma}{j}, j\geq 2, \tag{9}$$

and

$$|d_j|\ \leq\frac{1-\sigma}{j}, j\geq 1. \tag{10}$$

Lemma 3. [3] *Let $f=\varpi+\overline{\varkappa}$ where ϖ and $\varkappa$ are given by (2), and suppose that $0\le\sigma<1$. Then $f\in\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$ if*

$$\sum_{j=2}^{\infty} j^2|c_j|+\sum_{j=1}^{\infty} j^2|d_j|\le 1-\sigma. \tag{11}$$

Moreover, if $f\in\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$, then

$$|c_j|\le\frac{1-\sigma}{j^2},\ j\ge 2 \tag{12}$$

and

$$|d_j|\le\frac{1-\sigma}{j^2},\ j\ge 1. \tag{13}$$

Lemma 4. [9] *If $f=\varpi+\overline{\varkappa}\in\mathcal{S}^*_{\mathbb{H}}$ where ϖ and $\varkappa$ are given by (1) with $d_1=0$,then*

$$|c_j|\le\frac{(2j+1)(j+1)}{6}\ \textit{and}\ |d_j|\le\frac{(2j-1)(j-1)}{6}. \tag{14}$$

Lemma 5. [9] *If $f=\varpi+\overline{\varkappa}\in\mathcal{K}_{\mathbb{H}}$ where ϖ and $\varkappa$ are given by (1) with $d_1=0$, then*

$$|c_j|\le\frac{j+1}{2}\ \textit{and}\ |d_j|\le\frac{j-1}{2}. \tag{15}$$

For convenience throughout in the sequel, we use the following notations:

$$\sum_{j=2}^{\infty}\frac{t^{j-1}}{(j-1)!}=e^t-1$$

and

$$\sum_{j=j}^{\infty}\frac{t^{j-1}}{(j-j)!}=t^{j-1}e^t,\quad j\ge 2.$$

3. Inclusion relations of the class $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$

In this section we will prove the inclusion relations of the harmonic class $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$ with the classes $\mathcal{K}_{\mathbb{H}}$ and $\mathcal{S}^*_{\mathbb{H}}$ associated of the operator Ξ defined by (3).

Theorem 6. *Let $t_1,t_2>0$, $\varepsilon\ge 0$, $\sigma\in[0,1)$ and $\xi\in\mathbb{C}$. If*

$$\begin{aligned}&\left[2\varepsilon\left(t_1^4+t_2^4\right)+(21\varepsilon+2)t_1^3+(54\varepsilon+15)\,t_1^2+(30\varepsilon+24)\,t_1+6\left(1-e^{-t_1}\right)\right.\\&\left.+(2-3\varepsilon)t_2^3+(24\varepsilon-11)\,t_2^2+(6\varepsilon+6)\,t_2\right]\\&\le 6\,|\xi|\,,\end{aligned} \tag{16}$$

then

$$\Xi(\mathcal{S}^*_{\mathbb{H}})\subset\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi).$$

Proof. Let $f=\varpi+\overline{\varkappa}\in\mathcal{S}^*_{\mathbb{H}}$ where ϖ and $\varkappa$ are of the form (2) with $d_1=0$. We need to show that $\Xi(f)=\varpi+\varkappa\in\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$, where ϖ and $\varkappa$ defined by (4) with $d_1=0$ are analytic functions in E. In view of Lemma 1, we need to prove that

$$Q(t_1,t_2,\varepsilon)\le|\xi|\,,$$

where

$$Q(t_1,t_2,\varepsilon)=\sum_{j=2}^{\infty} j\,(1-\varepsilon+j\varepsilon)\left|\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}c_j\right|+\sum_{j=2}^{\infty} j\,(1-\varepsilon+j\varepsilon)\left|\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}d_j\right|. \tag{17}$$

Using the inequalities (14) of Lemma 4, we get

$$\begin{aligned}
&Q(t_1,t_2,\varepsilon)\\
&\leq \frac{1}{6}\left[\sum_{j=2}^{\infty}(2j+1)(j+1)\left(j\left(1-\varepsilon+j\varepsilon\right)\right)\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.\\
&\left.+\sum_{j=2}^{\infty}(2j-1)(j-1)\left(j\left(1-\varepsilon+j\varepsilon\right)\right)\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]\\
&=\frac{1}{6}\left[\sum_{j=2}^{\infty}\left[2\varepsilon j^4+(2+\varepsilon)j^3+(3-2\varepsilon)j^2+(1-\varepsilon)j\right]\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.\\
&\left.+\sum_{j=2}^{\infty}\left[2\varepsilon j^4+(2-5\varepsilon)j^3+(4\varepsilon-3)j^2+(1-\varepsilon)j\right]\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]
\end{aligned} \tag{18}$$

Writing

$$j=(j-1)+1, \tag{19}$$

$$j^2=(j-1)(j-2)+3(j-1)+1, \tag{20}$$

and

$$j^3=(j-1)(j-2)(j-3)+6(j-1)(j-2)+7(j-1)+1, \tag{21}$$

$$j^4=(j-1)(j-2)(j-3)(j-4)+10(j-1)(j-2)(j-3)+25(j-1)(j-2)+15(j-1)+1, \tag{22}$$

in (18), we have

$$\begin{aligned}
&Q(t_1,t_2,\varepsilon)\\
&\leq \frac{1}{6}\left[\sum_{j=2}^{\infty}[2\varepsilon(j-1)(j-2)(j-3)(j-4)+(21\varepsilon+2)\,(j-1)\,(j-2)\,(j-3)\right.\\
&+(54\varepsilon+15)\,(j-1)\,(j-2)+(30\varepsilon+24)\,(j-1)+6]\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\\
&+\sum_{j=2}^{\infty}[2\varepsilon(j-1)(j-2)(j-3)(j-4)+(2-3\varepsilon)\,(j-1)\,(j-2)\,(j-3)\\
&\left.+(24\varepsilon-11)\,(j-1)\,(j-2)+(6\varepsilon+6)\,(j-1)]\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]
\end{aligned}$$

$$= \frac{1}{6}\left[2\varepsilon\sum_{j=5}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-5)!} + (21\varepsilon+2)\sum_{j=4}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-4)!} + (54\varepsilon+15)\sum_{j=3}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-3)!}\right.$$
$$+ (30\varepsilon+24)\sum_{j=2}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-2)!} + 6\sum_{j=2}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-1)!} + 2\varepsilon\sum_{j=5}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-5)!} + (2-3\varepsilon)\sum_{j=4}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-4)!}$$
$$\left.+ (24\varepsilon-11)\sum_{j=3}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-3)!} + (6\varepsilon+6)\sum_{j=2}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-2)!}\right]$$
$$= \frac{1}{6}\left[2\varepsilon t_1^4 + (21\varepsilon+2)t_1^3 + (54\varepsilon+15)\,t_1^2 + (30\varepsilon+24)\,t_1 + 6\left(1-e^{-t_1}\right)\right.$$
$$\left.+2\varepsilon t_2^4 + (2-3\varepsilon)t_2^3 + (24\varepsilon-11)\,t_2^2 + (6\varepsilon+6)\,t_2\right].$$

But this last expression is bounded above by $|\xi|$ if (16) holds. □

Theorem 7. *Let $t_1, t_2 > 0$, $\varepsilon \geq 0$, $\sigma \in [0,1)$ and $\xi \in \mathbb{C}$. If*

$$\left[\varepsilon\left(t_1^3+t_2^3\right) + (6\varepsilon+1)\,t_1^2 + (6\varepsilon+4)\,t_1 + 2\left(1-e^{-t_1}\right) + (4\varepsilon+1)\,t_2^2 + (2\varepsilon+2)\,t_2\right]$$
$$\leq 2\,|\xi|, \tag{23}$$

then

$$\Xi(\mathcal{K}_{\mathbb{H}}) \subset \mathcal{T}_{\mathbb{H}}(\varepsilon,\xi).$$

Proof. Let $f = \varpi + \overline{\varkappa} \in \mathcal{K}_{\mathbb{H}}$ where ϖ and $\varkappa$ are of the form (2) with $d_1 = 0$. We need to show that $\Xi(f) = \varpi + \varkappa \in \mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$, where ϖ and $\varkappa$ defined by (4) with $d_1 = 0$ are analytic functions in E. In view of Lemma 1, we need to prove that

$$Q(t_1,t_2,\varepsilon) \leq |\xi|,$$

where $Q(t_1,t_2,\varepsilon)$ as given in(17). Using the inequalities (15) of Lemma 5, we get

$$Q(t_1,t_2,\varepsilon)$$
$$\leq \frac{1}{2}\left[\sum_{j=2}^{\infty}(j+1)\left(j\left(1-\varepsilon+j\varepsilon\right)\right)\frac{e^{-t_1}t_1^{j-1}}{(j-1)!} + \sum_{j=2}^{\infty}(j-1)\left(j\left(1-\varepsilon+j\varepsilon\right)\right)\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]$$
$$= \frac{1}{2}\left[\sum_{j=2}^{\infty}\left[\varepsilon j^3 + j^2 + (1-\varepsilon)j\right]\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.$$
$$\left.+\sum_{j=2}^{\infty}\left[\varepsilon j^3 + (1-2\varepsilon)j^2 - (1-\varepsilon)j\right]\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]. \tag{24}$$

Using the equations (19)-(21) in (24), we have

$$
\begin{aligned}
&Q(t_1,t_2,\varepsilon)\\
&\leq \frac{1}{2}\left[\sum_{j=2}^{\infty}\left[\varepsilon\left(j-1\right)\left(j-2\right)\left(j-3\right)+\left(6\varepsilon+1\right)\left(j-1\right)\left(j-2\right)+\left(6\varepsilon+4\right)\left(j-1\right)+2\right]\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.\\
&\left.+\sum_{j=2}^{\infty}\left[\varepsilon\left(j-1\right)\left(j-2\right)\left(j-3\right)+\left(4\varepsilon+1\right)\left(j-1\right)\left(j-2\right)+\left(2\varepsilon+2\right)\left(j-1\right)\right]\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]\\
&=\frac{1}{2}\left[\varepsilon\sum_{j=4}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-4)!}+\left(6\varepsilon+1\right)\sum_{j=3}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-3)!}+\left(6\varepsilon+4\right)\sum_{j=2}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-2)!}+2\sum_{j=2}^{\infty}\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.\\
&\left.+\varepsilon\sum_{j=4}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-4)!}+\left(4\varepsilon+1\right)\sum_{j=3}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-3)!}+\left(2\varepsilon+2\right)\sum_{j=2}^{\infty}\frac{e^{-t_2}t_2^{j-1}}{(j-2)!}\right]\\
&=\frac{1}{2}\left[\varepsilon t_1^3+\left(6\varepsilon+1\right)t_1^2+\left(6\varepsilon+4\right)t_1+2\left(1-e^{-t_1}\right)+\varepsilon t_2^3+\left(4\varepsilon+1\right)t_2^2+\left(2\varepsilon+2\right)t_2\right].
\end{aligned}
$$

But this last expression is bounded above by $|\xi|$ if (23) holds. □

Next we determine connection between $\mathcal{T}\mathrm{j}_{\mathbb{H}}(\sigma)$ and $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$.

Theorem 8. *Let $t_1,t_2>0$, $\varepsilon\geq 0$, $\sigma\in[0,1)$ and $\xi\in\mathbb{C}$. If*

$$(1-\sigma)[\varepsilon\left(t_1+t_2\right)+\left(2-e^{-t_1}-e^{-t_2}\right)]\leq|\xi|-|d_1|\,,$$

then

$$\Xi(\mathcal{T}\Psi_{\mathbb{H}}(\sigma))\subset\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi).$$

Proof. Let $f=\varpi+\overline{\varkappa}\in\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$ where ϖ and $\varkappa$ are given by (2). In view of Lemma 1, it is enough to show that $L(t_1,t_2,\varepsilon)\leq|\xi|$, where

$$L(t_1,t_2,\varepsilon)=\sum_{j=2}^{\infty}j\left(1-\varepsilon+j\varepsilon\right)\left|\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}c_j\right|+|d_1|+\sum_{j=2}^{\infty}j\left(1-\varepsilon+j\varepsilon\right)\left|\frac{e^{-t_2t}t_2^{j-1}}{(j-1)!}d_j\right|. \tag{25}$$

Using the inequalities (9) and (10) of Lemma 2, it follows that

$$
\begin{aligned}
L(t_1,t_2,\varepsilon)&\leq(1-\sigma)\left[\sum_{j=2}^{\infty}\left(1-\varepsilon+j\varepsilon\right)\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}+\sum_{j=2}^{\infty}\left(1-\varepsilon+j\varepsilon\right)\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]+|d_1|\\
&=(1-\sigma)\left[\sum_{j=2}^{\infty}\left(\varepsilon\left(j-1\right)+1\right)\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}+\sum_{j=2}^{\infty}\left(\varepsilon\left(j-1\right)+1\right)\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]+|d_1|\\
&=(1-\sigma)\left[\varepsilon t_1+\left(1-e^{-t_1}\right)+\varepsilon t_2+\left(1-e^{-t_2}\right)\right]+|d_1|\\
&\leq|\xi|\,,
\end{aligned}
$$

by the given hypothesis, this completes the proof of Theorem 8. □

Next we find the relationship between the classes $\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$ and $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$.

Theorem 9. *Let* $t_1, t_2 > 0$, $\varepsilon \geq 0$, $\sigma \in [0,1)$ *and* $\xi \in \mathbb{C}$. *If*

$$\begin{aligned}
&(1-\sigma)\left[\varepsilon(2-e^{-t_1}-e^{-t_2})+\frac{1-\varepsilon}{t_1}(1-e^{-t_1}-t_1e^{-t_1})\right.\\
&\left.+\frac{1-\varepsilon}{t_2}(1-e^{-t_2}-t_2e^{-t_2})\right]\\
&\leq |\xi|-|d_1|,
\end{aligned}$$

then

$$\Xi(\mathcal{T}\Phi_{\mathbb{H}}(\sigma))\subset\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi).$$

Proof. Making use of Lemma 1, we need only to prove that $L(t_1,t_2,\varepsilon)\leq|\xi|$, where $L(t_1,t_2,\varepsilon)$ as given in (25). Using the inequalities (12) and (13) of Lemma 3, it follows that

$$\begin{aligned}
L(t_1,t_2,\varepsilon) &= \sum_{j=2}^{\infty} j\,(1-\varepsilon+j\varepsilon)\left|\frac{e^{-t1}t_1^{j-1}}{(j-1)!}c_j\right|\\
&+|d_1|+\sum_{j=2}^{\infty} j\,(1-\varepsilon+j\varepsilon)\left|\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}d_j\right|\\
&\leq (1-\sigma)\left[\sum_{j=2}^{\infty}\frac{(1-\varepsilon+j\varepsilon)}{j}\frac{e^{-t_1}t_1^{j-1}}{(j-1)!}\right.\\
&\left.+\sum_{j=2}^{\infty}\frac{(1-\varepsilon+j\varepsilon)}{j}\frac{e^{-t_2}t_2^{j-1}}{(j-1)!}\right]+|d_1|\\
&=(1-\sigma)\left[\sum_{j=0}^{\infty}\left(\varepsilon+\frac{1-\varepsilon}{j+2}\right)\frac{e^{-t_1}t_1^{j+1}}{(j+1)!}\right.\\
&\left.+\sum_{j=0}^{\infty}\left(\varepsilon+\frac{1-\varepsilon}{j+2}\right)\frac{e^{-t_2}t_2^{j+1}}{(j+1)!}\right]+|d_1|\\
&=(1-\sigma)\left[\varepsilon(1-e^{-t_1})+\frac{1-\varepsilon}{t_1}(1-e^{-t_1}-t_1e^{-t_1})\right.\\
&\left.+\varepsilon(1-e^{-t_2})+\frac{1-\varepsilon}{t_2}(1-e^{-t_2}-t_2e^{-t_2})\right]+|d_1|\\
&\leq|\xi|,
\end{aligned}$$

by given hypothesis. □

Theorem 10. *Let* $t_1, t_2 > 0$, $\varepsilon \geq 0$ *and* $\xi \in \mathbb{C}$. *If*

$$e^{-t_1}+e^{-t_2}\geq 1+\frac{|d_1|}{|\xi|}$$

then

$$\Xi(\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)) \subset \mathcal{T}_{\mathbb{H}}(\varepsilon,\xi).$$

Proof. Using the same technique as in the proof of Theorem 3.2.6, Lemma 1 and the inequalities (6) and (7) of Lemma 1, we obtain

$$\begin{aligned} L(t_1,t_2,\varepsilon) &\leq |\xi| \left[\sum_{j=2}^{\infty} \frac{e^{-t_1} t_1^{j-1}}{(j-1)!} + \sum_{j=2}^{\infty} \frac{e^{-t_2} t_2^{j-1}}{(j-1)!} \right] + |d_1| \\ &= |\xi| \left[\sum_{j=0}^{\infty} \frac{e^{-t_1} t_1^{j+1}}{(j+1)!} + \sum_{j=0}^{\infty} \frac{e^{-t_2} t_2^{j+1}}{(j+1)!} \right] + |d_1| \\ &= |\xi| \left[e^{-t_1}(e^{t_1}-1) + e^{-t_2}(e^{t_2}-1) \right] + |d_1| \\ &= |\xi| \left[2 - e^{-t_1} - e^{-t_2} \right] + |d_1| \\ &\leq |\xi|, \end{aligned}$$

by the given condition and this completes the proof of the theorem. □

4. Corollaries and Consequences

In this section, we apply our main results in order to deduce each of the following new corollaries and consequences.

Corollary 11. *Let* $t_1, t_2 > 0$ *and* $\xi \in \mathbb{C}$. *If*

$$2t_1^3 + 15t_1^2 + 24t_1 + 6\left(1 - e^{-t_1}\right) + 2t_2^3 - 11t_2^2 + 6t_2 \leq 6\,|\xi|,$$

then

$$\Xi(\mathcal{S}^*_{\mathbb{H}}) \subset \mathcal{T}_{\mathbb{H}}(\xi).$$

Corollary 12. *Let* $t_1, t_2 > 0$ *and* $\xi \in \mathbb{C}$. *If*

$$t_1^2 + 4t_1 + 2\left(1 - e^{-t_1}\right) + t_2^2 + 2t_2 \leq 2\,|\xi|,$$

then

$$\Xi(\mathcal{K}_{\mathbb{H}}) \subset \mathcal{T}_{\mathbb{H}}(\xi).$$

Corollary 13. *Let* $t_1, t_2 > 0$, $\sigma \in [0,1)$ *and* $\xi \in \mathbb{C}$. *If*

$$(1-\sigma)\left(2 - e^{-t_1} - e^{-t_2}\right) \leq |\xi| - |d_1|,$$

then

$$\Xi(\mathcal{T}\Psi_{\mathbb{H}}(\sigma)) \subset \mathcal{T}_{\mathbb{H}}(\xi).$$

Corollary 14. *Let* $t_1, t_2 > 0$, $\sigma \in [0, 1)$ *and* $\xi \in \mathbb{C}$. *If*

$$(1-\sigma)\left(\frac{1}{t_1}(1-e^{-t_1}-t_1e^{-t_1})+\frac{1}{t_2}(1-e^{-t_2}-t_2e^{-t_2})\right)\leq|\xi|-|d_1|,$$

then

$$\Xi(\mathcal{T}\Phi_{\mathbb{H}}(\sigma))\subset\mathcal{T}_{\mathbb{H}}(\xi).$$

5. Conclusions

The main purpose of this article is to find some inclusion relations of the harmonic class $\mathcal{T}_{\mathbb{H}}(\varepsilon,\xi)$ with the classes $\mathcal{S}^*_{\mathbb{H}}$ of harmonic starlike functions and $\mathcal{K}_{\mathbb{H}}$ of harmonic convex functions, also for the harmonic classes $\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$ and $\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$ associated with the operator Ξ defined by Poisson distribution series. Making use of the operator Ξ could inspire researchers to find new inclusion relations for new harmonic classes of analytic functions with the classes $\mathcal{S}^*_{\mathbb{H}}$, $\mathcal{K}_{\mathbb{H}}$, $\mathcal{T}\Psi_{\mathbb{H}}(\sigma)$ and $\mathcal{T}\Phi_{\mathbb{H}}(\sigma)$.

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