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A WEAK SOLUTION TO A NON-LOCAL PROBLEM IN FRACTIONAL ORLICZ-SOBOLEV SPACES

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Abstract. In this paper, we present the Fractional Orlicz−Sobolev spaces and we deal with the existence to a result for the following elliptic system:

$$\begin{cases} (-\Delta)^s_{a_1(.)} v = \dfrac{\partial F}{\partial v}(x, \zeta_1, \zeta_2) & in\, \Omega, \\ (-\Delta)^s_{a_2(.)} u = \dfrac{\partial F}{\partial u}(x, \zeta_1, \zeta_2) & in\, \Omega, \\ v = u = 0 & on\; \mathbb{R}^N \backslash \Omega, \end{cases}$$

where $\Omega \subset \mathbb{R}^N$, with Lipschitz boundary $\partial\Omega$, $s \in (0,1)$ and $(-\Delta)^s_{a_i(.)}$ stands the non-local Fractional $a(.)$-Laplacian operator of elliptic type with $i = 1, 2$. Employing the Mountain Pass Theorem we obtain that system above has a non-trivial solution in Fractional Orlicz−Sobolev spaces.

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1. Introduction

Let $\Omega \subset \mathbb{R}^N$, be an open set, we start by fixing $s \in (0,1)$. For any $p \in [1, \infty)$, the Fractional Sobolev space is defined as follows,

$$X^{s,p}(\Omega) := \left\{ \zeta \in L^p(\Omega) : \frac{|\zeta(y) - \zeta(x)|}{|y-x|^{\frac{N}{p}+s}} \in L^p(\Omega^2) \right\},$$

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endowed with the norm

$$||\zeta||_{s,p} = \left(\int_{\Omega} |\zeta|^p \, dx + \int_{\Omega} \int_{\Omega} \frac{|\zeta(y) - \zeta(x)|^p}{|y - x|^{sp+N}} dx \, dy \right)^{\frac{1}{p}}. \tag{1.1}$$

After the well-known studies of Caffarelli et al. ([12]), many scientists wrote about difficulties involving the non-local Fractional diffusion operator $(-\Delta)^s$, with p-structure because of its accurate description of patterns that involve unusual scattering. Furthermore, some non-local phenomena that do not follow a power type of growth law have been seen in various fields of study. For example, see [4, 5] and their references. The Fractional $m(.)$-Laplacian operator, developed in [17], is used to characterize these kinds of events,

$$(-\Delta)^s_{a(.)}\zeta(x) = p.v \int_{\mathbb{R}^N} a\left(\frac{|\zeta(y) - \zeta(x)|}{|y - x|^s} \right) \frac{\zeta(y) - \zeta(x)}{|y - x|^s} \frac{dy}{|y - x|^N}, \tag{1.2}$$

for all $x \in \mathbb{R}^N$, where $p.v$ is the principal value and $a\colon \mathbb{R}^{\geq 0} \to \mathbb{R}^{\geq 0}$ is a non-decreasing and right continuous function, with

$$\lim_{\zeta \to \infty} a(\zeta) = \infty, \; a(0) = 0, \quad a(\zeta) > 0, \; \text{for } \zeta > 0. \tag{1.3}$$

It given that the following representation:

$$A_{i=1;2}(\zeta) := \int_0^{|\zeta|} r a_i(r) dr \quad \forall \, \zeta \in \mathbb{R}, \tag{1.4}$$

exist, and they are an Orlicz functions. Using the operator (1.2) we introduce the non-local elliptic system as follows:

$$\begin{cases} (-\Delta)^s_{a_1(.)}\zeta_1 = \dfrac{\partial F}{\partial \zeta_1}(x, \zeta_1, \zeta_2) & in \, \Omega, \\ (-\Delta)^s_{a_2(.)}\zeta_2 = \dfrac{\partial F}{\partial \zeta_2}(x, \zeta_1, \zeta_2) & in \, \Omega, \\ \zeta_1 = \zeta_2 = 0 & on \; \mathbb{R}^N \backslash \Omega, \end{cases} \tag{1.5}$$

where $\frac{\partial F}{\partial \zeta_1}$ and $\frac{\partial F}{\partial \zeta_2}$ denotes the partial derivative of F which satisfy:
(F_1): $F : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is a function of C^1, such that $F(x, 0, 0) = 0$ a.e $x \in \Omega$ and there exist an Orlicz functions $\Psi_{i=1,2} : \mathbb{R} \to \mathbb{R}^{\geq 0}$, $\Psi_i \ll A_i^*$, with

$$n_i < l_{\Psi_i} \leq \frac{s\zeta\psi_i(\zeta)}{\Psi_i(\zeta)} \leq n_{\Psi_i} < \infty, \tag{1.6}$$

where $\Psi_i(\zeta) := \int_0^{|\zeta|} \psi_i(r)dr, \quad for\ all\ \zeta \in \mathbb{R}$. Moreover,

$$\begin{cases} \Big|\dfrac{\partial F}{\partial \zeta_1}(x,\zeta_1,\zeta_2)\Big| \leq c_1\Big(\psi_1(|\zeta_1|) + \overline{\Psi}_1^{-1}(\Psi_2(\zeta_2))\Big), \\ \Big|\dfrac{\partial F}{\partial \zeta_2}(x,\zeta_1,\zeta_2)\Big| \leq c_1\Big(\psi_2(|\zeta_2|) + \overline{\Psi}_2^{-1}(\Psi_1(\zeta_1))\Big), \end{cases} \tag{1.7}$$

where $c_1 > 0$, $\overline{\Psi}_i$ denote the complements of $\Psi_{i=1,2}$.

(F_2):

$$\lim_{|(v,v)| \to +\infty} \frac{F(x,\zeta_1,\zeta_2)}{A_1(\zeta_1) + A_2(\zeta_2)} = \infty, \qquad \text{uniformly } \forall x \in \Omega. \tag{1.8}$$

(F_3): $\gamma : \mathbb{R}^{\geq 0} \to \mathbb{R}$ is a continuous function such that

$$+\infty > n_\Gamma \geq \frac{s\zeta\gamma(\zeta)}{\Gamma(\zeta)} \geq l_\Gamma > s, \tag{1.9}$$

where $\Gamma(\zeta) := \int_0^{|\zeta|} \gamma(r)\ dr, \quad$ a.e $\zeta \in \mathbb{R}$, is an Orlicz function and $\Phi_i(\zeta) := |\zeta|^{\frac{l_i' l_\Gamma'}{l_\Gamma' - 1}}, \zeta \in \mathbb{R}$ with $\Phi_i \ll A_i^*$, respectively, such that

$$\Gamma\Big(\frac{F(x,\zeta_1,\zeta_2)}{|\zeta_1|^{l_1'} + |\zeta_2|^{l_2'}}\Big) \leq c_2\overline{F}(x,\zeta_1,\zeta_2), \quad |(\zeta_1,\zeta_2)| \geq r, \tag{1.10}$$

where $c_2, r > 0$ and

$$\overline{F}(x,\zeta_1,\zeta_2) := sn_1^{-1}\frac{\partial F}{\partial \zeta_1}(x,\zeta_1,\zeta_2)v + sn_2^{-1}\frac{\partial F}{\partial \zeta_2}(x,\zeta_1,\zeta_2)u - F(x,\zeta_1,\zeta_2).$$

When we take $a_1(\zeta) = |\zeta|^{p-2}$, $a_2(\zeta) = |\zeta|^{q-2}$ $(p,q > 1)$. Then the system (1.5) reduces to the following fractional (p,q)-Laplacian system :

$$\begin{cases} (-\Delta)_p^s u = \frac{\partial F}{\partial v}(x,\zeta_1,\zeta_2) & in\, \Omega, \\ (-\Delta)_q^s v = \frac{\partial F}{\partial u}(x,\zeta_1,\zeta_2) & in\, \Omega, \\ u = v = 0 & on\ \mathbb{R}^N \backslash \Omega. \end{cases} \tag{1.11}$$

The existence of solutions for systems like (1.11) have also received a wide range of interests ([13,18]). Therefore, we have found in the published papers many researchers who studied this type of systems with some important methods, such as three critical points theorem, Nehari manifold, fibering method and variational method, , see for instance [2,3,15,16].
It makes sense to think of the results that can be obtained if the $a(.)$-Laplace operator is transformed into a fractional $a(.)$-Laplacian operator. As far as we know, a certain numbers of results have been obtained on fractional Orlicz-Sobolev (F.O.S) spaces ([9,19]). In this

work we generalize the system (1.11) based on Orlicz functions in a general class of functional areas called F.O.S spaces in the whole space $\mathbb{R}^N$.

This paper is organized as follows. In the second Section, we recall some well-known properties and results on Orlicz and F.O.S spaces. Third Section is devoted to present the existence of a result and its proof which relies on the Mountain Pass Theorem.

2. A short list of tentative results and hypotheses

We refer the reader to [1,8,17,21] for some properties of the F.O.S spaces.

Let $A\colon \mathbb{R}^+ \to \mathbb{R}^+$ be an Orlicz function (N-function), i. e., A is an convex function such that

$$A(\zeta) > 0 \quad \text{for } \zeta \neq 0, \quad \lim_{t\to 0} \frac{A(\zeta)}{\zeta} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{A(\zeta)}{\zeta} = \infty.$$

Equivalently, A allow the expression of:

$$A(\zeta) = \int_0^{|\zeta|} a(s)\,\mathrm{d}s, \tag{2.1}$$

The Orlicz function $\overline{A}$ complementary to A is defined by $\overline{A}(\zeta) = \int_0^{|\zeta|} \overline{a}(r)\,\mathrm{d}r$. The relationship related A and $\overline{A}$ given by

$$\overline{A}(\zeta) := \sup_{r\geq 0}\{\zeta r - A(r)\}. \tag{2.2}$$

Definition 2.1. The Orlicz function $A \in \Delta_2$, if for some constant $M > 2$,

$$A(2\zeta) \leq M\,A(\zeta), \qquad \forall \zeta > 0. \tag{2.3}$$

The notation $\Phi \ll A$ means that Φ grows essentially less rapidly than A, i.e., for each $\varepsilon > 0$,

$$\frac{\Phi(\zeta)}{A(\varepsilon\zeta)} \to 0 \quad \text{as } \zeta \to \infty. \tag{2.4}$$

The Orlicz space $L_A(\Omega)$ is a set of mesurable function such that,

$$\int_\Omega A\Big(k|\zeta(x)|\Big)\,\mathrm{d}x < +\infty, \quad \text{for some } k > 0. \tag{2.5}$$

The space $L_A(\Omega)$ is a Banach space under the norm

$$\|\zeta\|_A = \inf\Big\{k > 0\,/\,\int_\Omega A\Big(\frac{|\zeta(x)|}{k}\Big)\,\mathrm{d}x \leq 1\Big\}. \tag{2.6}$$

Also, the Hölder inequality holds

$$\int_\Omega |\zeta_1(x)\zeta_2(x)|\,\mathrm{d}x \leq ||\zeta_1||_A||\zeta_2||_{(\overline{A})} \text{ for all } \zeta_1 \in L_A(\Omega) \text{ and } \zeta_2 \in L_{\overline{A}}(\Omega).$$

and the Young inequality reads as follows:

(2.7) $$s\zeta \leq A(s) + \overline{A}(\zeta) \;\; \forall \zeta,\, s \geq 0.$$

The fact that $A \in \Delta_2$ implies that

(2.8) $$\zeta_k \to \zeta \; in \; L_A(\Omega) \Leftrightarrow \int_\Omega A(\zeta_k - \zeta)dx \to 0.$$

We assume that

$$(H_0) \; \int_0^1 \frac{A_i^{-1}(\zeta)}{\zeta^{1+\frac{s}{N}}} \mathrm{d}\zeta < \infty \; \text{ and } (H_\infty) \int_1^{+\infty} \frac{A_i^{-1}(\zeta)}{\zeta^{1+\frac{s}{N}}} \mathrm{d}\zeta = +\infty, \text{for } s \in (0,1\;).$$

Under the assumptions (H_0) and (H_∞), we have the possibility to introduced an Orlicz function A^*, by the expression of its inverse in $\mathbb{R}^{\geq 0}$:

(2.9) $$(A_i^*)^{-1}(\zeta) = \int_0^t \frac{A_i^{-1}(r)}{r^{\frac{N+s}{N}}} dr \; for \, \zeta \geq 0.$$

The following lemmas are proved in [14].

Lemma 2.2. *Let* $\xi_0(\zeta) = \min\{\zeta^{l_i'}, \zeta^{n_i'}\}$, $\xi_1(\zeta) = \max\{\zeta^{l_i'}, \zeta^{n_i'}\}$ *where* $l_i' := \frac{l_i}{s}$, $n_i' := \dfrac{n_i}{s}$, $\zeta \geq 0$, $s \in (0,1)$ *and* A_i *is an Orlicz function then these assertions are equivalent:*

1)

(2.10) $$s < l_i := \inf_{\zeta > 0} \frac{s\zeta a_i(\zeta)}{A_i(\zeta)} \leq \sup_{\zeta > 0} \frac{s\zeta a_i(\zeta)}{A_i(\zeta)} := n_i < \infty.$$

2)

(2.11) $$\xi_0(\zeta) A_i(\rho) \leq A_i(\rho\zeta) \leq \xi_1(\zeta) A_i(\rho), \; \forall \zeta, \rho \geq 0.$$

3) $A_i \in \Delta_2$.

Remark 2.3. i) According to last Lemma, assumptions (F_2) and (F_3) infer that

$$\lim_{|(\zeta_1,\zeta_2)| \to +\infty} \overline{F}(x, \zeta_1, \zeta_2) \to +\infty, \qquad \text{uniformly } \forall x \in \Omega.$$

ii) Based on (2.7), $F(x,0,0) = 0$ and the reason that

$$F(x, \zeta_1, \zeta_2) = \int_0^{\zeta_1} \frac{\partial F}{\partial r}(x, r, 0)dr + \int_0^{\zeta_2} \frac{\partial F}{\partial t}(x, 0, t)dt + F(x, 0, 0), \; x \in \Omega, \zeta_1 \in \mathbb{R}, \zeta_2 \in \mathbb{R}.$$

By (1.7) and last Lemma, there exists a constant $c_4 > 0$ such that

(2.12) $$|F(x, \zeta_1, \zeta_2)| \leq c_4(\Psi_1(\zeta_1) + \Psi_2(\zeta_2)), \;\; x \in \Omega, \zeta_1 \in \mathbb{R}, \zeta_2 \in \mathbb{R}.$$

Lemma 2.4. *If A_i is an Orlicz function satisfies (2.10) then we have*

$$\xi_0(||\zeta||_{A_i}) \leq \int_\Omega A_i(|\zeta|)dx \leq \xi_1(||\zeta||_{A_i}),\ \forall \zeta \in L_{A_i}(\Omega). \tag{2.13}$$

Lemma 2.5. *Let $\overline{A}_i$ be the complement of A_i and $\xi_2(\zeta) = \min\{\zeta^{\overline{l'}_i}, \zeta^{\overline{n'}_i}\}$, $\xi_3(\zeta) = \max\{\zeta^{\overline{l'}_i}, \zeta^{\overline{n'}_i}\}$, $t \geq 0$ where $\overline{l'}_i = \frac{l'_i}{l'_i-1}$ and $\overline{n'}_i = \frac{n'_i}{n'_i-1}$, If A_i is an Orlicz function and (2.10) hold, then $\overline{A}_i$ satisfies:*
1)

$$\overline{n}_i\overline{A}_i(r) \leq r\overline{A}'_i(r) \leq \overline{l}_i\overline{A}_i(r) \tag{2.14}$$

2)

$$\zeta_2(\zeta)\overline{A}_i(r) \leq \overline{A}_i(r\zeta) \leq \xi_3(\zeta)\overline{A}_i(r),\ \forall \zeta, r \geq 0. \tag{2.15}$$

3)

$$\zeta_2(||r||_{\overline{A}_i}) \leq \int_\Omega \overline{A}_i(r)dx \leq \xi_3(||r||_{\overline{A}_i}),\ \forall r \in L_{\overline{A}_i}(\Omega). \tag{2.16}$$

Lemma 2.6. *Let $\xi_4(\zeta) = \min\{t^{l_i^*}, t^{n_i^*}\}$, $\xi_5(\zeta) = \max\{t^{l_i^*}, t^{n_i^*}\}$, $t \geq 0$ where $l_i^* = \frac{l'_iN}{N-sl'_i}$ and $n_i^* = \frac{n'_iN}{N-sn'_i}$ If A_i is an N-function and (2.10) hold, then A_i^* satisfies:*
1)

$$\xi_4(\zeta)A_i^*(\rho) \leq A_i^*(\rho t) \leq \xi_5(\zeta)A_i^*(\rho),\ \forall \zeta, \rho \geq 0. \tag{2.17}$$

2)

$$\xi_4(||w||_{A_i^*}) \leq \int_\Omega A_i^*(|w|)dx \leq \xi_5(||w||_{A_i^*}),\ \forall w \in L_{A_i^*}(\Omega), \tag{2.18}$$

where A_i is satisfying (H_0) and (H_∞).

Fractional Orlicz-Sobolev Spaces. The F.O.S spaces $X^{s,A_i}(\Omega)$ is defined (see [11]) as follows:

$$X^{s,A_i}(\Omega) = \left\{\zeta \in L_{A_i}(\Omega) : \int_\Omega\int_\Omega A_i\left(\frac{k|\zeta(y)-\zeta(x)|}{|y-x|^s}\right)|y-x|^{-N}dydx < \infty,\ k > 0\right\}.$$

which equipped with the norm,

$$||\zeta||_{s,A_i} = [\zeta]_{s,A_i} + ||\zeta||_{A_i} \tag{2.19}$$

where $[.]_{s,A_i}$ is defined by

$$[\zeta]_{s,A_i} = \inf\left\{k > 0 : \int_\Omega\int_\Omega A_i\left(\frac{|\zeta(y)-\zeta(x)|}{k|y-x|^s}\right)|y-x|^{-N}dydx \leq 1\right\}.$$

We choose

$$X_0^{s,A_i}(\Omega) = \{\zeta \in X^{s,A_i}(\mathbb{R}^N) : \zeta = 0\ a.e\ \mathbb{R}^N \setminus \Omega\},$$

which eqqiped with $[.]_{s,A_i}$. In F.O.S spaces the generalized poincaré inequality hold (see [6]). i.e, there exists $\mu > 0$ such that,

$$||w||_{A_i} \leq \mu[w]_{s,A_i},\ \forall w \in X_0^{s,A_i}(\Omega). \tag{2.20}$$

and we can see that, if Ω is bounded then $[\zeta]_{s,A_i} \sim ||\zeta||_{s,A_i}$ in $X_0^{s,A_i}(\Omega)$.

Remark 2.7. [17] $(X_0^{s,A_i}(\Omega), [.]_{s,A_i})$ is a separable, reflexive Banach space.

Theorem 2.8. ([6]) *Assume that* (H_0), (H_∞) *and* (2.10) *hold then the embedding* $X^{s,A_i}(\Omega) \hookrightarrow L_{A_i^*}(\Omega)$ *is continuous, and the embedding* $X^{s,A_i}(\Omega) \hookrightarrow L_N(\Omega)$, *is compact for any orlicz function* $N \ll A_i^*$.

Remark 2.9. By (F_1) and (F_3), employing Theorem 2.8 the following embeddings $X_0^{s,A_i}(\Omega) \hookrightarrow L^{\Psi_i}(\Omega)$, $X_0^{s,A_i}(\Omega) \hookrightarrow L^{l_i' \overline{l'}_\Gamma}(\Omega)$ and $X_0^{s,A_i}(\Omega) \hookrightarrow L^{l_i' \overline{n'}_\Gamma}(\Omega)$ are compact, where $\overline{l'}_\Gamma = \frac{l'_\Gamma}{l'_\Gamma - 1}$ and $\overline{n'}_\Gamma = \frac{n'_\Gamma}{n'_\Gamma - 1}$.

Examples [23]. Now, we set some exemples to an Orlicz function:

1) $a(\zeta) = \zeta^{p-1}, t > 0, 1 < p+1 < N.$
2) $a(\zeta) = 2p(1+\zeta^2)^{p-1}, \zeta > 0, 1 \leq p < \dfrac{N}{2}.$
3) $a(\zeta) = \dfrac{log(1+|\zeta|)}{|\zeta|},\ \zeta, \in \mathbb{R}^*.$

3. Main results

We present the following existence result by using Mountain Pass Theorem, see [22].

Theorem 3.1. *Assume that,* (F_1), (F_2) *and* (F_3) *hold. Then the system* (1.5) *possesses a nontrivial weak solution.*

In order to proved our result we defined the following working space $X := X_0^{s,A_1}(\Omega) \times X_0^{s,A_2}(\Omega)$ with the norm

$$\begin{aligned} ||(\zeta_1,\zeta_2)|| &:= [\zeta_1]_{s,A_1} + [\zeta_2]_{s,A_2} \\ &\sim ||\zeta_1||_{s,A_1} + ||\zeta_2||_{s,A_2}. \end{aligned}$$

By Remark (2.7), we are able to show that X is a reflexive separable Banach space. We notice that the energy function I on X corresponding to system (1.5) is

$$I(\zeta_1,\zeta_2) := \int_{\Omega^2} A_1\left(\frac{|\zeta_1(y)-\zeta_1(x)|}{|y-x|^s}\right)\frac{dydx}{|y-x|^N} + \int_{\Omega^2} A_2\left(\frac{|\zeta_2(y)-\zeta_2(x)|}{|y-x|^s}\right)\frac{dydx}{|y-x|^N}$$

$$-\int_{\Omega} F(x,\zeta_1,\zeta_2)dx, \;\; for \;\; all \;\; (\zeta_1,\zeta_2) \in X.$$

Denote by $I_i (i = 1, 2) : X \to \mathbb{R}$ the functionals

$$I_1(\zeta_1,\zeta_2) = \int_{\Omega^2} A_1\left(\frac{|\zeta_1(y)-\zeta_1(x)|}{|y-x|^s}\right)\frac{dydx}{|y-x|^N} + \int_{\Omega^2} A_2\left(\frac{|\zeta_2(y)-\zeta_2(x)|}{|y-x|^s}\right)\frac{dydx}{|y-x|^N}$$

and $I_2(\zeta_1,\zeta_2) = \int_{\Omega} F(x,\zeta_1,\zeta_2)dx$. Then $I(\zeta_1,\zeta_2) = I_1(\zeta_1,\zeta_2) - I_2(\zeta_1,\zeta_2)$.

We define the functional $\mathcal{F}_{i=1;2.} : X_0^{s,A_i}(\Omega) \to \mathbb{R}$ by

$$\mathcal{F}_i(\zeta) = \int_{\Omega^2} A_i\left(\frac{|\zeta(y)-\zeta(x)|}{|y-x|^s}\right)\frac{dydx}{|y-x|^N}.$$

Lemma 3.2. ([8] *Lemma 4.1*) *We have the following properties:*

1)

$$\mathcal{F}_i\left(\frac{\zeta}{[\zeta]_{s,A_i}}\right) \le 1, \; for\, all\, \zeta \in X_0^{s,A_i} \setminus \{0\},$$

2)

$$\xi_0([\zeta]_{s,A_i}) \le \mathcal{F}(\zeta) \le \zeta_1([\zeta]_{s,A_i}), \;\; for\, all\, \zeta \in X_0^{s,A_i}.$$

Lemma 3.3. [6] *The functional $\mathcal{F}_i$ is weak lower semi-continuous.*

Lemma 3.4. *Suppose that the sequence $\zeta_k \rightharpoonup \zeta$ in $X_0^{s,A_1}(\Omega)$ and*

$$\limsup \langle \mathcal{F}_1'(\zeta_k), \zeta_k - \zeta\rangle \le 0. \tag{3.1}$$

Then $(\zeta_k) \to \zeta \in X_0^{s,A_1}(\Omega)$.

The Fractional $a(.)$-Laplacian defined in (1.2) has been well established between $X_0^{s,A_i}(\Omega)$ and $(X_0^{s,A_i}(\Omega))^*$. In fact, in ([17], Theorem 6.12) we can see that,

$$\begin{aligned} \langle(\mathcal{F}_i'(\zeta), v\rangle &= \int_{\Omega^2} a_i\left(\frac{|\zeta(y)-\zeta(x)|}{|y-x|^s}\right)\frac{(\zeta(y)-\zeta(x))(v(y)-v(x))}{|y-x|^{2s}}\frac{dydx}{|y-x|^N} \\ &= \langle(-\Delta)^s_{a_i}\zeta, v\rangle \end{aligned} \tag{3.2}$$

for every $u, v \in X_0^{s,A_i}(\Omega)$. Under the assumptions (2.10) and (F_1), by similar arguments as ([7], lemma 3.2; [17], Theorem 6.12) we show that I_1 is well-defined and of class $C^1(X,\mathbb{R})$ and

$$\begin{aligned} \langle I'(\zeta_1,\zeta_2),(\overline{\zeta_1},\overline{\zeta_2})\rangle &= \int_{\Omega^2} a_1(|h_{\zeta_1}|)h_{\zeta_1}h_{\overline{\zeta_1}}d\mu + \int_{\Omega^2} a_2(|h_{\zeta_2}|)h_{\zeta_2}h_{\overline{\zeta_2}}d\mu \\ &\quad - \int_{\Omega}\frac{\partial F}{\partial v}(x,\zeta_1,\zeta_2)\overline{\zeta_1}dx - \int_{\Omega}\frac{\partial F}{\partial \zeta_2}(x,\zeta_1,\zeta_2)\overline{\zeta_2}dx \end{aligned}$$

for all $(\overline{\zeta_1},\overline{\zeta_2}) \in X$, $h_\zeta = \frac{\zeta(x)-\zeta(y)}{|y-x|^s}$ and $d\mu = \frac{dydx}{|y-x|^N}$. Then, the critical points of I on X are weak solutions of system (1.5).

Definition 3.5. Let $J : X_0^{s,A_i}(\Omega) \to \mathbb{R}$ is a class C^1. We say that ζ_k is a Cerami sequence (at the level $c \in \mathbb{R}$) for the functional J when

$$J(\zeta_k) \to c \quad and \quad \|J'(\zeta_k)\|(1 + \|\zeta_k\|) \to 0.$$

Theorem 3.6. *([20], Theorem 6) Let E^* be the dual of the Banach space E. Suppose that $J \in C^1(E, \mathbb{R})$ satisfies*

$$\inf_{||\zeta_2||=\rho} J(u) \geq \alpha > \beta \geq \max\{J(0), J(e)\},$$

for some $\rho > 0$ and $e \in E$ with $||e|| > \rho$. Let $c \geq \beta$ be characterized by

$$c = \inf_{\gamma \in \Gamma} max_{\zeta \in [0,1]} J(\gamma(\zeta))$$

where $\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}$. Then there exists a sequence $\{w_k\} \subset E$ such that

$$J(w_k) \to c \geq \beta \;\; and \;\; ||J'(w_k)||_{E^*}(1 + ||w_k||) \to 0 \; as \; k \to \infty. \tag{3.3}$$

Remark 3.7. By arguments in [10], we can replaced the (PS)-condition denoted $(PS)_c$ by Cerami sequence condition ($(C)_c$-condition).

Lemma 3.8. *([[22], Theorem 2.2]) Let J be a function of class C^1 and E be a Banach Space. Suppose that J satisfy $(PS)_c$, $J(0) = 0$ and*
(J_1) $\exists \rho, \alpha > 0$ such that $J|_{\partial B_\rho} \geq \alpha$.
(J_2) $\exists e \in E \backslash B_\rho$ such that $J(e) \leq 0$.
Then J owns a critical value $c \geq \alpha$.

Proof. This Lemma is proved in several steps.
Step 1: Showed that the energy function J satisfied (J_1) and (J_2). For a given $\epsilon \in (0,1)$ we have

$$\begin{aligned} I(\zeta_1, \zeta_2) = & \int_{\Omega^2} A_1(|h_{\zeta_1}|)d\mu + \int_{\Omega^2} A_2(|h_{\zeta_2}|)d\mu - \int_\Omega F(x, \zeta_1, \zeta_2)dx \\ > & \;\epsilon \int_{\Omega^2} A_1(|h_{\zeta_1}|)d\mu + \epsilon \int_{\Omega^2} A_2(|h_{\zeta_2}|)d\mu - \int_\Omega |F(x, \zeta_1, \zeta_2)|dx \end{aligned}$$

Using (2.12) and when $||(\zeta_1, \zeta_2)|| \leq 1$ by Lemma 3.2 we obtain

$$\begin{aligned} I(\zeta_1, \zeta_2) > & \;\epsilon \min\{[\zeta_1]_{s,A_1}^{l_1'}, [\zeta_2]_{s,A_1}^{n_1'}\} + \epsilon \min\{[\zeta_2]_{s,A_2}^{l_2'}, [\zeta_2]_{s,A_2}^{n_2'}\} \\ - & \;c_4 \int_\Omega \Psi_1(\zeta_1)dx - c_4 \int_\Omega \Psi_2(\zeta_2)dx \\ \geq & \;\epsilon[\zeta_1]_{s,A_1}^{n_1'} + \epsilon[\zeta_2]_{s,A_2}^{n_2'} - c_4 \max\{||\zeta_1||_{\Psi_1}^{l'_{\Psi_1}}; ||\zeta_1||_{\Psi_1}^{n'_{\Psi_1}}\} - c_4 \max\{||\zeta_2||_{\Psi_2}^{l'_{\Psi_2}}; ||\zeta_2||_{\Psi_2}^{n'_{\Psi_2}}\} \\ \geq & \;\epsilon[\zeta_1]_{s,A_1}^{n_1'} + \epsilon[\zeta_2]_{s,A_2}^{n_2'} - c_4||\zeta_1||_{\Psi_1}^{l'_{\Psi_1}} - c_4||\zeta_2||_{\Psi_2}^{l'_{\Psi_2}} \\ \geq & \;\epsilon[\zeta_1]_{s,A_1}^{n_1'} + \epsilon[\zeta_2]_{s,A_2}^{n_2'} - c_5||\zeta_1||_{A_1}^{l'_{\Psi_1}} - c_5||\zeta_2||_{A_2}^{l'_{\Psi_2}} \end{aligned}$$

According to Poincaré's Inequality we infer that

$$\begin{aligned} I(\zeta_1,\zeta_2) > \ & [\zeta_1]^{n_1'}_{s,A_1}(\epsilon - c_5||\zeta_1||^{l'_{\Psi_1}-n'_1}_{A_1}) + [\zeta_2]^{n_2'}_{s,A_2}(\epsilon - c_5||\zeta_2||^{l'_{\Psi_2}-n'_2}_{A_2}) \\ \geq \ & [\zeta_1]^{n_1'}_{s,A_1}(\epsilon - \overline{c}_5[\zeta_1]^{l'_{\Psi_1}-n'_1}_{s,A_1}) + [\zeta_2]^{n_2'}_{s,A_2}(\epsilon - \overline{c}_5[\zeta_2]^{l'_{\Psi_2}-n'_2}_{s,A_2}), \end{aligned}$$

where $c_5, \overline{c}_5 > 0$ achieved by Poincaré inequality and compact embedding, since $1 < n_i' < l'_{\Psi_i}$, then whenever

$$\rho \leq \min\left\{1, \left(\frac{\epsilon}{2\overline{c}_5}\right)^{\frac{1}{l'_{\Psi_1}-n'_1}}, \left(\frac{\epsilon}{2\overline{c}_5}\right)^{\frac{1}{l'_{\Psi_2}-n'_2}}\right\}$$

there exist $\alpha > 0$ sufficiently small such that $I(\zeta_1,\zeta_2) > \alpha$ for every $(\zeta_1,\zeta_2) \in X$ with $||(\zeta_1,\zeta_2)|| = \rho$. Therfore (J_1) is satisfied. Now, to show (J_2), again using (F_2) together with the continuous of F, then for any constant given $G > 0$, there is a constant $C_G > 0$ such that

$$F(x,\zeta_1,\zeta_2) \geq G(A_1(\zeta_1) + A_2(\zeta_2) - C_G \quad \forall (x,\zeta_1,\zeta_2) \in \Omega\times\mathbb{R}\times\mathbb{R}. \tag{3.4}$$

Now, choose $u_0 \in C_c^\infty(\Omega)\setminus\{0\}$ with $0 \leq u_0(x) \leq 1$. So by Theorem 7 in [6] we have $(u_0, 0) \in X$, and by (3.4) and Lemma 2.2, when $t > 0$ we have

$$\begin{aligned} I(tu_0,0) &= \int_{\Omega^2} A_1(|h_{tu_0}|d\mu - \int_\Omega F(x,tu_0,0)dx \\ &\leq A_1(t)\int_{\Omega^2}\max\{|h_{u_0}|^{l_1'}; |h_{u_0}|^{n_1'}\}d\mu - GA_1(t)\int_{\Omega^2}\min\{|u_0|^{l_1'}; |u_0|^{n_1'}\}dx \\ &+ C_G|\Omega|. \\ &\leq A_1(t)\Bigg(\int_{\Omega^2}\left|\frac{u_0(y)-u_0(x)}{|y-x|^s}\right|^{l_1'}d\mu + \int_{\Omega^2}\left|\frac{u_0(y)-u_0(x)}{|y-x|^s}\right|^{n_1'}d\mu - G||u_0||^{n_1'}_{n_1'}\Bigg) \\ &+ C_G|\Omega|. \\ &\leq A_1(t)\big(||u_0||^{l_1'}_{s,l_1'} + ||u_0||^{n_1'}_{s,n_1'} - G||u_0||^{n_1'}_{n_1'}\big) + C_G|\Omega|, \end{aligned}$$

Because $G > 0$ is given arbitrarily and $A_1(t) \to +\infty$ as $t \to \infty$, we can choose $G > \frac{||u_0||^{l_1'}_{s,l_1'} + ||u_0||^{n_1'}_{s,n_1'}}{||u_0||^{n_1'}_{n_1'}}$ and large t such that $I(tu_0,0) \leq 0$ and $||(tu_0,0)|| > \rho$. Thus end the prove of Lemma 3.8

Step 2: Showed that, any $(C)_c$-sequence in X is bounded. Let $\{\zeta_{1,k},\zeta_{2,k}\}$ be a $(C)_c$-sequence of

I in X. For n big enough by (3.3) and (2.10), we obtain

$$
\begin{aligned}
c+1 \geq\ & I(\zeta_{1,k},\zeta_{2,k}) - \langle I'(\zeta_{1,k},\zeta_{2,k}), (\frac{s}{n_1}\zeta_{1,k}, \frac{s}{n_2}\zeta_{2,k})\rangle \\
=\ & \int_{\Omega^2} A_1(|h_{\zeta_{1,k}}|)d\mu + \int_{\Omega^2} A_2(|h_{\zeta_{2,k}}|)d\mu - \int_{\Omega} F(x,\zeta_{1,k},\zeta_{2,k})dx \\
-\ & \frac{s}{n_1}\int_{\Omega^2} a_1(|h_{\zeta_{1,k}}|)h^2_{\zeta_{1,k}}d\mu - \frac{s}{n_2}\int_{\Omega^2} a_2(|h_{\zeta_{2,k}}|)h^2_{\zeta_{2,k}}d\mu \\
+\ & \frac{s}{n_1}\int_{\Omega}\frac{\partial F}{\partial \zeta_1}(x,\zeta_{1,k},\zeta_{2,k})\zeta_{1,k}dx + \frac{s}{n_2}\int_{\Omega}\frac{\partial F}{\partial \zeta_2}(x,\zeta_{1,k},\zeta_{2,k})\zeta_{2,k}dx \\
=\ & \int_{\Omega^2}\big(A_1(|h_{\zeta_{1,k}}|) - \frac{s}{n_1}a_1(|h_{\zeta_{1,k}}|)h^2_{\zeta_{1,k}}\big)d\mu \\
+\ & \int_{\Omega^2}\big(A_2(|h_{\zeta_{2,k}}|) - \frac{s}{n_2}a_2(|h_{\zeta_{2,k}}|)h^2_{\zeta_{2,k}}\big)d\mu \\
+\ & \int_{\Omega}\Big(\frac{s}{n_1}\frac{\partial F}{\partial v}(x,\zeta_{1,k},\zeta_{2,k})\zeta_{1,k}dx + \frac{s}{n_2}\frac{\partial F}{\partial u}(x,\zeta_{1,k},\zeta_{2,k})\zeta_{2,k} - F(x,\zeta_{1,k},\zeta_{2,k})\Big)dx \\
\geq\ & \int_{\Omega}\overline{F}(x,\zeta_{1,k},\zeta_{2,k})dx,
\end{aligned} \tag{3.5}
$$

through contradiction, we are able to prove the bounded character of the sequence $\{(\zeta_{1,k},\zeta_{2,k})\}$. Consider a sub-sequence of $\{(\zeta_{1,k},\zeta_{2,k})\}$, still referred to as $\{(\zeta_{1,k},\zeta_{2,k})\}$, such that

$$||(\zeta_{1,k},\zeta_{2,k})|| = ||\zeta_{1,k}||_{s,A_1} + ||\zeta_{2,k}||_{s,A_2} \to +\infty.$$

Next, we examine the problem in two cases.

Case1: Suppose that $||\zeta_{1,k}||_{s,A_1} \to +\infty$ or $||\zeta_{2,k}||_{s,A_2} \to +\infty$. Let $\overline{\zeta}_{2,k} = \dfrac{\zeta_{2,k}}{||\zeta_{2,k}||_{s,A_1}}$ and $\overline{\zeta}_{1,k} = \dfrac{\zeta_{1,k}}{||\zeta_{1,k}||_{s,A_2}}$. Then $\{(\overline{\zeta}_{2,k},\overline{\zeta}_{1,k})\}$ is bounded in separable, reflexive Banach space X. Switch to a subsequence which is less denoted by $\{(\overline{\zeta}_{2,k},\overline{\zeta}_{1,k})\}$. Using Remark 2.9, $\exists(\overline{\zeta}_1,\overline{u}) \in X$ such that:

(a) $\overline{\zeta}_{1,k} \rightharpoonup \overline{\zeta}_1$ in $X_0^{s,A_1}(\Omega)$; $\overline{\zeta}_{1,k} \to \overline{\zeta}_1$ in $L^{l_1'\overline{l}'_\Gamma}(\Omega)$ and $L^{l_2'\overline{n}'_\Gamma}(\Omega)$; $\overline{\zeta}_{1,k} \to \overline{\zeta}_1$ in a.e in Ω.

(b) $\overline{\zeta}_{2,k} \rightharpoonup \overline{\zeta}_2$ in $X_0^{s,A_2}(\Omega)$; $\overline{\zeta}_{2,k} \to \overline{\zeta}_2$ in $L^{l_2'\overline{l}'_\Gamma}(\Omega)$ and $L^{l_2'\overline{n}'_\Gamma}(\Omega)$; $\overline{\zeta}_{2,k} \to \overline{\zeta}_2$ in a.e in Ω.

Firstly, we suppose that $\overline{\zeta}_1 \neq 0$ or $\overline{\zeta}_2(x) \neq 0$ a.e $x \in \Omega$, has nonzero Lebesgue measure. We can see that

$$|\zeta_{1,k}| = |\overline{\zeta}_{1,k}|\,||\zeta_{1,k}||_{s,A_1} \to +\infty \; in \; [\overline{\zeta}_1 \neq 0],$$

and

$$|\zeta_{2,k}| = |\overline{\zeta}_{2,k}|\,||\zeta_{2,k}||_{s,A_2} \to +\infty \; in \; [\overline{\zeta}_2 \neq 0].$$

Then, by Fatou's Lemma ,(3.5) and Remark 2.3, we have

$$c+1 \geq \int_{\Omega}\overline{F}(x,\zeta_{1,k},\zeta_{2,k})dx \to +\infty.$$

This is a contradiction. Next, we assumed that the two $[\overline{\zeta}_1 \neq 0]$ and $[\overline{\zeta}_2 \neq 0]$ have a null Lebesgue measure, that is $\overline{\zeta}_1 = 0$ in $X_0^{s,A_1}(\Omega)$ and $\overline{\zeta}_2 = 0$ in $X_0^{s,A_2}(\Omega)$. By (2.13), we have

$$\begin{aligned} \min\{||\zeta_{1,k}||_{s,A_1}^{l_1'}, ||\zeta_{1,k}||_{s,A_1}^{n_1'}\} + & \min\{||\zeta_{2,k}||_{s,A_2}^{l_2'}, ||\zeta_{2,k}||_{s,A_2}^{n_2'}\} \\ \leq & \int_{\Omega^2} A_1(|h_{\zeta_{1,k}}|)d\mu + \int_{\Omega^2} A_2(|h_{\zeta_{2,k}}|)d\mu \\ = & I(\zeta_{1,k}, \zeta_{2,k}) + \int_\Omega F(x, \zeta_{1,k}, \zeta_{2,k})dx. \end{aligned} \tag{3.6}$$

When k is sufficiently large,

$$||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'} \leq I(\zeta_{1,k}, \zeta_{2,k}) + \int_\Omega F(x, \zeta_{1,k}, \zeta_{2,k})dx,$$

which is equivalent to

$$\begin{aligned} 1 \leq & \frac{I(\zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} + \int_{|\zeta_{1,k},\zeta_{2,k}| \leq R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} dx \\ + & \int_{|\zeta_{1,k},\zeta_{2,k}| > R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} dx \end{aligned} \tag{3.7}$$

where $R > 0$ is a constant such that $R > r$ (see (F_3)). Since $\{\zeta_{1,k}, \zeta_{2,k}\}$ is $(C)_c$-sequence of I in X, then By equation 3.3 we can see that $I(\zeta_{1,k}, \zeta_{2,k}) \to c$ as $k \to \infty$, then we infer that

$$\frac{I(\zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} \leq o_k(1). \tag{3.8}$$

By (3.7) and (3.8) we have that

$$1 \leq o_k(1) + \int_{|\zeta_{1,k},\zeta_{2,k}| \leq R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} dx + \int_{|\zeta_{1,k},\zeta_{2,k}| > R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} dx, \tag{3.9}$$

keeping in account that $|(\zeta_1, \zeta_2)| > R$ and by (F_2) we have

$$F(x, \zeta_1, \zeta_2) \geq 0, \ x \in \Omega \ .$$

For $|(\zeta_1, \zeta_2)| \leq R$ and since F is continuous, a constant $C_R > 0$ exists such that

$$|F(x, \zeta_1, \zeta_2)| < C_R, \ \forall x \in \Omega, \tag{3.10}$$

then

$$\int_{|\zeta_{1,k},\zeta_{2,k}| \leq R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} dx \leq \frac{C_R|\Omega|}{||\zeta_{1,k}||_{s,A_1}^{l_1'} + ||\zeta_{2,k}||_{s,A_2}^{l_2'}} = o_k(1). \tag{3.11}$$

In addition, it follows from Höder's inequality that

$$\begin{aligned}\int_{|\zeta_{1,k},\zeta_{2,k}|>R} \frac{F(x,\zeta_{1,k},\zeta_{2,k})}{||\zeta_{1,k}||^{l'_1}_{s,A_1}+||\zeta_{2,k}||^{l'_2}_{s,A_2}}dx &= \int_{|\zeta_{1,k},\zeta_{2,k}|>R} \frac{F(x,\zeta_{1,k},\zeta_{2,k})}{\frac{|\zeta_{1,k}|^{l'_1}}{|\overline{\zeta}_{1,k}|^{l'_1}}+\frac{|\zeta_{2,k}|^{l'_2}}{|\overline{\zeta}_{2,k}|^{l'_2}}}dx \\ &\le \int_{|\zeta_{1,k},\zeta_{2,k}|>R} \frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l'_1}+|\zeta_{2,k}|^{l'_2}}(|\overline{\zeta}_{1,k}|^{l'_1}+|\overline{\zeta}_{2,k}|^{l'_2})dx \\ &\le 2\left\|\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l'_1}+|\zeta_{2,k}|^{l'_2}}\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}\right\|_{\Gamma}||(|\overline{\zeta}_{1,k}|^{l'_1}+|\overline{\zeta}_{2,k}|^{l'_2})\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}||_{\overline{\Gamma}},\end{aligned} \tag{3.12}$$

where χ is the characteristic function which satisfies the

$$\chi_{\{|(v_k(x),u_k(x))|>R\}} = \begin{cases} 1 & for\, x\in\{x\in\Omega:|(v_k(x),u_k(x))|>R\}, \\ 0 & for\, x\in\{x\in\Omega:|(v_k(x),u_k(x))|\le R\}. \end{cases}$$

For k large enough, by (1.10), (3.5) and the following fact that $\overline{F}$ is continuous, we get

$$\int_\Omega \Gamma\Big(\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l'_1}+|\zeta_{2,k}|^{l'_2}}\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}\Big)dx \le c_2\int_\Omega \overline{F}(x,\zeta_{1,k},\zeta_{2,k})dx + C \le c_2(c+1)+C$$

So, for k big enough, by (2.13), there is a constant $c_6>0$ such that

$$\left\|\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l'_1}+|\zeta_{2,k}|^{l'_2}}\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}\right\|_{\Gamma} \le c_6. \tag{3.13}$$

Furthermore, it is obvious that

$$||(|\overline{\zeta}_{1,k}|^{l'_1}+|\overline{\zeta}_{2,k}|^{l'_2})\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}||_{\overline{\Gamma}} \le \|\,|\overline{\zeta}_{1,k}|^{l'_1}+|\overline{\zeta}_{2,k}|^{l'_2}||_{\overline{\Gamma}} \le \|\,|\overline{\zeta}_{1,k}|^{l'_1}||_{\overline{\Gamma}} + \|\,|\overline{\zeta}_{2,k}|^{l'_2}||_{\overline{\Gamma}}.$$

According to the lemma 2.2, the lemma 2.5 and (F_3) involves that the N-function $\overline{\Gamma}$ satisfy a Δ_2-condition globally. Then by (2.8), $||\zeta_1||_{\overline{\Gamma}}\to 0$ as $\int_\Omega \overline{\Gamma}(|\zeta_1|)dx \to 0$. It follows from Lemma 2.5 , (a) and (b) in case 1 that

$$\begin{aligned}\int_\Omega \overline{\Gamma}(|\overline{\zeta}_{1,k}|^{l'_1})dx + \int_\Omega \overline{\Gamma}(|\overline{\zeta}_{1,k}|^{l'_2})dx &\le \overline{\Gamma}(1)\int_\Omega \max\{|\overline{\zeta}_{1,k}|^{l'_1\overline{l'}_\Gamma}, |\overline{\zeta}_{1,k}|^{l'_1\overline{n}_\Gamma})dx \\ &+ \overline{\Gamma}(1)\int_\Omega \max\{|\overline{\zeta}_{1,k}|^{l'_2\overline{l'}_\Gamma}, |\overline{\zeta}_{1,k}|^{l'_2\overline{n'}_\Gamma})dx \\ &\le \overline{\Gamma}(1)\Big(\int_\Omega |\overline{\zeta}_{1,k}|^{l'_1\overline{l'}_\Gamma}dx + \int_\Omega |\overline{\zeta}_{1,k}|^{l'_1\overline{n'}_\Gamma}dx \\ &+ \int_\Omega |\overline{\zeta}_{1,k}|^{l'_2\overline{l'}_\Gamma}dx + \int_\Omega |\overline{\zeta}_{1,k}|^{l'_2\overline{n'}_\Gamma}dx\Big) \\ &= o_k(1),\end{aligned}$$

which implies,

$$||(|\overline{\zeta}_{1,k}|^{l'_1}+|\overline{\zeta}_{2,k}|^{l'_2})\chi_{\{|(\zeta_{1,k},\zeta_{2,k})|>R\}}||_{\overline{\Gamma}} \le \|\,|\overline{\zeta}_{1,k}|^{l'_1}||_{\overline{\Gamma}} + \|\,|\overline{\zeta}_{2,k}|^{l'_2}||_{\overline{\Gamma}} = o_k(1). \tag{3.14}$$

If we combine (3.11), (3.12), (3.13), (3.14) with (3.7), we get a contradiction.

Case2. Supposing that $||\zeta_{1,k}||_{s,A_1} \leq C$ or $||\zeta_{2,k}||_{s,A_2} \leq C$ for a certain $C > 0$ and every $k \in \mathbb{N}$. With no loss of generality, we assume that $||\zeta_{1,k}||_{s,A_1} \to +\infty$ and $||\zeta_{2,k}||_{s,A_2} \leq C$, for some $C > 0$ and for all $k \in \mathbb{N}$. Let $\overline{\zeta}_{1,k} = \dfrac{\zeta_{1,k}}{||\zeta_{1,k}||_{s,A_1}}$ and $\overline{\zeta}_{2,k} = \dfrac{\zeta_{2,k}}{||\zeta_{2,k}||_{s,A_1}}$ then $||\overline{\zeta}_{1,k}||_{s,A_2} \to 0$ and $||\overline{\zeta}_{2,k}||_{s,A_1} \to 1$. According to the remark 2.9, there is a point $(\overline{u}, \overline{\zeta}_1) \in X$ such that:

(c) $\overline{\zeta}_{2,k} \rightharpoonup \overline{u}$ in $X_0^{s,A_1}(\Omega)$, $\overline{\zeta}_{2,k} \to \overline{u}$ in $L^{l_1'\overline{l'}_\Gamma}(\Omega)$ and $L^{l_2'\overline{n'}_\Gamma}(\Omega)$ $\overline{\zeta}_{2,k} \to \overline{u}$ in a.e in Ω,

(d) $\overline{\zeta}_{1,k} \rightharpoonup \overline{\zeta}_1$ in $X_0^{s,A_2}(\Omega)$; $\overline{\zeta}_{1,k} \to \overline{\zeta}_1$ in $L^{l_2'\overline{l'}_\Gamma}(\Omega)$ and $L^{l_2'\overline{n'}_\Gamma}(\Omega)$; $\overline{\zeta}_{1,k} \to \overline{\zeta}_1$ in a.e in Ω. Likewise, we first guess that $[\overline{u} \neq 0\ ]$ has a non null Lebesgue measure. We can then see that

$$|\zeta_{2,k}| = |\overline{\zeta}_{2,k}|\ ||\zeta_{2,k}||_{s,A_1} \to +\infty,\ in\ [\overline{u} \neq 0].$$

Further, by (3.5), the remark 2.3 and Fatou's lemma, we get a contradiction by

$$c + 1 \geq \int_\Omega \overline{F}(x, \zeta_{1,k}, \zeta_{2,k})dx \to +\infty.$$

Next, we assume that $[\overline{u} \neq 0]$ has a zero Lebesgue measure, i.e. $\overline{u} = 0$ in $X_0^{s,A_1}(\Omega)$. By Lemma 2.5 and (c) and (d) in case 2, we have

$$\begin{aligned} \min\left\{ \|\, |\zeta_{1,k}|^{l_2'} \|_{\overline{\Gamma}}^{\overline{l}_\Gamma},\ \|\, |\zeta_{1,k}|^{l_2'} \|_{\overline{\Gamma}}^{\overline{n}_\Gamma} \right\} &\leq \int_\Omega \overline{\Gamma}(|\zeta_{1,k}|^{l_2'})dx \\ &\leq \overline{\Gamma}(1) \int_\Omega \max\{(|\zeta_{1,k}|^{l_2'\overline{l'}_\Gamma}, |\zeta_{1,k}|^{l_2'\overline{n'}_\Gamma}\}dx \\ &\leq \overline{\Gamma}(1)\left(\int_\Omega |\zeta_{1,k}|^{l_2'\overline{l'}_\Gamma}dx + \int_\Omega |\zeta_{1,k}|^{l_2'\overline{n'}_\Gamma} \right)dx \to C, \end{aligned}$$

Then a constant $L > 0$ exists such that

$$\|\, |\zeta_{1,k}|^{l_2'} \|_{\overline{\Gamma}} \leq L,\ \forall k \in \mathbb{N}. \tag{3.15}$$

If k is big according to the following equation, (3.6) turns into

$$\|v_k\|_{s,A_1}^{l_1'} + K \leq I(\zeta_{1,k}, \zeta_{2,k}) + \int_\Omega F(x, \zeta_{1,k}, \zeta_{2,k}) + K,$$

where K is a constant and $K > 4Lc_6$. (see (3.13) and (3.15)). Then by (3.8), (3.10) , (3.13), (3.14) , (3.15) and Höder's Inequality, above estimate means

$$\begin{aligned} 1 &\leq \frac{I(\zeta_{1,k}, \zeta_{2,k}) + K}{\|v_k\|_{s,A_1}^{l_1'} + K} + \int_\Omega \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{\|v_k\|_{s,A_1}^{l_1'} + K}dx \\ &= o_k(1) + \int_{|(\zeta_{1,k}, \zeta_{2,k})| \leq R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{\|v_k\|_{s,A_1}^{l_1'} + K}dx + \int_{|(\zeta_{1,k}, \zeta_{2,k})| > R} \frac{F(x, \zeta_{1,k}, \zeta_{2,k})}{\|v_k\|_{s,A_1}^{l_1'} + K}dx \end{aligned}$$

$$
\begin{aligned}
&= o_k(1)+\int_{|(\zeta_{1,k},\zeta_{2,k})|>R}\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{\|v_k\|_{s,A_1}^{l_1'}+K}dx\\
&\le o_k(1)+\int_{|(\zeta_{1,k},\zeta_{2,k})|>R}\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l_1'}+|\zeta_{2,k}|^{l_2'}}\Big(|\overline{\zeta}_{1,k}|^{l_1'}+\frac{1}{K}|\overline{\zeta}_{1,k}|^{l_2'}\Big)dx\\
&\le o_k(1)+\Big\|\frac{F(x,\zeta_{1,k},\zeta_{2,k})}{|\zeta_{1,k}|^{l_1'}+|\zeta_{2,k}|^{l_2'}}\chi\{\,|(\zeta_{1,k},\zeta_{2,k})|>R\}\Big\|_{\Gamma}\\
&\times\ \Big\|\Big(|\overline{\zeta}_{1,k}|^{l_1'}+\frac{1}{K}|\overline{\zeta}_{2,k}|^{l_2'}\Big)\chi\{|(\zeta_{1,k},\zeta_{2,k})|>R\}_{\overline{\Gamma}}\\
&\le o_k(1)+2c_6\Big(\|\,|\overline{\zeta}_{1,k}|^{l_1'}\|_{\overline{\Gamma}}+\frac{1}{K}\|\,|\overline{\zeta}_{2,k}|^{l_2'}\|_{\overline{\Gamma}}\Big)\\
&\le o_k(1)+2c_6\big(o_k(1)+\frac{L}{K}\big)=o_k(1)+\frac{2Lc_6}{K}<o_k(1)+\frac{1}{2},
\end{aligned}
$$

which is a contradiction.

Step 3: Shown that, the energy function I satisfy the condition $(C)_c$. Let $\{(\zeta_{1,k},\zeta_{2,k})\}$ be some $(C)_c$-sequence of I in X. In the proof of step 2, we have shown that $\{(\zeta_{1,k},\zeta_{2,k})\}$ is bounded. Passing to a subsequence denote by $\{(\zeta_{1,k},\zeta_{2,k})\}$, by Remark 2.9, there exists a point $(\zeta_1,\zeta_2)\in X$ such that:

(e) $\zeta_{1,k}\rightharpoonup\zeta_1$ in $X_0^{s,A_1}(\Omega)$, $\zeta_{1,k}\to\zeta_1$ in $L^{\Psi_1}(\Omega)$, $\zeta_{1,k}\to\zeta_1$ a.e Ω.

(f) $\zeta_{2,k}\rightharpoonup\zeta_2$ in $X_0^{s,A_2}(\Omega)$, $\zeta_{2,k}\to\zeta_2$ in $L^{\Psi_2}(\Omega)$, $\zeta_{2,k}\to\zeta_2$ a.e Ω. then we have

$$
\begin{aligned}
&\langle[\mathcal{F}_1'(\zeta_{1,k}),\zeta_{1,k}-\zeta_1\rangle\\
&=\int_{\Omega^2}a_1(|h_{\zeta_{1,k}}|)h_{\zeta_{1,k}-\zeta_1}d\mu\\
&=\langle I'(\zeta_{1,k},\zeta_{2,k}),(\zeta_{1,k}-\zeta_1,0)\rangle+\int_\Omega\frac{\partial F}{\partial\zeta_1}(x,\zeta_{1,k},\zeta_{2,k})(\zeta_{1,k}-\zeta_1)dx.
\end{aligned}
\tag{3.16}
$$

Equation (3.3) shows that

$$
\Big|\langle I'(\zeta_{1,k},\zeta_{2,k}),(\zeta_{1,k}-\zeta_1,0)\rangle\Big|\le\|I'(\zeta_{1,k},\zeta_{2,k})\|_{(X_0^{s,A_1})^*}\|\zeta_{1,k}-\zeta_1\|_{X_0^{s,A_1}}\to 0.
\tag{3.17}
$$

By (F_1) and Höder's inequality, we get

$$
\begin{aligned}
&\Big|\int_\Omega\frac{\partial F}{\partial\zeta_1}(x,\zeta_{1,k},\zeta_{2,k})(\zeta_{1,k}-\zeta_1)dx\Big|\\
\le\ & c_1\int_\Omega(\psi_1(|\zeta_{1,k}|)+\overline{\Psi}_1^{-1}(\Psi_2(|\zeta_{2,k}|)))|\zeta_{1,k}-\zeta_1|dx\\
\le\ & 2c_1||\psi_1(|\zeta_{1,k}|)+\overline{\Psi}_1^{-1}(\Psi_2(|\zeta_{2,k}|))||_{\overline{\Psi}_1}||\zeta_{1,k}-\zeta_1||_{\Psi_1}.
\end{aligned}
\tag{3.18}
$$

The condition (F_1) proves that the functions Ψ_1 and $\overline{\Psi}_1$ are Orlicz functions globally fulfilling the condition Δ_2, which together with the convexity of the N-function, (2.13), the remark 2.9

and the boundedness of $\{(\zeta_{1,k}, \zeta_{2,k})\}$, imply that

$$\int_\Omega \overline{\Psi}_1(\psi_1(|\zeta_{1,k}|) + \overline{\Psi}_1^{-1}(\Psi_2(|\zeta_{2,k}|)))dx \leq C \int_\Omega (\Psi_1(\zeta_{1,k}) + \Psi_2(\zeta_{2,k}))dx \leq C,$$

which together with (2.13) shows again that

$$||\psi_1(|\zeta_{1,k}|) + \overline{\Psi}_1^{-1}(\Psi_2(|\zeta_{2,k}|))||_{\overline{\Psi}_1} \leq C, \tag{3.19}$$

for a certain $C > 0$. Furthermore, (e) and (f) show that

$$||\zeta_{1,k} - \zeta_1||_{\Psi_1} \to 0. \tag{3.20}$$

then, combining (3.16), (3.17), (3.18), (3.19) and (3.20) we obtain

$$\langle \mathcal{F}_1'(\zeta_{1,k}), \zeta_{1,k} - \zeta_1 \rangle \to 0 \ \ when \ \ k \to \infty.$$

By Lemma 3.4 we infer that $\{(\zeta_{1,k}, \zeta_{2,k})\} \to (\zeta_1, \zeta_2)$ in X. □

Conclusion (Proof of Theorem 3.1). The energy function I satisfy all conditions of Lemma 3.2 and the obvious fact $I(0) = 0$. So the system (1.5) has a weak non-trivial solution that is a critical point of I.

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