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EUROPEAN OPTION PRICING MODEL: A NOVEL SEMI-ANALYTICAL SOLUTION

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Abstract. This study proposes a new analytical solution to the Black-Scholes Model (BSM) for European call option pricing. This approach is based on the reduced differential transform method (RDTM). The European call option prices calculated using RDTM were consistent with those derived using the BSM. Furthermore, the RDT approach can be concluded to be very effective and dependable. As a result, it is assumed that the assets are driven by geometric Brownian motion and do not pay dividends.

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Key words and phrases. geometric Brownian motion; homogeneous Black-Scholes equation; reduced differential transform method; valuation; vanilla call option.

1. Introduction

In various mathematical issues, including those involving fluid mechanics, thermodynamics, and mathematical physics, differential equations have been applied. The analytical method has recently become very popular for solving linear and non-linear partial differential equations (PDEs). Zhou [1] introduced the differential transform method as one of the analytical techniques to solve linear or non-linear PDEs and scales down computational labor. It is possible to acquire precise solutions using this method without the need for any additional challenging computations, making it a helpful tool for hybrid solutions; for more information, see [2]. Hybrid methods of two-dimensional differential transform and least squares approaches were employed by [3] to solve two-dimensional non-linear wave equations. The projected differential transform method was considered by [4] as a new modified variation of the differential transform approach for solving non-linear Klein-Gordon and Schrödinger

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equations. See [5,7–10] for further information on the theory and uses of the RDTM. The pricing of the vanilla option with a European flavor via RDTM is given in this work as a convergent power series solution of the homogeneous BSM "governing equation". The remainder of the study is organized as follows: Some preliminary materials used in this paper are included in Section 2. The homogeneous BSM for the pricing of a vanilla option with a European flavor is described in Section 3. The applications of the RDTM are covered in Section 4, along with a discussion of the findings and closing remarks.

2. Preliminaries

Definition 1. *With regard to time t and space x in the area of interest, if $u(x,t)$ is analytical and continuously differentiable, the reduced differential transform (RDT) is defined as [7,8]*

$$\Omega_k(x) = \frac{1}{k!}\left[\frac{\partial^k u(x,t)}{\partial t^k}\right]_{t=0}, \tag{1}$$

On the other hand, RDT inversion of $\Omega_k(x)$ has the following definition [7]:

$$\omega(x,t) = \sum_{k=0}^{\infty} \Omega_k(x) t^k. \tag{2}$$

The following results show some of the attributes of RDTM that are relevant to the current investigation [7].

Theorem 1. *If $\beta(x,t) = \zeta(x,t) \pm \eta_(x,t)$, then $\beta_k^*(x) = \zeta_k^*(x) \pm \eta_k^*(x)$.*

Theorem 2. *If $\beta(x,t) = q\zeta(x,t)$, then $\beta_k^*(x) = q\zeta_k^*(x)$.*

Theorem 3. *If $\beta(x,t) = \frac{\partial \zeta(x,t)}{\partial x}$, then $\beta_k^*(x) = \frac{\partial \zeta_k^*(x)}{\partial x}$.*

Theorem 4. *If $\beta(x,t) = \frac{\partial^r \zeta(x,t)}{\partial t^r}$, then $\beta_k^*(x) = \frac{(k+r)!\zeta_{k+r}^*(x)}{k!}$.*

3. European Option Pricing Model

Out of the different varieties of financial derivatives, the option happens to be one of the most crucial financial tool. Option is among the asset class and if used correctly, it produces more effective advantages that trading stocks and Exchange Trust Funds (ETFs) alone cannot produce. Option traders increasingly used the Black-Scholes model to predict and evaluate option prices over time, and it is now virtually generally recognized. By including certain assumptions about option markets, Black-Scholes is modeled with stock price changes as a stochastic process [11].

Theorem 5. *Take into account the homogeneous BSM for the cost of a vanilla call option with a European flavor, denoted by $U_c(S,t)$, with a non-dividend yield of the following form:*

$$\frac{\partial U_c(S,t)}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 U_c(S,t)}{\partial S^2} + rS\frac{\partial U_c(S,t)}{\partial S} - rU_c(S,t) = 0, \tag{3}$$

subject to the conditions:

$$\lim_{S\to\infty} U_c(S,t) = S, \lim_{S\to 0} U_c(S,t) = 0, 0 \le t \le T \tag{4}$$

$$U_c(S,T) = \max(S_T - K, 0), S \in (0,\infty). \tag{5}$$

where the price of the underlying asset is S, the strike price is K, the volatility of the underlying asset is σ, the period to expiration is T, the time now is t, and the risk-free interest rate is r. The variables transformation allows (3) to be reduced to a standard boundary value problem.

Proof. To transform (3) into a common boundary value problem, we get rid of the terms $\frac{\partial U_c(S,t)}{\partial t}$ and $\frac{\partial^2 U_c(S,t)}{\partial S^2}$ using the following variables transformation

$$S = Ke^x \Rightarrow x = \ln\left(\frac{S}{K}\right), \tag{6}$$

$$\tau = \frac{\sigma^2}{2}(T-t), \tag{7}$$

and

$$U_c(S,t) = Ku(x,\tau). \tag{8}$$

Thus, (3) and (4) reduce to

$$\frac{\partial u}{\partial \tau} = \frac{\partial^2 u}{\partial x^2} + (w-1)\frac{\partial u}{\partial x} - wu, \tag{9}$$

and

$$u(x,0) = \max(e^x - 1, 0) = (e^x - 1)^+. \tag{10}$$

respectively, with $w = \frac{2r}{\sigma^2}$. This completes the proof. □

4. Applications of the Method, Discussion of the Findings, and Final Remarks

This section presents applications of the method, discussion of the findings and final remarks.

4.1. Applications of the Method.

Here, the applications of the method are presented.

4.1.1. *Application 1.*

The solution of (9) subject to (10) via the reduced differential transform method is presented in the following result as follows.

Theorem 6. *Consider the common boundary value problem presented in equations (9) and (10). Then, the RDTM's derived solution is provided by*

$$u(x,\tau) = \max(e^x - 1, 0)\exp(-w\tau) + \max(e^x, 0)(1 - \exp(-w\tau)),$$

Proof. By definition 1, (9) and (10) become

$$(m+1)U_{m+1}(x) = \frac{\partial^2 U_m(x)}{\partial x^2} + (w-1)\frac{\partial U_m(x)}{\partial x} - wU_m(x) \tag{11}$$

and

$$U_0(x) = \max(e^x - 1, 0) = (e^x - 1)^+, x > 0. \tag{12}$$

respectively. Using (11) and (12), we obtain the following relations

$$U_1(x) = w((e^x)^+ - (e^x - 1)^+), \tag{13}$$

$$U_2(x) = -\frac{w^2}{2!}\left((e^x)^+ - (e^x - 1)^+\right), \tag{14}$$

$$U_3(x) = \frac{w^3}{3!}\left((e^x)^+ - (e^x - 1)^+\right), \tag{15}$$

$$U_4(x) = -\frac{w^4}{4!}\left((e^x)^+ - (e^x - 1)^+\right), \tag{16}$$

$$U_5(x) = \frac{w^5}{5!}\left((e^x)^+ - (e^x - 1)^+\right). \tag{17}$$

Continuing in this way and by means of the RDTM inversion formula in definition 1,

$$u(x,\tau) = \max(e^x - 1, 0)e^{-w\tau} + \max(e^x, 0)(1 - e^{-w\tau}), x > 0. \tag{18}$$

□

The following remarks are deduced from subsection 4.1.1.

Remark 1. *Equation (18) is the required solution of (9) subject to (10).*

Remark 2. *Under the geometric Brownian motion, the price of the vanilla call option with a European flair is calculated by*

$$U_c(S,\tau) = Ku(x,\tau), \tag{19}$$

with $u(x,\tau)$ given by (18),

$$x = \ln\left(\frac{S}{K}\right), w = \frac{2r}{\sigma^2}, \tau = \frac{\sigma^2}{2}(T-t).$$

Remark 3. *The famous valuation formula of vanilla call option of European flair is given by [11]*

$$U_c(S,t) = S\mathcal{N}(d_1) - K\exp(-r(T-t))\mathcal{N}(d_2), \tag{20}$$

where, $\mathcal{N}$ is the cumulative normal density function,

$$d_1 = \frac{\ln(\frac{S}{K}) + \left(r + \frac{\sigma^2}{2}\right)(T-t)}{\sigma\sqrt{T-t}}, \tag{21}$$

and

$$d_2 = d_1 - \sigma\sqrt{T-t}. \tag{22}$$

Remark 4. *The standard boundary problem for the European put option is solved using the RDT approach as shown in the ensuing result.*

Theorem 7. *Consider the standard boundary value problem for the European put option given by*

$$\frac{\partial v}{\partial \tau} = \frac{\partial^2 v}{\partial x^2} + (w-1)\frac{\partial v}{\partial x} - wv, \tag{23}$$

and

$$v(x,0) = \max(1-e^x, 0) = (1-e^x)^+. \tag{24}$$

respectively, with $w = \frac{2r}{\sigma^2}$. Then the required solution is given by

$$v(x,\tau) = e^{-w\tau} - e^x + \max(e^x - 1, 0)e^{-w\tau} + \max(e^x, 0)(1 - e^{-w\tau}),$$

Remark 5. *The European put option price is given in the following result using the RDTM.*

Theorem 8. *Using the put-call parity of the form*

$$V_p(S,\tau) = Ke^{-w\tau} - Ke^x + U_c(S,\tau) \tag{25}$$

and the European call option price given by (19). The European put option price is obtained as

$$\begin{aligned} V_p(S,\tau) &= Kv(x,\tau), \\ &= Ke^{-w\tau} - Ke^x \\ &+ K\max(e^x - 1, 0)e^{-w\tau}, \\ &+ K\max(e^x, 0)(1 - e^{-w\tau}), \end{aligned} \tag{26}$$

with

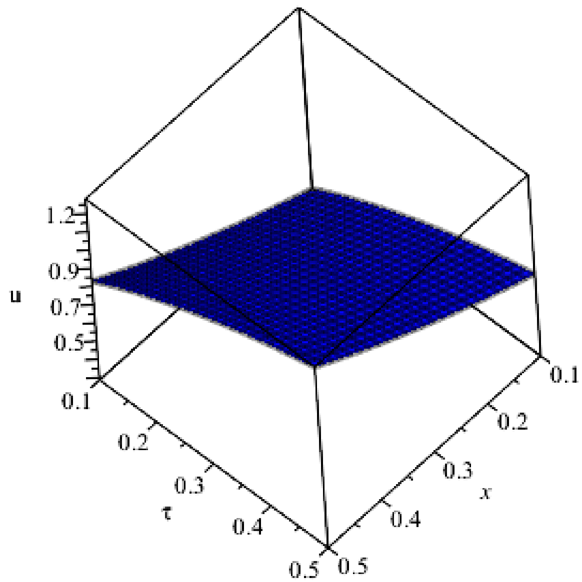
$$x = \ln\left(\frac{S}{K}\right), w = \frac{2r}{\sigma^2}, \tau = \frac{\sigma^2}{2}(T-t).$$

Remark 6. *The European-style vanilla put option is valued using the Black-Scholes model, which is computed as [11]*

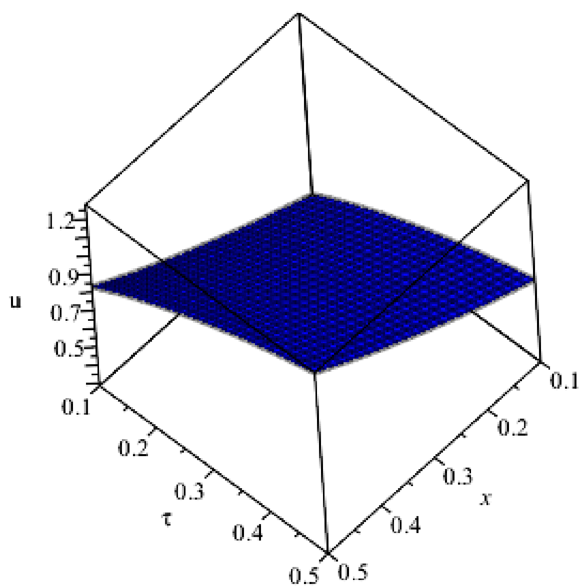
$$V_p(S,t) = K\exp(-r(T-t))\mathcal{N}(d_2) - S\mathcal{N}(d_1), \tag{27}$$

where, d_1 and d_2 are given by (21) and (22), respectively.

Remark 7. *The plots of RDTM (18) with different values of x and τ are displayed in Figures 1, 2 and 3.*

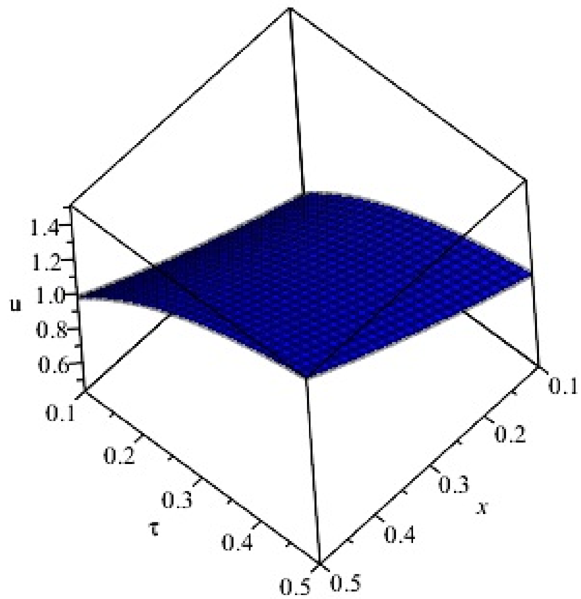


(A) RDTM

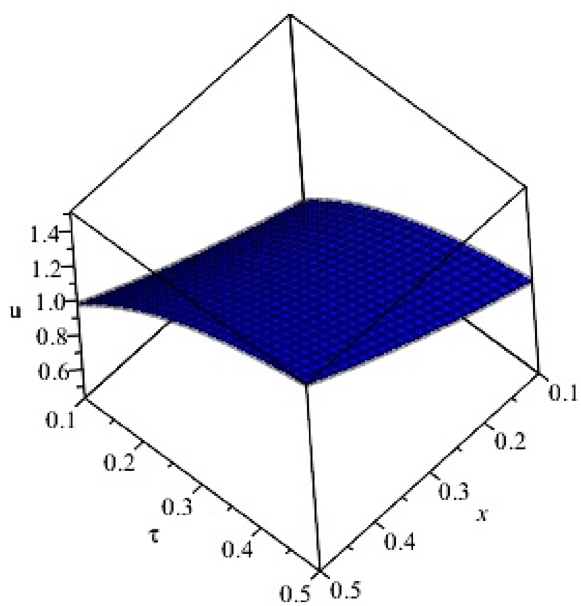


(B) Exact solution

FIGURE 1. RDTM (18) versus exact solution with fixed $w = 2$ and different values of x and τ

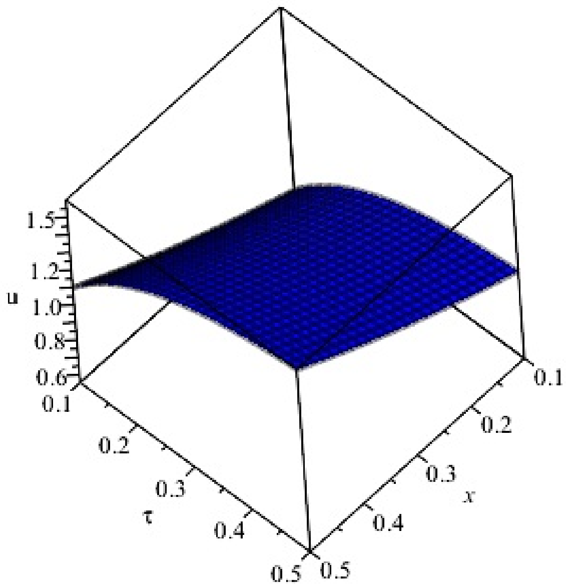


(A) RDTM

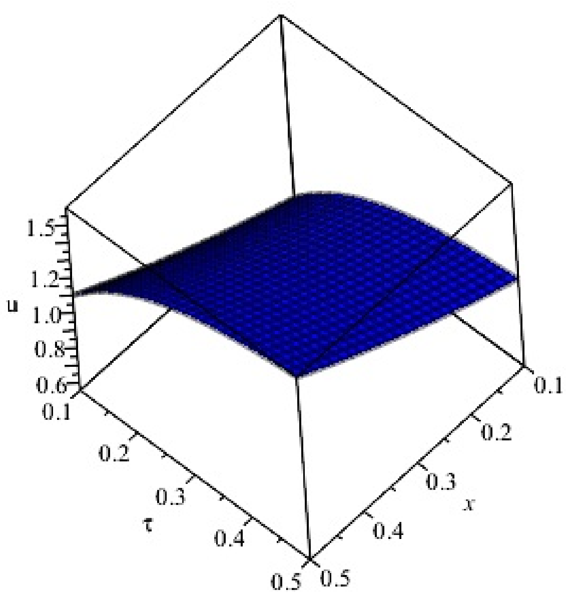


(B) Exact solution

FIGURE 2. RDTM (18) versus exact solution with fixed $w = 4$ and different values of x and τ



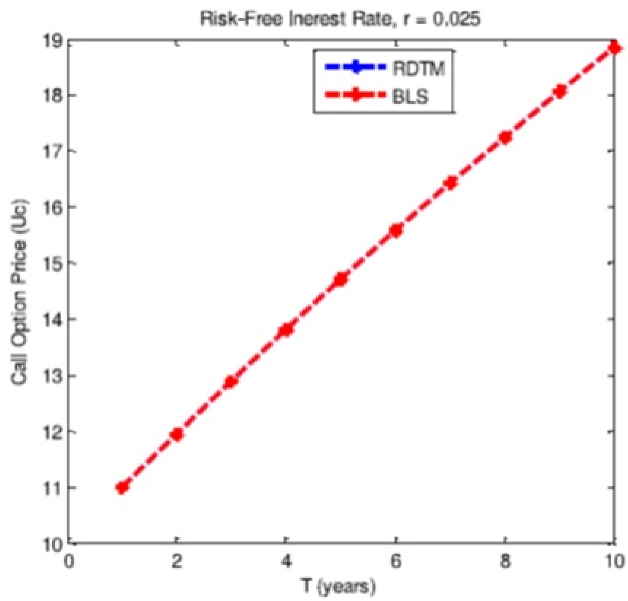
(A) RDTM



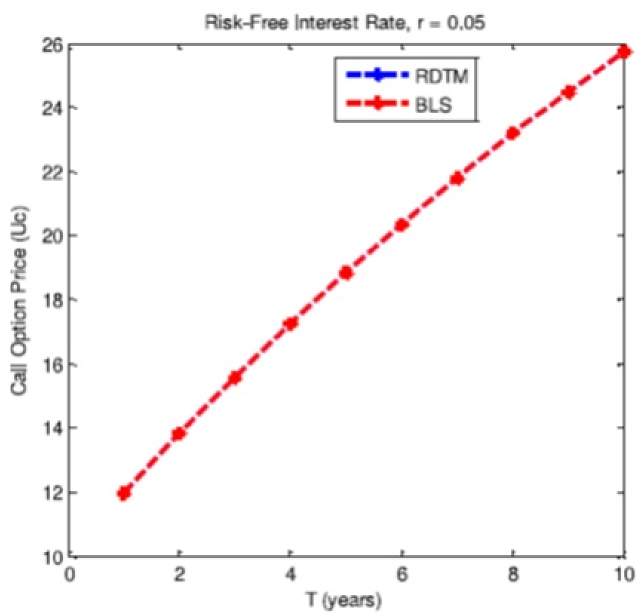
(B) Exact solution

FIGURE 3. RDTM (18) versus exact solution with fixed $w = 6$ and different values of x and τ

4.1.2. *Application 2.* Consider the governing PDE (9) with the following inputs: $S = 50, K = 40$, and $\sigma = 0.02$ for the cost of the plain call option with a European flavor. Figure 4 illustrates a comparison of the call option pricing calculated using the Black-Scholes model (BLS) [11] and the RDTM (19), with various values for the period to expiration (T) and risk-free interest rate (r).



(A) RDTM and BLS



(B) RTDM and BLS

FIGURE 4. The comparative study of RDTM Solution and BLS [11] with $r = 0.025, 0.05$.

4.1.3. *Application 3.* Let us have a look at the pricing of a European call option with 12 months left before expiry, the strike price, $K = 0.8$, the risk-neutral interest rate is $r = 0.03$ and the volatility is $\sigma = 0.25$. Table 1 displays the exact solution [11] and RDTM's results.

TABLE 1. RDTM's results and exact solution [11]

σ	r	T	K	S	RDTM	BLS
25%	3%	1	0.8	21.74	20.96	20.96
25%	3%	1	0.8	26.62	25.84	25.84
25%	3%	1	0.8	30.56	29.78	29.78
25%	3%	1	0.8	34.90	34.12	34.12
25%	3%	1	0.8	39.38	38.60	38.60
25%	3%	1	0.8	43.86	43.08	43.08
25%	3%	1	0.8	48.20	47.42	47.42
25%	3%	1	0.8	52.14	51.34	51.34
25%	3%	1	0.8	56.48	55.70	55.70
25%	3%	1	0.8	60.96	60.18	60.18

4.2. **Discussion of the Findings, and Final Remarks.** The homogeneous Black-Scholes PDE of vanilla option with a European flavor and non-dividend yield is described with a novel analytical solution in this study. The Black-Scholes PDE has been transformed into a common boundary value issue using the variable transformation. The RDTM was used to resolve the common boundary value problem. The obtained result matched the original one exactly. Figures 1, 2, and 3 show that the results produced using RDTM and the precise solution are comparable. Additionally, it has been noted that the value of $u(x, \tau)$ grows for a range of x and t values. Figure 4 shows that the European call option prices produced using the reduced differential transform approach and the Black-Scholes model valuation formula are in accord. In other words, the reduced differential transform method's curves elegantly mimic the Black-Scholes curves. Figure 4 makes it quite evident that longer expiration dates result in higher prices for European call options. Figure 4 also shows that the risk-free interest rate (r) affects the pricing of European call options; that is, as r rises, the prices of vanilla call options likewise rise. Additionally, the RDT approach fixes the flaw that is primarily brought on by unsatisfactory conditions. As a result, it can be said that the RDTM is extremely effective, converges quickly to the exact solution, is dependable, and requires less computational effort to find analytical and numerical solutions to differential equations, particularly the Black-Scholes PDE discussed in this paper as seen in Table 1. In order to value vanilla options with a European flair, the RDTM is an excellent tool to include in the class of differential transform methods.

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