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APPROXIMATING FINITE FAMILIES OF RESOLVENTS OF CONVEX FUNCTIONS AND FIXED POINT PROBLEM OF DEMICONTRACTIVE TYPE MAPPINGS

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Abstract. In this paper, we introduce and study a modified Halpern iterative method for approximating a common solution of finite families of the resolvents of convex functions and fixed point problems of demicontractive-type mappings in Hadamard spaces. Under some mild assumptions, we prove a strong convergence theorem of the sequence generated by the modified Halpern method to an element in the intersection of the set of solutions of the aforementioned problems. We present some consequences and applications of our main result. Our results improves and generalizes many recent results in the literature.

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1. Introduction

Let C be a nonempty, closed and convex subset of a Hadamard space X, and let $T : C \to C$ be a nonlinear mapping. The fixed point problem (FPP) is to find a point $x \in C$ such that

$$x = Tx. \tag{1}$$

Throughout this article, we denote by $Fix(T)$ the fixed point sets of the mapping T. Fixed point theory is an area of nonlinear analysis that has been extensively studied by mathematicians. Fixed point theorem, in particular, applies in proving the existence of solutions of differential equations, and the existence of solutions of optimization problems. For instance, a solution of an equilibrium problem is a fixed point of the resolvent of the monotone operator associated with the monotone equilibrium problems, also a solution of a minimization problem is the fixed point of resolvent of convex function associated with the convex minimization problems. Thus, the fixed point theorem is very important

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tool for solving optimization problems.

Let X be an Hadamard space and $f : X \to (-\infty, \infty]$ be a proper, convex and lower-semicontinuous function. One of the most important problems in convex analysis is the Convex Minimization Problem (in short, CMP), which is to find $x^* \in X$ such that

$$f(x^*) = \min_{y \in X} f(y). \tag{2}$$

We denote by $arg \min_{y \in X} f(y)$ the set of minimizers of f in X. For $\lambda > 0$, the resolvent of a lower semi-continuous function f in X is defined as

$$J_\lambda^f(x) = arg \min_{y \in X} \big[f(y) + \frac{1}{2\lambda} d^2(y, x)\big]. \tag{3}$$

It is also known that $Fix(J_\lambda^f)$ coincides with $arg \min_{y \in X} f(y)$. CMP provides us with algorithms for solving a variety of problem which may appear in science and engineering, and one of the most popular methods for approximation of a minimizer of a convex function is the $proximal\ point\ algorithm$ (PPA), which was introduced by Martinet [25] and Rockafellar [30] in Hilbert spaces. Indeed, let f be a proper, convex and lower semicontinuous function on a real Hilbert space H which attains its minimum. The PPA is find by $x_1 \in H$ such that

$$x_{n+1} = arg \min_{y \in H} \big(f(y) + \frac{1}{2\lambda_n} \|y - x_n\|^2\big), \ \lambda_n > 0, \ \forall n \geq 1. \tag{4}$$

It was proved that the sequence $\{x_n\}$ converges weakly to a minimizer of f provided $\sum_{n=1}^{\infty} \lambda_n = \infty$. However, as shown by $Güer$ [16], the PPA does not necessarily converges strongly (i.e., convergence in metric) in general. To obtain a strong convergence of the PPA, Xu [34], Kamimura and Takahashi [21] introduced a Halpern-type regularization of the PPA in Hilbert space. They proved a strong convergence of Halpern PPA under some certain conditions on the parameters. Recently, many convergence results of PPA for solving optimization problems have been extended from the classical linear spaces such as Euclidean spaces, Hilbert spaces and Banach spaces to the setting of Hadamard spaces [1,5,15,24,28,33]. The minimizers of the objective convex functionals in the spaces with nonlinearity play a crucial role in the branch of analysis and geometry.

In 2013, *Bačák* [6] studied the MP (2) in $CAT(0)$ spaces using the following iterative algorithm. Let $x_1 \in X$, then

$$x_{n+1} = arg \min_{y \in X} \big[f(y) + \frac{1}{2\lambda} d^2(y, x_n)\big], \tag{5}$$

where $\lambda > 0$ for all $n \in \mathbb{N}$. He proved that $\{x_n\}$ is Δ-convergent to the minimizer of f under the conditions that f has a minimizer in X and $\sum_{n=1}^{\infty} \lambda_n = \infty$. Suparatulatorn et al. [32] introduced the following modified Halpern iteration process for solving CMP (2) and nonexpansive mapping in the framework of $CAT(0)$ spaces.

Suppose that $u, x_1 \in X$ are arbitrary chosen and $\{x_n\}$ is generated in the following manner:

$$\begin{cases} y_n = arg \min_{y\in X} \left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right]; \\ x_{n+1} = \alpha_n u \oplus (1-\alpha_n) T y_n; \end{cases} \tag{6}$$

for each $n \in \mathbb{N}$, where $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the following conditions:

(1) $\lim\limits_{n\to\infty} \alpha_n = 0$;

(2) $\sum\limits_{n=1}^{\infty} \alpha_n = \infty$;

(3) $\sum\limits_{n=1}^{\infty} |\alpha_n - \alpha_{n+1}| < \infty$;

(4) $\sum\limits_{n=1}^{\infty} |\lambda_n - \lambda_{n+1}| < \infty$.

Then $\{x_n\}$ strongly converges to $z \in \Gamma := Fix(T) \cap arg \min\limits_{y\in X} f(y) \neq \emptyset$, which is the nearest point of *Γ to u.*

In 2015, Cholamjiak et al. [9] introduced the following modified PPA involving fixed point iterates for two nonexpansive mappings and proved that the sequence generated by their iterative process converges to a minimizer of a convex function and a fixed point problem of two nonexpansive mappings. Let $\{x_n\}$ be generated in the following manner:

$$\begin{cases} z_n = arg \min_{y\in X} \left[f(y) + \frac{1}{2\lambda_n} d^2(y, x_n)\right]; \\ y_n = \big(1-\beta_n\big) x_n \oplus \beta_n T_1 z_n; \\ x_{n+1} = \big(1-\alpha_n\big) T_1 x_n \oplus \alpha_n T_2 y_n; \end{cases} \tag{7}$$

for each $n \in \mathbb{N}$. Then, the sequence $\{x_n\}$ Δ-converges to an element of Ω, where $\Omega := Fix(T_1) \cap Fix(T_2) \cap arg \min\limits_{y\in X} f(y)$.

In 2017, Asawathep Cuntavepanit and Withun Phuenggrattana [10] proposed on iterative method for solving the common solution of convex minimization problem and fixed point problem for a finite family of nonexpansive mappings in $CAT(0)$ spaces. They proved the following theorem:

Theorem 1.1: Let C be a nonempty closed convex subset of a complete $CAT(0)$ space (X, d) and $f : C \to (-\infty, \infty]$ be a proper convex and lower semi-continuous function. Let $\{T_i\}_{i=1}^{N}$ be a finite family of nonexpansive mappings of C into itself with $\Omega = \bigcap\limits_{i=1}^{N} Fix(T_i) \cap arg \min\limits_{y\in C} f(y)$ is nonempty. Assume that $\{\lambda_n\}$ is a sequence such that $\lambda_n > 0$ for all $n \in \mathbb{N}$. Let $\{x_n\}$ be generated in the following manner:

$$y_n = arg \min \left[f(y) + \frac{1}{2\lambda_n} d(y, x_n)^2\right], \tag{8}$$

$$z_n = \bigoplus_{i=n}^{\mathbb{N}} y_n^{(i)} T_i y_n,$$

$$x_{n+1} = (1-\alpha_n) x_n \oplus \alpha_n z_n, \; n \in \mathbb{N},$$

where $T_0 = I$ (identity mapping), $\{\alpha_n\} \subset (0,1),\ and\ \{\gamma_n^{(i)}\} \subset (0,1) for\ all\ i = 0,1,\cdots,N$ with $\sum_{i=0}^{N} \gamma_n^{(1)} = 1\ for\ all\ n \in \mathbb{N}$. Then $\lim_{n\to\infty} d(x_n, p)$ exists. In this article, we consider the following problem: Let C be a nonempty, closed and convex subset of an Hadamard space X and $h_j : X \to \mathbb{R},\ j = 1,2\cdots,N$ be proper, convex and lower semi-continuous function. For $\lambda > 0$, define the Moreau-Yosida resolvent of $h_j\ on\ C$ by

$$J_\lambda^{h_j}(x) = arg \min_{y\in C}(h_j(y) + \frac{1}{2\lambda} d^2(y,x)),\ j = 1,2,\cdots,N, \tag{9}$$

and denote by

$$\prod_{j=1}^{N} J_\lambda^{h_j} = J_\lambda^{h_j} \circ J_\lambda^{h_{j-1}} \circ, \cdots \circ, J_\lambda^{h_2} \circ J_\lambda^{h_1},\ j = 1,2,\cdots,N. \tag{10}$$

Let $S : X \to X$ be a μ-demicontractive-type mapping (see chapter 2 for definitions) with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0,1)$ such that S is Δ-demiclosed at 0. The problem is to find:

$$x \in Fix(S) \cap \bigcap_{j=1}^{N} Fix(J_\lambda^{h_j}). \tag{11}$$

We denote by Ω, the solution set of (11) and it is assumed to be nonempty. Inspired by the aforementioned results, using the fixed point approach, we propose a modified Halpern method for solving finite families of resolvents of convex functions and fixed point of demicontractive type mappings in the framework of an Hadamard space. We present some of the consequences of our results and give applications for finding the Fréchet mean, mean of probability and convex feasibility problems. The result discuss in this article extends and complements many related results in the literature.

We highlight some of our contributions in this article as follows:

(1) The classes of mapping considered in this study generalized the classes of nonexpansive, quasi-nonexpansive and demicontractive.
(2) The problem discussed in Suparatulatorn [32] and Cholamjiak [9] are special case of the problem solved in this manuscript.
(3) We obtain a strong convergence result desirable to the weak counterpart obtained in Cholamjiak [9] and Cuntavepanit [10].

2. Preliminaries

Let X be a metric space and $x, y \in X$. A geodesic from x to y is a map (or a curve) c from the closed interval $[0, d(x,y)] \subset \mathbb{R}$ to X such that $c(0) = x,\ c(d(x,y)) = y$ and $d(c(t), c(t')) = |t - t'|$ for all $t, t' \in [0, d(x,y)]$. The image of c is called a geodesic segment joining from x to y. When it is unique, this geodesic segment is denoted by $[x,y]$. The space (X,d) is said to be a geodesic space if every two points of X are joined by a geodesic, and X is said to be uniquely geodesic if there is exactly one geodesic joining x and y for each $x, y \in X$. A subset D of a geodesic space X is said to be convex, if for any two points $x, y \in D$, the geodesic joining x and y is contained in D, that is, if $c : [0, d(x, y)] \to X$ is a geodesic such that $x = c(0)$ and $y = c(d(x, y))$, then $c(t) \in D \; \forall \; t \in [0, d(x, y)]$. A geodesic triangle $\Delta(x_1, x_2, x_3)$ in a geodesic metric space (X, d) consists of three vertices (points in X) with unparameterized geodesic segments between each pair of vertices. For any geodesic triangle there is comparison (Alexandrov) triangle $\bar{\Delta} \subset \mathbb{R}^2$, such that $d(x_i, x_j) = d_{\mathbb{R}^2}(\bar{x}_i, \bar{x}_j)$, for $i, j \in \{1, 2, 3\}$. A geodesic space X is a CAT(0) space if the distance between an arbitrary pair of points on a geodesic triangle Δ does not exceed the distance between its corresponding pair of points on its comparison triangle $\bar{\Delta}$. If Δ and $\bar{\Delta}$ are geodesic and comparison triangles in X respectively, then Δ is said to satisfy the CAT(0) inequality for all points x, y of Δ and $\bar{x}, \bar{y}$ of $\bar{\Delta}$ if

$$d(x, y) = d_{\mathbb{R}^2}(\bar{x}, \bar{y}). \tag{12}$$

Let x, y, z be points in X and y_0 be the midpoint of the segment $[y, z]$, then the CAT(0) inequality implies

$$d^2(x, y_0) \leq \frac{1}{2}d^2(x, y) + \frac{1}{2}d^2(x, z) - \frac{1}{4}d(y, z). \tag{13}$$

Berg and Nikolaev [7] introduced the notion of quasi-linearization in a CAT(0) space as follows: Let a pair $(a, b) \in X \times X$ denoted by $\overrightarrow{ab}$, be called a vector. Then, the quasilinearization map $\langle ., . \rangle$: $(X \times X) \times (X \times X) \to \mathbb{R}$ is defined by

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \frac{1}{2}(d^2(a, d) + d^2(b, c) - d^2(a, c) - d^2(b, d)), \text{ for all } a, b, c, d \in X. \tag{14}$$

It is easy to see that $\langle \overrightarrow{ab}, \overrightarrow{ab} \rangle = d^2(a, b)$, $\langle \overrightarrow{ba}, \overrightarrow{cd} \rangle = -\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle$, $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{ae}, \overrightarrow{cd} \rangle + \langle \overrightarrow{eb}, \overrightarrow{cd} \rangle$ and $\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle = \langle \overrightarrow{cd}, \overrightarrow{ab} \rangle$, for all $a, b, c, d, e \in X$. Furthermore, a geodesic space X is said to satisfy the Cauchy-Schwartz inequality, if

$$\langle \overrightarrow{ab}, \overrightarrow{cd} \rangle \leq d(a, b)d(c, d),$$

for all $a, b, c, d \in X$. It is well known that a geodesically connected space is a CAT(0) space if and only if it satisfies the Cauchy-Schwartz inequality [14]. Also, it is known that complete CAT(0) spaces are called Hadamard spaces.

In 2010, Kakavandi and Amini [20] introduced the dual space of a Hadamard space X as follows: Consider the map $\Theta : \mathbb{R} \times X \times X \to C(X, \mathbb{R})$ define by

$$\Theta(t, a, b)(x) = t\langle \overrightarrow{ab}, \overrightarrow{ax} \rangle, \; (t \in \mathbb{R}, \; a, b, x \in X),$$

where $C(X, \mathbb{R})$ is the space of all continuous real-valued functions on X. Then the Cauchy-Schwartz inequality implies that $\Theta(t, a, b)$ is a Lipschitz function with Lipschitz semi-norm $L(\Theta(t, a, b)) = |t|d(a, b)$

$(t \in \mathbb{R}, a, b \in X)$, where $L(\phi) = \sup\{(\phi(x) - \phi(y))/d(x,y) : x, y \in X,\ x \neq y\}$ is the Lipschitz semi-norm for any function $\phi : X \to \mathbb{R}$. A pseudometric $\mathcal{D}$ on $\mathbb{R} \times X \times X$ is defined by

$$\mathcal{D}((t,a,b),(s,c,d)) = L(\Theta(t,a,b) - \Theta(s,c,d)),\ (t, s \in \mathbb{R}, a, b, c, d \in X).$$

In an Hadamard space X, the psuedometric space $(\mathbb{R} \times X \times X, \mathcal{D})$ can be considered as a subset of the pseudometric space of all real-valued Lipschitz functions $(Lip(X, \mathbb{R}), L)$. It is well known from [20] that $\mathcal{D}((t,a,b),(s,c,d)) = 0$ if and only if $t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle = s\langle \overrightarrow{cd}, \overrightarrow{xy} \rangle$, for all $x, y \in X$. Thus, $\mathcal{D}$ induces an equivalence relation on $\mathbb{R} \times X \times X$, where the equivalence class of (t, a, b) is defined as

$$[t\overrightarrow{ab}] := \{s\overrightarrow{cd} : \mathcal{D}((t,a,b),(s,c,d)) = 0\}.$$

The set $X^* = \{[t\overrightarrow{ab}] : (t,a,b) \in \mathbb{R} \times X \times X\}$ is a metric space with the metric $\mathcal{D}([t\overrightarrow{ab}], [s\overrightarrow{cd}) := \mathcal{D}((t,a,b),(s,c,d))$. The pair $(X^*, \mathcal{D})$ is called the dual space of the metric space (X, d). It is shown in [20] that the dual of a closed and convex subset of a Hilbert space H with nonempty interior is H and $t(b - a) \equiv [t\overrightarrow{ab}]$ for all $t \in \mathbb{R},\ a, b \in H$. We also note that X^* acts on $X \times X$ by

$$\langle x^*, \overrightarrow{xy} \rangle = t\langle \overrightarrow{ab}, \overrightarrow{xy} \rangle,\ (x^* = [t\overrightarrow{ab}] \in X^*,\ x, y \in X).$$

Let $\{x_n\}$ be a bounded sequence in X and $r(., \{x_n\}) : X \to [0, \infty)$ be a continuous mapping defined by

$$r(x, \{x_n\}) = \limsup_{n \to \infty} d(x, x_n).$$

The asymptotic radius of $\{x_n\}$ is given by

$$r(\{x_n\}) : \inf\{r(x, x_n) : x \in X\},$$

while the asymptotic center of $\{x_n\}$ is the set

$$A(\{x_n\}) = x \in X : r(x, \{x_n\}) = r(\{x_n\}).$$

It is well known from [12, 22] that in a complete CAT(0) space X, $A(\{x_n\})$ consists of exactly one point. A sequence $\{x_n\}$ in X is said to be Δ-convergent to a point $x \in X$ if $A(\{x_{n_k}\}) = \{x\}$ for every subsequence $\{x_{n_k}\}$ of $\{x_n\}$. In this case, we write $\Delta - \lim_{n \to \infty} x_n = x$.

Definition 2.1 Let X be a Hadamard space. A nonlinear mapping T is said to be:

(1) a contraction, if there exists $k \in (0, 1)$ such that

$$d(Tx, Ty) \leq kd(x, y), \forall x, y \in X,$$

if $k = 1$, then T is called nonexpansive,

(2) quasi-nonexpansive, if $Fix(T) \neq \emptyset$ and

$$d(Tx, y) \leq d(x, y), \forall x \in X, y \in Fix(T),$$

(3) k-demicontractive, if $Fix(T) \neq \emptyset$ and there exits $k \in [0,1)$ such that

$$d^2(Tx,y) \leq d^2(x,y) + kd^2(Tx,x)\ \forall x \in X, y \in Fix(T).$$

Clearly, nonexpansive mappings (with nonempty fixed point set) $\subset$ quasi-nonexpansive mappings $\subset$ demicontractive mappings. There are several examples in literature which show that these inclusion are proper (see for example [8,17], and the references therein). Furthermore, the class of demicontractive mappings is known to be of central importance in optimization theory since it contains many common types of operators that are useful in solving optimization problems (see [3,19,26] and the reference therein).

Definition 2.2 [18] Let (X,d) be a metric space. A mapping $T : X \to X$ is called k-demicontractive-type, if $Fix(T) \neq \emptyset$ and there exists $k \in (-\infty, 1)$ such that

$$d^2(Tx,y) \leq d^2(x,y) + kd^2(Tx,x)\ \forall\ x \in X, y \in Fix(T). \tag{15}$$

Also, by definition of quasilinearization mapping (see (14)), we obtain that

$$2\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle = d^2(x,y) + d^2(Tx,x) - d^2(Tx,y).$$

That is,

$$d^2(Tx,y) = d^2(x,y) + d^2(Tx,x) - 2\langle \overrightarrow{xTx}, \overrightarrow{xy} \rangle,$$

which implies from (15) that

$$\langle \overrightarrow{xy}, \overrightarrow{xTx} \rangle \geq \frac{1-k}{2} d^2(x,Tx). \tag{16}$$

Example 2.1 [18] Let $T : [0,1] \to \mathbb{R}$ be defined by $Tx = x - x^2$. Then T is k-demicontractive with $k = -1$. Indeed, it is clear that $Fix(T) = \{0\}$, and for all $x \in [0,1]$, we obtain that

$$\begin{aligned} |Tx - 0|^2 = |x - x^2|^2 &= |x|^2 - 2|x||x^2| + |x^2|^2 \\ &\leq |x|^2 - 2|x^2||x^2| + |x^2|^2 = |x - 0|^2 - |x - Tx|^2. \end{aligned}$$

Example 2.2 Let X be a geodesic space. A mapping $f : X \to (-\infty, \infty)$ is said to be convex, if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y)\ \forall x,y \in X,\ \lambda \in (0,1).$$

f is proper, if $D(f) := \{x \in X : f(x) < +\infty\} \neq \emptyset$, where D denotes the domain of f. The mapping $f : D(f) \to (-\infty, \infty]$ is lower semicontinuous at a point $x \in D$, if

$$f(x) \leq \liminf_{n\to\infty} f(x_n). \tag{17}$$

Lemma 2.1 [11,14] Let X be a $CAT(0)$ space. Then for all $w,x,y,z \in X$ and for all t $\in [0,1]$, we have

(1) $d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z)$,

(2) $d^2(tx \oplus (1-t)y, z) \leq td^2(x,z) + (1-t)d^2(y,z) - t(1-t)d^2(x,y)$,

(3) $d^2(z, tx \oplus (1-t)y) \leq t^2 d^2(z,x) + (1-t)^2 d^2(z,y) + 2t(1-t)\langle \overrightarrow{zx}, \overrightarrow{zy} \rangle$,

(4) $d(tx \oplus (1-t)y, tw \oplus (1-t)z) \leq td(x,w) + (1-t)d(y,z)$.

Lemma 2.2 [4] Let X be a Hadamard space. Then, for all $v, w, x, y, z \in X$ and $\alpha, \beta, \gamma \in (0,1)$ with $\alpha + \beta + \gamma = 1$ we have

$$d^2(\alpha x \oplus \beta y \oplus \gamma z, v) \leq \alpha d^2(x,v) + \beta d^2(y,v) + \gamma d^2(z,v).$$

Lemma 2.3 [23] Let X be a Hadamard space and $h : X \to (-\infty, \infty]$ be a proper, convex and semi-continuous function. Then for all $x, z \in X$ and $\lambda > 0$, we have

$$\frac{1}{2\lambda} d^2(J_\lambda^h x, z) - \frac{1}{2\lambda} d^2(x,z) + \frac{1}{2\lambda} d^2(x, J_\lambda^h x) + h(J_\lambda^h x) \leq h(z).$$

Lemma 2.4 [18] Let X be a Hadamard space and $T : X \to X$ be a k-demicontrative mapping with $k \in (-\infty, \gamma]$ and $\gamma \in (0,1)$. Let $T_\gamma x = \gamma x \oplus (1-\gamma)Tx$, then T_γ is quasi-nonexpansive and $Fix(T_\gamma) = Fix(T)$.

Lemma 2.5 [31] Let $\{a_n\}$ be a sequence of non-negative real number, $\{\gamma_n\}$ be a sequence of real numbers in $(0,1)$ with conditions $\sum_{n=1}^{\infty} \gamma_n$ and $\{d_n\}$ be a sequence of real numbers. Assume that

$$a_{n+1} \leq (1-\gamma_n)a_n + \gamma_n d_n, n \geq 1. \tag{18}$$

If $\limsup_{k\to\infty} d_{n_k} \leq 0$ for every subsequence $\{a_{n_k}\}$ of $\{a_n\}$ satisfies the condition:

$$\liminf_{k\to\infty} (a_{n_{k+1}} - a_{n_k}) \geq 0, \ then \ \lim_{n\to\infty} a_n = 0. \tag{19}$$

Lemma 2.6 Let X be a complete $CAT(0)$ space and $f_j : X \to (-\infty, \infty], j = 1, 2, \cdots, m$ be finite family of proper, convex and lower semi-continuous functions. For $0 < \lambda \leq \mu$, we have that

$$Fix\bigg(\prod_{j=1}^{m} J_\mu^{(j)}\bigg) \subseteq \bigg(\bigcap_{j=1}^{m} Fix(J_\lambda^{(j)})\bigg),$$

where $\prod_{j=1}^{m} J_\mu^{(j)} = J_\mu^{(1)} \circ J_\mu^{(2)} \circ \cdots \circ J_\mu^{(m)}$.

Lemma 2.7 [13] Let $S : X \to X$ be a nonexpansive mapping. Then the condition $\{x_n\}$ Δ-converges to p and $d(x_n, Sx_n) \to 0$ imply $p = Sp$.

Lemma 2.8 [14] Every bounded sequence in an Hadamard space has a Δ-convergence subsequence.

Lemma 2.9 [13] Let X be a Hadamard space and $T : X \to X$ be a nonexpansive mapping. Then, T is Δ-demiclosed.

Lemma 2.10 [29] Let X be a Hadamard space and $\{x_n\}$ be a subsequence in X. If there exists a nonempty subset E in which

(i) $\lim_{n\to\infty} d(x_n, z)$ exists for every $z \in E$, and

(ii) If $\{x_{n_k}\}$ is a subsequence of $\{x_n\}$ which is Δ-convergent to $p \in E$.

(iii) Then, there is $p \in E$ such that $\{x_n\}$ which is Δ-convergent to $p \in X$.

Lemma 2.11 [20] Let X be a Hadamard space, $\{x_n\}$ be a bounded sequence in X and $x \in X$. Then $\{x_n\}$ Δ-converges to x if and only if $\limsup_{n\to\infty}\langle \overrightarrow{x_n x}, \overrightarrow{yx}\rangle \le 0 \; \forall y \in X$.

3. Main Results

In this article, we introduce a modified Halpern method to approximating finite families of resolvents of convex functions. We also establish a strong convergence result and present some consequences of our result. Our main result is stated as follows:

Theorem 3.1 Let X be an Hadamard space and X^* be its dual space. Let $h_j : X \to \mathbb{R}, j = 1, 2, \cdots, N$ be a proper, convex and lower semi-continuous function, and $S : X \to X$ be a μ-demicontractive-type mapping with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0, 1)$ such that S is $\Delta - demiclosed$ at 0. Assume that $\Omega := Fix(S) \cap \bigcap_{j=1}^{N} argmin\; h_j(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = \prod_{j=1}^{N} J_{\lambda_n}^{(h_j)} x_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(N)} x_n \\ v_n = (1-\beta_n)u_n \oplus \beta_n S_\gamma u_n \\ x_{n+1} = \alpha_n u \; \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{20}$$

where $S_\gamma x$ = γx $\oplus$(1-γ)Sx, $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

a) $0 < \lambda < \lambda_n, \; \forall\, n \ge 1,$
b) $0 < a \le \beta_n \le b < 1,$
c) $\alpha_n + \Theta_n + \delta_n = 1,$
d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then, $\{x_n\}$ converges strongly to an element z in Ω.

Proof: Let $x^* \in \Omega$, then by applying Lemma 2.1, Lemma 2.4 and (20), we have

$$\begin{aligned} d^2(v_n, x^*) &= d^2((1-\beta_n)u_n \; \oplus \beta_n S_\gamma u_n, x^*) \\ &\le (1-\beta_n)d^2(u_n, x^*) + \beta_n d^2(S_\gamma u_n, x^*) - \beta_n(1-\beta_n)d^2(u_n, S_\gamma u_n) \\ &\le (1-\beta_n)d^2(u_n, x^*) + \beta_n d^2(u_n, x^*) - \beta_n(1-\beta_n)d^2(u_n, S_\gamma u_n) \\ &= d^2(u_n, x^*) - \beta_n(1-\beta_n)d^2(u_n, S_\gamma u_n) \\ &= d^2(\prod_{j=1}^{N} J_{\lambda_n}^{(j)} x_n, x^*) - \beta_n(1-\beta_n)d^2(u_n, S_\gamma u_n) \\ &\le d^2(\prod_{j=2}^{N} J_{\lambda_n}^{(j)} x_n, x^*) - \beta_n(1-\beta_n)d^2(u_n S_\gamma u_n) \end{aligned}$$

$$\begin{aligned} &\vdots \\ &\le d^2(x_n, x^*) - \beta_n(1-\beta_n)d^2(u_n, S_\gamma u_n) \qquad (21) \\ &\le d^2(x_n, x^*). \qquad (22) \end{aligned}$$

Thus, by using (20) and (22), we get

$$\begin{aligned} d(x_{n+1}, x^*) &= d(\alpha_n u \oplus \Theta_n v_n \oplus \delta_n v_n, x^*) \\ &\le \alpha_n d(u, x^*) + \Theta_n d(v_n, x^*) + \delta_n d(v_n, x^*) \\ &= \alpha_n d(u, x^*) + (1-\alpha_n) d(v_n, x^*) \\ &\le \alpha_n d(u, x^*) + (1-\alpha_n) d(x_n, x^*) \\ &\le \max\{d(u, x^*),\ d(x_n, x^*)\} \\ &\vdots \\ &\le \max\{d(u, x^*),\ d(x_1, x^*)\}. \end{aligned}$$

Hence, by induction, we have that $\{x_n\}$ is bounded. Consequently, $\{u_n\}$ and $\{v_n\}$ are all bounded. Let $w_n = \frac{\Theta_n}{1-\alpha_n} v_n + \frac{\delta_n}{1-\alpha_n} v_n$, then from (22), we have that

$$\begin{aligned} d^2(w_n, x^*) &\le \frac{\Theta_n}{1-\alpha_n} d^2(v_n, x^*) + \frac{\delta_n}{1-\alpha_n} d^2(v_n, x^*) \\ &= d^2(v_n, x^*) \qquad (23) \\ &\le d^2(x_n, x^*). \qquad (24) \end{aligned}$$

It is obvious that x_{n+1} in (20) can be re-written as $x_{n+1} = \alpha_n u \oplus (1-\alpha_n) w_n$

From Lemma 2.1(3), (21),(23), we obtain

$$\begin{aligned} d^2(x_{n+1}, x^*) &\le d^2(\alpha_n u \oplus (1-\alpha_n) w_n, x^*) \\ &\le \alpha_n^2 d^2(u, x^*) + (1-\alpha_n)^2 d^2(w_n, x^*) + 2\alpha_n(1-\alpha_n)\langle \overrightarrow{ux^*}, \overrightarrow{x^* w_n} \rangle \\ &\le (1-\alpha_n) d^2(w_n, x^*) + \alpha_n\big(\alpha_n d^2(u, x^*) + 2(1-\alpha_n)\langle \overrightarrow{ux^*}, \overrightarrow{x^* w_n} \rangle\big) \\ &\le (1-\alpha_n) d^2(v_n, x^*) + \alpha_n\big(\alpha_n d^2(u, x^*) + 2(1-\alpha_n)\langle \overrightarrow{ux^*}, \overrightarrow{x^* w_n} \rangle\big) \\ &\le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \Delta_n - (1-\alpha_n)\beta_n(1-\beta_n) d^2(u_n, S_\gamma u_n) \qquad (25) \\ &\le (1-\alpha_n) d^2(x_n, x^*) + \alpha_n \Delta_n, \qquad (26) \end{aligned}$$

where $\Delta_n = (\alpha_n d^2(u, x^*) + 2(1-\alpha_n)\langle \overrightarrow{ux^*}, \overrightarrow{x^*w_n}\rangle)$ From Lemma 2.5 it suffices that

$$\limsup_{k\to\infty}(d(x_{n_k}, x^*) - d(x_{n_{k+1}}, x^*)) \leq 0, \tag{27}$$

where $\{d(x_{n_k}, x^*)\}$ is a subsequence of $\{d(x_n, x^*)\}$, then from (27), we have

$$\begin{aligned}\limsup_{k\to\infty}((1-\alpha_n)\beta_{n_k}(1-\beta_{n_k})d^2(u_{n_k}, S_\gamma u_{n_k})) &\leq \limsup_{k\to\infty}((1-\alpha_{n_k})d^2(x_{n_k}, x^*) - d^2(x_{n_{k+1}}, x^*)) \\ &+ \limsup_{k\to\infty}(\alpha_{n_k}\Delta_{n_k}) \\ &\leq \limsup_{k\to\infty}((1-\alpha_{n_k})d^2(x_{n_k}, x^*)) \\ &= -\liminf_{k\to\infty}(d^2(x_{n_{k+1}}, x^*) - d^2(x_{n_k}, x^*)) \\ &\leq 0. \end{aligned}\tag{28}$$

Thus form condition (b) and (c) of (20), we obtain that

$$\lim_{k\to\infty} d^2(u_{n_k}, S_\gamma u_{n_k}) = 0. \tag{29}$$

Using Algorithm (20) and (29), we get

$$\lim_{k\to\infty} d(v_{n_k}, u_{n_k}) \leq (1-\beta_{n_k})d(u_{n_k}, u_{n_k}) + \beta_{n_k}d(S_\gamma u_{n_k}, u_{n_k}) \to 0 \; as \; k \to \infty. \tag{30}$$

In a similar way as in (30), we also have that

$$d(x_{n_{k+1}}, v_{n_k}) \leq \alpha_{n_k}d(u, v_{n_k}) + \Theta_{n_k}d(v_{n_k}, v_{n_k}) + \delta_{n_k}d(v_{n_k}, v_{n_k}) \to \; 0 \; as \; k \to \infty. \tag{31}$$

From (30) and (31), we have that

$$\lim_{k\to\infty} d(x_{n_{k+1}}, u_{n_k}) = 0 \tag{32}$$

On applying Lemma 2.3, we have that

$$\begin{aligned} d^2(u_n, \prod_{j=2}^{N} J_{\lambda_n}^{(j)} x_n) &= d^2(\prod_{j=1}^{N} J_{\lambda_n}^{(j)} x_n, \prod_{j=2}^{N} J_{\lambda_n}^{(j)} x_n) \\ &\leq d^2(\prod_{j=2}^{N} J_{\lambda_n}^{(j)} x_n, x^*) - d^2(u_n, x^*) \\ &\leq d^2(x_n, x^*) - d^2(u_n, x^*) \\ &\leq d^2(x_n, x^*) - d^2(v_n, x^*) \\ &\leq d^2(x_n, x^*) - d^2(x_{n+1}, x^*) + d^2(x_{n+1}, x^*) - d^2(v_n, x^*) \\ &\leq d^2(x_n, x^*) - d^2(x_{n+1}, x^*) + \alpha_n d^2(u, x^*) + (1-\alpha_n)d^2(v_n, x^*) - d^2(v_n, x^*). \end{aligned}\tag{33}$$

Let $\{u_{n_k}\}$ and $\{v_{n_k}\}$ be subsequences $\{u_n\}$ and $\{v_n\}$ respectively, thus we have from (33)

$$\limsup_{k\to\infty} d^2\big(u_{n_k}, \prod_{j=2}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}\big) \le \limsup_{k\to\infty}(d^2(x_{n_k}, x^*) - d^2(x_{n_{k+1}}, x^*)) + \limsup_{k\to\infty}(\alpha_{n_k} d^2(u, x^*))$$
$$= -\liminf_{k\to\infty}(d^2(x_{n_{k+1}}, x^*) - d^2(x_{n_k}, x^*)) \le 0. \tag{34}$$

Thus,

$$\lim_{k\to\infty} d\big(u_{n_k}, \prod_{j=2}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}\big) = 0. \tag{35}$$

It follows from (34) that

$$\lim_{k\to\infty} \big(d(u_{n_k}, \prod_{j=2}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k})\big) = d\big(\prod_{j=1}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}, \prod_{j=2}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}\big) = 0. \tag{36}$$

By following the same process as in (35)-(36), we obtain that

$$\lim_{k\to\infty} \Big(d\big(\prod_{j=2}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}, \prod_{j=3}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}\big) = d\big(\prod_{j=3}^{N} J^{(j)}_{\lambda_{n_k}}, \prod_{j=4}^{N} J^{(j)}_{\lambda_{n_k}} x_{n_k}\big) = \cdots = d(J^{(N)}_{\lambda_{n_k}} x_{n_k}, x_{n_k})\Big) = 0. \tag{37}$$

By summing (36) and (37), we have that

$$\lim_{k\to\infty} d(u_{n_k}, x_{n_k}) = 0. \tag{38}$$

Hence, from (30), (31) and (34), we obtain

$$\begin{cases} \lim_{k\to\infty} d(v_{n_k}, x_{n_k}) = 0, \\ \lim_{k\to\infty} d(v_{n_{k+1}}, x_{n_k}) = 0. \end{cases} \tag{39}$$

Since $\{x_{n_k}\}$ is bounded, then by Lemma 2.8, there exist a subsequence $\{x_{n_{k_i}}\}$ of $\{x_{n_k}\}$ such that $\Delta - \lim_{i\to\infty} x_{n_{k_i}} = q$. Thus, we obtain from (38) and (39) that $\Delta - \lim_{i\to\infty} u_{n_{k_i}} = q$ and $\Delta - \lim_{i\to\infty} v_{n_{k_i}} = q$ for some subsequences $\{u_{n_{k_i}}\}$ and $\{v_{n_{k_i}}\}$ of $\{u_{n_k}\}$ and $\{v_{n_k}\}$ respectively. Since $J^{h_j}_{\lambda_n}, j = 1, 2, \cdots, N$ is nonexpansive, we obtain from (38) and Lemma 2.6 that $q \in \bigcap_{j=1}^{N} Fix(J^{hj}_{\lambda_n})$. Also, using (29) and Lemma 2.7, we obtain that $q \in Fix(S_\gamma) = Fix(S)$. Therefore $q \in \Omega$. Next, we show $\{x_{n_k}\}$ converges strongly $z \in \Omega$. Since $\{x_{n_{k_i}}\}$ Δ-converges to $z \in \Omega$, it follows from Lemma 2.10 that there exists $z \in \Omega$ such that $\{x_{n_{k_i}}\}$ Δ-converges to z.

Thus, by Lemma 2.11, we obtain that

$$\limsup_{k\to\infty} \langle \overrightarrow{uz}, \overrightarrow{x_{n_k} z} \rangle \le 0. \tag{40}$$

Hence, we obtain from (26), (40) and condition (a) of Algorithm (20) that $\limsup_{k\to\infty} \Delta_{n_k} \leq 0$. Also we have from (24) that

$$d^2(x_{n_{k+1}}, z) \leq (1-\alpha_{n_k})d^2(x_{n_k}, z) + \alpha_{n_k}\Delta_{n_k}. \tag{41}$$

Thus, by Lemma 2.5 and condition of Algorithm (20), we obtain that $\{x_{n_k}\}$ converges strongly to $z \in \Omega$.

We now state some consequences of our main result.

Corollary 3.2 Let X be an Hadamard space and X^* be its dual space. Let $h_j : X \to \mathbb{R}, j = 1, 2, \cdots, N$ be a proper, convex and lower semi-continuous function, and $S : X \to X$ be a quasi-nonexpansive mapping such that S is $\Delta - demiclosed$ at 0. Assume that $\Omega := Fix(S) \cap \bigcap_{j=1}^{N} argmin\ h_j(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = \prod_{j=1}^{N} J_{\lambda_n}^{(h_j)} x_n = J_{\lambda_n}^{(1)} \circ J_{\lambda_n}^{(2)} \circ \cdots \circ J_{\lambda_n}^{(N)} x_n \\ v_n = (1-\beta_n)u_n \oplus \beta_n S u_n \\ x_{n+1} = \alpha_n u \ \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{42}$$

where $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

a) $0 < \lambda < \lambda_n, \ \forall n \geq 1,$
b) $0 < a \leq \beta_n \leq b < 1,$
c) $\alpha_n + \Theta_n + \delta_n = 1,$
d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty.$

Then, $\{x_n\}$ converges strongly to an element z in Ω.

Corollary 3.3 Let X be an Hadamard space and X^* be its dual space. Let $h : X \to \mathbb{R}$ be a proper, convex and lower semi-continuous function, and $S : X \to X$ be a μ-demicontractive-type mapping with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0, 1)$ such that S is $\Delta - demiclosed$ at 0. Assume that $\Omega := Fix(S) \cap argmin\ h(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = J_{\lambda_n}^{h} x_n \\ v_n = (1-\beta_n)u_n \oplus \beta_n S_\gamma u_n \\ x_{n+1} = \alpha_n u \ \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{43}$$

where $S_\gamma x = \gamma$x $\oplus$(1-γ)Sx, $\{\lambda_n\}$ is a sequence in $(0,\infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ such that the following conditions are satisfied:

a) $0 < \lambda < \lambda_n, \ \forall n \geq 1,$

b) $0 < a \leq \beta_n \leq b < 1,$

c) $\alpha_n + \Theta_n + \delta_n = 1,$

d) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty.$

Then, $\{x_n\}$ converges strongly to an element z in Ω.

Corollary 3.4 Let X be an Hadamard space and X^* be its dual space. Let $h : X \to \mathbb{R}$ be a proper, convex and lower semi-continuous function and assume that $\Omega := argmin\ h(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = J^h_{\lambda_n} x_n \\ x_{n+1} = \alpha_n u \ \oplus (1-\alpha_n) u_n, \end{cases} \tag{44}$$

where $\{\lambda_n\}$ is a sequence in $(0,\infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0,1)$ such that the following conditions are satisfied:

a) $0 < \lambda < \lambda_n, \ \forall\ n \geq 1,$

b) $0 < a \leq \beta_n \leq b < 1,$

c) $\lim\limits_{n\to\infty} \alpha_n = 0$ and $\sum\limits_{n=1}^{\infty} \alpha_n = \infty.$

Then, $\{x_n\}$ converges strongly to an element z in Ω.

4. Application

In this section, we apply our results to find Frechét mean of a finite elements of a Hadamard space, the mean of a probability, and convex feasibility problem.

5. Computing Frechét mean and the mean of a probability

Let $\{v_j\}_{j=1}^p \subset X\ and\ \{\alpha_j\}_{j=1}^p$ be positive weights satisfying $\sum\limits_{j=1}^{p} \alpha_j = 1$. Consider

$$h_1(v) := \sum_{j=1}^{p} \alpha_j (v, v_j),\ for\ every\ v \in X. \tag{45}$$

Suppose that $\mu \in P^2(V)$ is a probability measure and takes

$$h_2(v) := \int d^2(v,z) d\mu(z),\ for\ every\ v \in X. \tag{46}$$

Then the minimizers of $h_1\ and\ h_2$ are the *Frechét* mean of $\{v_i\}_{i=1}^p$ and mean of *probability* P , respectively. The two mean play significant role in both science and engineering (see, e.g. [27]). Moreover, it is not difficult to deduce using properties of metric d that $h_1 and\ h_2$ are convex, proper and lower semi-continuous functions on X. Hence we have the following results:

Theorem 4.1 Let X be an Hadamard space and X^* be its dual space. Let $S : X \to X$ be a μ-demicontractive-type mapping with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0,1)$ such that S is $\Delta - demiclosed$ at 0. Assume that $\Omega := Fix(S) \cap \bigcap_{j=1}^{N} argmin\ h_j(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = J_{\lambda_n}^{h_1} \\ v_n = (1-\beta_n)u_n \oplus \beta_n S_\gamma u_n \\ x_{n+1} = \alpha_n u \ \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{47}$$

where $S_\gamma x$ = γx $\oplus$(1-γ)Sx, $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

a) $\lambda_n \geq \ell$ for some positive number ℓ,

b) $0 < a \leq \beta_n \leq b < 1$,

c) $\alpha_n + \Theta_n + \delta_n = 1$,

d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to an element to the Frechét mean of $\{v_j\}_{j=1}^{p}$.

Theorem 4.2 Let X be an Hadamard space and X^* be its dual space. Let $S : X \to X$ be a μ-demicontractive-type mapping with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0,1)$ such that S is $\Delta - demiclosed$ at 0. Assume that $\Omega := Fix(S) \cap \bigcap_{j=1}^{N} argmin\ h_j(y)$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = J_{\lambda_n}^{h_2} \\ v_n = (1-\beta_n)u_n \oplus \beta_n S_\gamma u_n \\ x_{n+1} = \alpha_n u \ \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{48}$$

where $S_\gamma x$ = γx $\oplus$(1-γ)Sx, $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

a) $\lambda_n \geq \ell$ for some positive number ℓ,

b) $0 < a \leq \beta_n \leq b < 1$,

c) $\alpha_n + \Theta_n + \delta_n = 1$,

d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to the mean of the probability P with respect to λ.

6. Convex Feasibility Problem.

Let $\{\psi_j\}_{j=1}^{N}$ be a finite family of nonempty, closed and convex subsets of an Hadamard space X such that $\bigcap_{j=1}^{N} \psi_j \neq \emptyset$. The Convex Feasibility Problem (CFP) is to find $x^* \in \bigcap_{j=1}^{N} \psi_j$. For a nonempty, closed and convex subset ψ of an Hadamard space X, the indicator function

$$i_{\psi}(x) = \begin{cases} 0, x \in \psi \\ \infty, x \in X \setminus \psi, \end{cases}$$

is proper and lower semi-continuous and $J_{\lambda}^{i_{\psi}} = P_{\psi}$. Therefore, by letting $h_j = i_{\psi}, (j = 1, 2, \cdots, m)$, we have the following theorem:

Theorem 4.3 Let X be an Hadamard space and X^* be its dual space. Let $S : X \to X$ be a μ-demicontractive-type mapping with $\mu \in (-\infty, \gamma]$ and $\gamma \in (0, 1)$ such that S is $\Delta - demiclosed$ at 0. Assume that $\Phi := Fix(S) \cap \bigcap_{j=1}^{N} P_{\psi_j}$ is nonempty, then sequence $\{x_n\}$ is generated for arbitrary $x_1 \in X$ by

$$\begin{cases} u_n = \Delta_{j=1}^{N} P_{\psi} x_n = P_{\psi}^{(1)} \circ P_{\psi}^{(2)} \circ \cdots \circ P_{\psi}^{(N)} x_n \\ v_n = (1 - \beta_n) u_n \oplus \beta_n S_{\gamma} u_n \\ x_{n+1} = \alpha_n u \ \oplus \Theta_n v_n \oplus \delta_n v_n, \end{cases} \tag{49}$$

where $S_{\gamma}x = \gamma$x $\oplus$(1-γ)Sx, $\{\lambda_n\}$ is a sequence in $(0, \infty)$, $\{\alpha_n\}$, $\{\Theta_n\}$, $\{\beta_n\}$ and $\{\delta_n\}$ are sequences in $(0, 1)$ such that the following conditions are satisfied:

a) $0 < \lambda < \lambda_n, \ \forall n \geq 1$,
b) $0 < a \leq \beta_n \leq b < 1$,
c) $\alpha_n + \Theta_n + \delta_n = 1$,
d) $\lim_{n\to\infty} \alpha_n = 0$ and $\sum_{n=1}^{\infty} \alpha_n = \infty$.

Then, $\{x_n\}$ converges strongly to an element z in P_{Φ},where P_{Φ} is the metric projection of X onto Φ.

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