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BIPOLAR FUZZY IDEALS OF Γ-SEMIRINGS

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Abstract. This article explores the notion of bipolar fuzzy ideals of Γ-semirings. Later, we characterize bipolar fuzzy ideals of Γ-semirings to crisp Γ-semirings. Further, the relation between bipolar fuzzy ideals of Γ-semirings and their level cuts is investigated.

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1. Introduction

In 1965, Zadeh [13] established the idea of fuzzy subsets of a set. Fuzzy sets have several extensions, including intuitionistic fuzzy sets, interval-valued fuzzy sets, vague sets, neutrosophic sets, etc., which were developed. The idea of bipolar-valued fuzzy sets, which is a significant extension of fuzzy sets whose membership degree interval is extended from the interval $[0, 1]$ to the interval $[-1, 1]$, was first suggested by Zhang [14] in 1994. A generalization of both semirings and Γ-rings [2,11], the concept of Γ-semirings was first developed by Murali Krishna Rao [10] in 1995. The study of fuzzy ideals and bipolar fuzzy ideals continues as follows. In 1987, Mukherjee and Sen [9] studied fuzzy ideals of rings. In 1992, Malik and Mordeson [7] introduced the concept of fuzzy homomorphisms of rings. In 2009, Lee [6] introduced the notion of bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras. In 2011, Ghosh and Samanta [3] studied the relation between the fuzzy left (resp., right)

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ideals of Γ-semirings. In 2020, Ragamayi and Bhargavi [12] introduced the notion of homomorphism of vague ideals of Γ-nearrings. In 2022, Kalyani et al. [5] introduced and studied the theory of bipolar fuzzy sublattices and bipolar fuzzy ideals of lattices. Mohana Rupa et al. [8] introduced and studied the concept of bipolar fuzzy d-ideals of d-algebras and characterized bipolar fuzzy d-ideals to the crisp d-ideals. As a continuity of all these, we introduced the concept of bipolar fuzzy sets of Γ-semirings in 2023. Now, we are studying the concept of bipolar fuzzy ideals of Γ-semirings.

2. Preliminaries

First, we will review the definition of the Γ-semiring, which will be the space we will study in this article.

Definition 2.1. [1] Let M_S and Γ be two additive commutative semigroups. Then M_S is called a Γ-semiring if there exists a mapping $M_S \times \Gamma \times M_S \to M_S$, $(j, \ddot{\alpha}, n) \mapsto j\ddot{\alpha}n$ for $j, n \in M_S$ and $\ddot{\alpha} \in \Gamma$, satisfying the following conditions:
(i) $j\ddot{\alpha}(n + u) = j\ddot{\alpha}n + j\ddot{\alpha}u$
(ii) $(j + n)\ddot{\alpha}u = j\ddot{\alpha}u + n\ddot{\alpha}u$
(iii) $j(\ddot{\alpha} + \ddot{\beta})u = j\ddot{\alpha}u + j\ddot{\beta}u$
(iv) $j\ddot{\alpha}(n\ddot{\beta}u) = (j\ddot{\alpha}n)\ddot{\beta}u$, $\forall j, n, u \in M_S, \ddot{\alpha}, \ddot{\beta} \in \Gamma$.

Definition 2.2. [1] Let D be any non-empty set. A mapping $F : D \to [0, 1]$ is called a fuzzy subset of D.

Definition 2.3. [14] Let D be the universe of discourse. A bipolar-valued fuzzy set F in D is an object having the form $F := \{\ddot{d}, F^-(\ddot{d}), F^+(\ddot{d}) \mid \ddot{d} \in D\}$, where $F^- : D \to [-1, 0]$ and $F^+ : D \to [0, 1]$ are mappings.

For the sake of simplicity, we shall use the symbol $F = \{D; F^-, F^+\}$ for the bipolar-valued fuzzy set $F := \{\ddot{d}, F^-(\ddot{d}), F^+(\ddot{d}) \mid \ddot{d} \in D\}$, and use the notion of bipolar fuzzy sets instead of the notion of bipolar-valued fuzzy sets.

Definition 2.4. [14] Let $F = \{D; F^-, F^+\}$ be a bipolar fuzzy set and $s \times t \in [-1, 0] \times [0, 1]$, the sets $F_s^N = \{\ddot{d} \in D \mid F^-(\ddot{d}) \le s\}$ and $F_t^P = \{\ddot{d} \in D \mid F^+(\ddot{d}) \ge t\}$ are called negative s-cut and positive t-cut, respectively. For $s \times t \in [-1, 0] \times [0, 1]$, the set $F_{(s,t)} = F_s^N \cap F_t^P$ is called (s, t)-set of $F = \{D; F^-, F^+\}$.

Definition 2.5. [14] Let $F = \{D; F^-, F^+\}$ and $\varphi = \{D; \varphi^-, \varphi^+\}$ be two bipolar fuzzy sets of a universe of discourse D. The intersection of F and φ is defined as

$$(F^- \cap \varphi^-)(\ddot{d}) = \min\{F^-(\ddot{d}), \varphi^-(\ddot{d})\} \text{ and } (F^+ \cap \varphi^+)(\ddot{d}) = \min\{F^+(\ddot{d}), \varphi^+(\ddot{d})\}, \forall \ddot{d} \in D.$$

The union of F and φ is defined as

$$(F^- \cup \varphi^-)(\ddot{d}) = \max\{F^-(\ddot{d}), \varphi^-(\ddot{d})\} \text{ and } (F^+ \cup \varphi^+)(\ddot{d}) = \max\{F^+(\ddot{d}), \varphi^+(\ddot{d})\}, \forall \ddot{d} \in D.$$

A bipolar fuzzy set F is contained in another bipolar fuzzy set φ, written with $F \subseteq \varphi$ if

$$F^-(\ddot{d}) \geq \varphi^-(\ddot{d}) \text{ and } F^+(\ddot{d}) \leq \varphi^+(\ddot{d}), \forall \ddot{d} \in D.$$

Definition 2.6. [4] Let $g : C \to D$ be a homomorphism from a set C onto a set D and let $F = \{C; F^-, F^+\}$ be a bipolar fuzzy set of C and $\varphi = \{D; \varphi^-, \varphi^+\}$ be a bipolar fuzzy set of D, then the homomorphic image $g(F)$ of F is $g(F) = \{(g(F))^-, (g(F))^+\}$ defined as for all $\ddot{d} \in D$,

$$(g(F))^-(\ddot{d}) = \begin{cases} \min\{F^-(\ddot{u}) \mid \ddot{u} \in g^{-1}(\ddot{d})\}, & \text{if } g^{-1}(\ddot{d}) \neq \emptyset \\ 0, & \text{otherwise} \end{cases}$$

and

$$(g(F))^+(\ddot{d}) = \begin{cases} \max\{F^+(\ddot{u}) \mid \ddot{u} \in g^{-1}(\ddot{d})\}, & \text{if } g^{-1}(\ddot{d}) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

The pre-image $g^{-1}(\varphi)$ of φ under g is a bipolar fuzzy set defined as $\left(g^{-1}(\varphi)\right)^-(\ddot{u}) = \varphi^-(g(\ddot{u}))$ and $\left(g^{-1}(\varphi)\right)^+(\ddot{u}) = \varphi^+(g(\ddot{u})), \forall \ddot{u} \in C$.

Definition 2.7. [1] Let T be a subset of a Γ-semiring M_S. The characteristic function of T taking values in $[0,1]$ is a fuzzy set given by

$$\delta_T(\ddot{t}) = \begin{cases} 1, & \text{if } \ddot{t} \in T \\ 0, & \text{otherwise.} \end{cases}$$

Then δ_T is a fuzzy characteristic function of T in $[0,1]$.

Definition 2.8. [1] Let T be a subset of a Γ-semiring M_S. The bipolar fuzzy characteristic function of T is given by

$$\delta_T^+(\ddot{t}) = \begin{cases} 1, & \text{if } \ddot{t} \in T \\ 0, & \text{otherwise} \end{cases} \text{ and } \delta_T^-(\ddot{t}) = \begin{cases} -1, & \text{if } \ddot{t} \in T \\ 0, & \text{otherwise.} \end{cases}$$

Then δ_T is a bipolar fuzzy characteristic function of T.

Definition 2.9. [1] A Bipolar fuzzy set $F = \{M_S; F^-, F^+\}$ in M_S is called a bipolar fuzzy Γ-semiring of M_S if it satisfies the following properties: for all $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$,

(i) $F^-(\varrho + \varsigma) \leq \max\{F^-(\varrho), F^-(\varsigma)\}$

(ii) $F^-(\varrho\ddot{\gamma}\varsigma) \leq \max\{F^-(\varrho), F^-(\varsigma)\}$

(iii) $F^+(\varrho + \varsigma) \geq \min\{F^+(\varrho), F^+(\varsigma)\}$

(iv) $F^+(\varrho\ddot{\gamma}\varsigma) \geq \min\{F^+(\varrho), F^+(\varsigma)\}$.

Definition 2.10. [4] An additive subsemigroup B of a Γ-semiring M_S is called a right (resp., left) ideal of M_S if $\varrho\ddot{\gamma}\varsigma \in B$ (resp., $\varsigma\ddot{\gamma}\varrho \in B$) for all $\varrho \in B, \ddot{\gamma} \in \Gamma$ and $\varsigma \in M_S$. A left and right ideal of M_S is called an ideal of M_S.

Definition 2.11. [1] Let F be a fuzzy subset of a Γ-semiring M_S. Then F is called a fuzzy left (resp., right) ideal of M_S if for all $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$,
(i) $F(\varrho + \varsigma) \geq \min\{F(\varrho), F(\varsigma)\}$
(ii) $F(\varrho\ddot{\gamma}\varsigma) \geq F(\varsigma)$ (resp., $\geq F(\varrho)$).
Also, F is called a fuzzy ideal of M_S if it is both a fuzzy left ideal and a fuzzy right ideal of M_S.

Notations: Throughout the following session, we use the following notations:
(1) M_S for a Γ-semiring
(2) BF for bipolar fuzzy
(3) BFS for a bipolar fuzzy set
(4) BFGS for a bipolar fuzzy Γ-semiring
(5) BFI for a bipolar fuzzy ideal.

3. Bipolar Fuzzy Ideals of Γ-semirings

In this session, we introduce and study the notion of BFI of Γ-semirings, and we characterize and discuss a few properties related to BFI of Γ-semirings.

Definition 3.1. A BFS $F = \{M_S; F^-, F^+\}$ in M_S is called a BF left (resp., right) ideal of M_S if it satisfies the following properties: for any $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$,
(i) $F^-(\varrho + \varsigma) \leq \max\{F^-(\varrho), F^-(\varsigma)\}$
(ii) $F^-(\varrho\ddot{\gamma}\varsigma) \leq F^-(\varsigma)$ (resp., $\leq F^-(\varrho)$)
(iii) $F^+(\varrho + \varsigma) \geq \min\{F^+(\varrho), F^+(\varsigma)\}$
(iv) $F^+(\varrho\ddot{\gamma}\varsigma) \geq F^+(\varsigma)$ (resp., $\geq F^+(\varrho)$).

Also, a BFS F in M_S is called a BFI of M_S if it is both a BF left ideal and a BF right ideal of M_S.

Example 3.2. Let $\mathbb{N}$ be the set of all natural numbers with zero, and let $\mathbb{Z}^+$ be the set of all positive even integers. Then $\mathbb{N}$ and $\mathbb{Z}^+$ are additive commutative semigroups. Define the mapping $\mathbb{N} \times \mathbb{Z}^+ \times \mathbb{N} \to \mathbb{N}$ by $\ddot{a}\ddot{o}\ddot{b}$ usual product of $\ddot{a}, \ddot{o}, \ddot{b}, \forall \ddot{a}, \ddot{b} \in \mathbb{N}, \ddot{o} \in \mathbb{Z}^+$. Then $\mathbb{N}$ is a Γ-semiring.

Define a BFS $F = \{\mathbb{N}; F^-, F^+\}$, where $F^- : \mathbb{N} \to [-1, 0]$ and $F^+ : \mathbb{N} \to [0, 1]$ as follows:

$$F^-(\varrho) = \begin{cases} -0.8, & \text{if } \varrho \text{ is even or } 0 \\ -0.5, & \text{otherwise} \end{cases} \quad \text{and } F^+(\varrho) = \begin{cases} 0.8, & \text{if } \varrho \text{ is even or } 0 \\ 0.5, & \text{otherwise.} \end{cases}$$

Then F is a BFI of $\mathbb{N}$.

Theorem 3.3. *A BFS $F = \{M_S; F^-, F^+\}$ in M_S is a BFI of M_S if and only if the level cuts are ideals of M_S, i.e., for all $s \times t \in [-1, 0] \times [0, 1], \emptyset \neq F_s^N$ and $\emptyset \neq F_t^P$ are ideals of M_S.*

Proof. Suppose $F = \{M_S; F^-, F^+\}$ is a BFI of M_S. Let $s \times t \in [-1, 0] \times [0, 1]$ be such that $F_s^N \neq \emptyset$ and $F_t^P \neq \emptyset$. Let $\upsilon, \tau \in F_s^N, \varrho, \varsigma \in F_t^P$ and $\ddot{\gamma} \in \Gamma$. Then $F^-(\upsilon) \leq s, F^-(\tau) \leq s$ and $F^+(\varrho) \geq t, F^+(\varsigma) \geq t$. Since $F = \{M_S; F^-, F^+\}$ is a BFI of M_S, we have

(i) $F^{-}(\upsilon+\tau) \leq \max\{F^{-}(\upsilon), F^{-}(\tau)\} \leq s$
(ii) $F^{-}(\upsilon\ddot{\gamma}\tau) \leq F^{-}(\tau) \leq s$ (resp., $\leq F^{-}(\upsilon) \leq s$)
(iii) $F^{+}(\varrho+\varsigma) \geq \min\{F^{+}(\varrho), F^{+}(\varsigma)\} \geq t$
(iv) $F^{+}(\varrho\ddot{\gamma}\varsigma) \geq F^{+}(\varsigma) \geq t$ (resp., $\geq F^{+}(\varrho) \geq t$).
Then $(\upsilon+\tau) \in F_s^N, \upsilon\ddot{\gamma}\tau \in F_s^N$ and $\varrho+\varsigma \in F_t^P, \varrho\ddot{\gamma}\varsigma \in F_t^P$. Thus F_s^N and F_t^P are ideals of M_S.

Conversely, suppose that the level cuts F_s^N and F_t^P are ideals of M_S. Let $\upsilon, \tau \in F_s^N, \varrho, \varsigma \in F_t^P$ and $\ddot{\gamma} \in \Gamma$. Then $\upsilon+\tau \in F_s^N, \upsilon\ddot{\gamma}\tau \in F_s^N$ and $\varrho+\varsigma \in F_t^P, \varrho\ddot{\gamma}\varsigma \in F_t^P$. Choose $s = \max\{F^{-}(\upsilon), F^{-}(\tau)\}$ and $t = \min\{F^{+}(\varrho), F^{+}(\varsigma)\}$. Then
(i) $F^{-}(\upsilon+\tau) \leq s = \max\{F^{-}(\upsilon), F^{-}(\tau)\}$.
(ii) $F^{-}(\upsilon\ddot{\gamma}\tau) \leq s = \max\{F^{-}(\upsilon), F^{-}(\tau)\}$. If $F^{-}(\upsilon) < F^{-}(\tau)$, then $F^{-}(\upsilon\ddot{\gamma}\tau) \leq s = \max\{F^{-}(\upsilon), F^{-}(\tau)\} = F^{-}(\tau)$. If $F^{-}(\tau) < F^{-}(\upsilon)$, then$F^{-}(\upsilon\ddot{\gamma}\tau) \leq s = \max\{F^{-}(\upsilon), F^{-}(\tau)\} = F^{-}(\upsilon)$.
(iii) $F^{+}(\varrho+\varsigma) \geq t = \min\{F^{+}(\varrho), F^{+}(\varsigma)\}$.
(iv) $F^{+}(\varrho\ddot{\gamma}\varsigma) \geq t = \min\{F^{+}(\varrho), F^{+}(\varsigma)\}$. If $F^{+}(\varsigma) < F^{+}(\varrho)$, then $F^{+}(\varrho\ddot{\gamma}\varsigma) \geq t = \min\{F^{+}(\varrho), F^{+}(\varsigma)\} = F^{+}(\varsigma)$. If $F^{+}(\varrho) < F^{+}(\varsigma)$, then $F^{+}(\varrho\ddot{\gamma}\varsigma) \geq t = \min\{F^{+}(\varrho), F^{+}(\varsigma)\} = F^{+}(\varrho)$. Thus $F = \{M_S; F^{-}, F^{+}\}$ is a BFI of M_S. □

Theorem 3.4. *If $F = \{M_S; F^{-}, F^{+}\}$ and $\varphi = \{M_S; \varphi^{-}, \varphi^{+}\}$ are two BFIs of M_S, then $F \cap \varphi$ is a BFI of M_S.*

Proof. Assume that $F = \{M_S; F^{-}, F^{+}\}$ and $\varphi = \{M_S; \varphi^{-}, \varphi^{+}\}$ are BFIs of M_S. Let $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$. Then

$$\begin{aligned}(F^{-} \cap \varphi^{-})(\varrho+\varsigma) &= \min\{F^{-}(\varrho+\varsigma), \varphi^{-}(\varrho+\varsigma)\} \\ &\leq \min\{\max\{F^{-}(\varrho), F^{-}(\varsigma)\}, \max\{\varphi^{-}(\varrho), \varphi^{-}(\varsigma)\}\} \\ &\leq \min\{\max\{F^{-}(\varrho), \varphi^{-}(\varrho)\}, \max\{F^{-}(\varsigma), \varphi^{-}(\varsigma)\}\} \\ &\leq \max\{\min\{F^{-}(\varrho), \varphi^{-}(\varrho)\}, \min\{F^{-}(\varsigma), \varphi^{-}(\varsigma)\}\} \\ &= \max\{(F^{-} \cap \varphi^{-})(\varrho), (F^{-} \cap \varphi^{-})(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(F^{-} \cap \varphi^{-})(\varrho\ddot{\gamma}\varsigma) &= \min\{F^{-}(\varrho\ddot{\gamma}\varsigma), \varphi^{-}(\varrho\ddot{\gamma}\varsigma)\} \\ &\leq \min\{F^{-}(\varsigma), \varphi^{-}(\varsigma)\} \text{ (resp., } \leq \min\{F^{-}(\varrho), \varphi^{-}(\varrho)\}) \\ &= (F^{-} \cap \varphi^{-})(\varsigma),\end{aligned}$$

$$\begin{aligned}(F^{+} \cap \varphi^{+})(\varrho+\varsigma) &= \min\{F^{+}(\varrho+\varsigma), \varphi^{+}(\varrho+\varsigma)\} \\ &\geq \min\{\max\{F^{+}(\varrho), F^{+}(\varsigma)\}, \max\{\varphi^{+}(\varrho), \varphi^{+}(\varsigma)\}\} \\ &\geq \min\{\max\{F^{+}(\varrho), \varphi^{+}(\varrho)\}, \max\{F^{+}(\varsigma), \varphi^{+}(\varsigma)\}\} \\ &\geq \max\{\min\{F^{+}(\varrho), \varphi^{+}(\varrho)\}, \min\{F^{+}(\varsigma), \varphi^{+}(\varsigma)\}\} \\ &= \max\{(F^{+} \cap \varphi^{+})(\varrho), (F^{+} \cap \varphi^{+})(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(F^+\cap\varphi^+)(\varrho\ddot{\gamma}\varsigma) &= \min\{F^+(\varrho\ddot{\gamma}\varsigma),\varphi^+(\varrho\ddot{\gamma}\varsigma)\}\\ &\geq \min\{F^+(\varsigma),\varphi^+(\varsigma)\}\ (\text{resp., } \geq \min\{F^+(\varrho),\varphi^+(\varrho)\})\\ &= (F^+\cap\varphi^+)(\varsigma).\end{aligned}$$

Hence, $F\cap\varphi$ is a BFI of M_S. □

Corollary 3.5. *The intersection of an arbitrary family of BFIs of M_S is a BFI of M_S. In general, the union of two BFIs of M_S is not a BFI of M_S.*

Example 3.6. Consider the additive Abelian groups $Z_4=\{0,1,2,3\}$ and $\Upsilon=\{0,2\}$. Define $Z_4\times\Upsilon\times Z_4\to Z_4$ by $\varrho\ddot{\alpha}\varsigma$ usual product of $\varrho,\ddot{\alpha},\varsigma,\forall\varrho,\varsigma\in Z_4,\ddot{\alpha}\in\Upsilon$. Then Z_4 is a Γ-semiring. Define a BFS $F=\{Z_4;F^-,F^+\}$, where $F^-:Z_4\to[-1,0]$ and $F^+:Z_4\to[0,1]$ as follows:

$$F^-(\varrho)=\begin{cases}-0.8, & \text{if } \varrho=0\\ -0.6, & \text{if } \varrho=1\\ -0.4, & \text{otherwise}\end{cases}\quad\text{and } F^+(\varrho)=\begin{cases}0.9, & \text{if } \varrho=0\\ 0.7, & \text{if } \varrho=1\\ 0.5, & \text{otherwise.}\end{cases}$$

Define a BFS $\varphi=\{Z_4;\varphi^-,\varphi^+\}$, where $\varphi^-:Z_4\to[-1,0]$ and $\varphi^+:Z_4\to[0,1]$ as follows:

$$\varphi^-(\varrho)=\begin{cases}-0.7, & \text{if } \varrho=0\\ -0.6, & \text{if } \varrho=2\\ -0.4, & \text{otherwise}\end{cases}\quad\text{and } \varphi^+(\varrho)=\begin{cases}0.8, & \text{if } \varrho=0\\ 0.6, & \text{if } \varrho=2\\ 0.4, & \text{otherwise.}\end{cases}$$

Then F and φ are BFIs of Z_4, but $F\cup\varphi$ is not a BFI of Z_4.

Theorem 3.7. *Let F and φ be two BFIs of M_S. If $F\subseteq\varphi$ or $\varphi\subseteq F$, then $F\cup\varphi$ is a BFI of M_S.*

Proof. Suppose $F\subseteq\varphi$. Let $\varrho,\varsigma\in M_S$ and $\ddot{\gamma}\in\Gamma$. Then

$$\begin{aligned}(F^-\cup\varphi^-)(\varrho+\varsigma) &= \max\{F^-(\varrho+\varsigma),\varphi^-(\varrho+\varsigma)\}\\ &= F^-(\varrho+\varsigma)\\ &\leq \max\{F^-(\varrho),F^-(\varsigma)\}\\ &= \max\{\max\{F^-(\varrho),\varphi^-(\varrho)\},\max\{F^-(\varsigma),\varphi^-(\varsigma)\}\}\\ &= \max\{(F^-\cup\varphi^-)(\varrho),(F^-\cup\varphi^-)(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(F^-\cup\varphi^-)(\varrho\ddot{\gamma}\varsigma) &= \max\{F^-(\varrho\ddot{\gamma}\varsigma),\sigma^-(\varrho\ddot{\gamma}\varsigma)\}\\ &= F^-(\varrho\ddot{\gamma}\varsigma)\\ &\leq F^-(\varsigma)\ (\text{resp., } \leq F^-(\varrho))\\ &= \max\{F^-(\varsigma),\varphi^-(\varsigma)\}\\ &= (F^-\cup\varphi^-)(\varsigma),\end{aligned}$$

$$\begin{aligned}(F^+ \cup \varphi^+)(\varrho + \varsigma) &= \max\{F^+(\varrho + \varsigma), \varphi^+(\varrho + \varsigma)\} \\ &= \varphi^+(\varrho + \varsigma) \\ &\geq \min\{\varphi^+(\varrho), \varphi^+(\varsigma)\} \\ &= \min\{\max\{F^+(\varrho), \varphi^+(\varrho)\}, \max\{F^+(\varsigma), \varphi^+(\varsigma)\}\} \\ &= \min\{(F^+ \cup \varphi^+)(\varrho), (F^+ \cup \varphi^+)(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(F^+ \cup \varphi^+)(\varrho\ddot{\gamma}\varsigma) &= \max\{F^+(\varrho\ddot{\gamma}\varsigma), \varphi^+(\varrho\ddot{\gamma}\varsigma)\} \\ &= \varphi^+(\varrho\ddot{\gamma}\varsigma) \\ &\geq \varphi^+(\varsigma) \text{ (resp., } \geq \varphi^+(\varrho)) \\ &= \max\{F^+(\varsigma), \varphi^+(\varsigma)\} \\ &= (F^+ \cup \varphi^+)(\varsigma).\end{aligned}$$

Hence, $F \cup \varphi$ is a BFI of M_S. Similarly, if $\varphi \subseteq F$, we get $F \cup \varphi$ is a BFI of M_S. □

Theorem 3.8. *Let κ be a homomorphism from a Γ-semiring M_S onto a Γ-semiring N_S. If φ is a BFI of N_S, then the pre-image $\kappa^{-1}(\varphi)$ of φ is a BFI of M_S.*

Proof. Assume that φ is a BFI of N_S. Let $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$. Then

$$\begin{aligned}(\kappa^{-1}(\varphi)^-)(\varrho + \varsigma) &= \varphi^-(\kappa(\varrho + \varsigma)) \\ &= \varphi^-(\kappa(\varrho) + \kappa(\varsigma)) \\ &\leq \max\{\varphi^-(\kappa(\varrho)), \varphi^-(\kappa(\varsigma))\} \\ &= \max\{\kappa^{-1}(\varphi^-(\varrho)), \kappa^{-1}(\varphi^-(\varsigma))\},\end{aligned}$$

$$\begin{aligned}(\kappa^{-1}(\varphi)^-)(\varrho\ddot{\gamma}\varsigma) &= \varphi^-(\kappa(\varrho\ddot{\gamma}\varsigma)) \\ &= \varphi^-(\kappa(\varrho) * \kappa(\varsigma)) \\ &\leq \varphi^-(\kappa(\varsigma)) \text{ (resp., } \leq \varphi^-(\kappa(\varrho))) \\ &= (\kappa^{-1}(\varphi)^-)(\varsigma),\end{aligned}$$

$$\begin{aligned}(\kappa^{-1}(\varphi)^+)(\varrho + \varsigma) &= \varphi^+(\kappa(\varrho + \varsigma)) \\ &= \varphi^+(\kappa(\varrho) + \kappa(\varsigma)) \\ &\geq \min\{\varphi^+(\kappa(\varrho)), \varphi^+(\kappa(\varsigma))\} \\ &= \min\{\kappa^{-1}(\varphi^+(\varrho)), \kappa^{-1}(\varphi^+(\varsigma))\},\end{aligned}$$

$$\begin{aligned}(\kappa^{-1}(\varphi)^{+})(\varrho\ddot{\gamma}\varsigma) &= \varphi^{+}(\kappa(\varrho\ddot{\gamma}\varsigma))\\ &= \varphi^{+}(\kappa(\varrho) * \kappa(\varsigma))\\ &\geq \varphi^{+}(\kappa(\varsigma)) \text{ (resp., } \geq \varphi^{+}(\kappa(\varrho)))\\ &= (\kappa^{-1}(\varphi)^{+})(\varsigma).\end{aligned}$$

Hence, $\kappa^{-1}(\varphi)$ is a BFI of M_S. □

Theorem 3.9. *Let κ be a homomorphism from a Γ-semiring M_S onto a Γ-semiring N_S. If F is a BFI of M_S, then the homomorphic image $\kappa(F)$ of F is a BFI of N_S.*

Proof. Assume that F is a BFI of M_S. Let $\varrho, \varsigma \in N_S$ and $\ddot{\gamma} \in \Gamma$. Suppose neither $\kappa^{-1}(\varrho)$ nor $\kappa^{-1}(\varsigma)$ is non-empty. Since κ is onto, there exist $\upsilon, \tau \in M_S$ such that $\kappa(\upsilon) = \varrho$ and $\kappa(\tau) = \varsigma$ and it follows that $\upsilon + \tau \in \kappa^{-1}(\varrho + \varsigma)$ and $\upsilon\ddot{\gamma}\tau \in \kappa^{-1}(\varrho\ddot{\gamma}\varsigma)$. Thus

$$\begin{aligned}(\kappa(F))^{-}(\varrho+\varsigma) &= \min\{F^{-}(\ddot{z}) \mid \ddot{z} \in \kappa^{-1}(\varrho+\varsigma)\}\\ &= \min\{F^{-}(\upsilon+\tau) \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in \kappa^{-1}(\varsigma)\}\\ &\leq \min\{\max\{F^{-}(\upsilon), F^{-}(\tau)\} \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in \kappa^{-1}(\varsigma)\}\\ &= \min\{\max\{F^{-}(\upsilon) \mid \upsilon \in \kappa^{-1}(\varrho)\}, \max\{F^{-}(\tau) \mid \tau \in \kappa^{-1}(\varsigma)\}\}\\ &\leq \max\{\min\{F^{-}(\upsilon) \mid \upsilon \in \kappa^{-1}(\varrho)\}, \min\{F^{-}(\tau) \mid \tau \in \kappa^{-1}(\varsigma)\}\}\\ &= \max\{(\kappa(F))^{-}(\varrho), (\kappa(F))^{-}(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(\kappa(F))^{-}(\varrho\ddot{\gamma}\varsigma) &= \min\{F^{-}(\ddot{z}) \mid \ddot{z} \in \kappa^{-1}(\varrho\ddot{\gamma}\varsigma)\}\\ &= \min\{F^{-}(\upsilon\ddot{\gamma}\tau) \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in \kappa^{-1}(\varsigma)\}\\ &\leq \min\{F^{-}(\tau) \mid \tau \in \kappa^{-1}(\varsigma)\} \text{ (resp., } \leq \min\{F^{-}(\upsilon) \mid \upsilon \in \kappa^{-1}(\varrho)\})\\ &= (\kappa(F))^{-}(\varsigma),\end{aligned}$$

$$\begin{aligned}(\kappa(F))^{+}(\varrho+\varsigma) &= \max\{F^{+}(\ddot{z}) \mid \ddot{z} \in \kappa^{-1}(\varrho+\varsigma)\}\\ &= \max\{F^{+}(\upsilon+\tau) \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in g^{-1}(\varsigma)\}\\ &\geq \max\{\min\{F^{+}(\upsilon), F^{+}(\tau)\} \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in \kappa^{-1}(\varsigma)\}\\ &\geq \min\{\max\{F^{+}(\upsilon) \mid \upsilon \in \kappa^{-1}(\varrho)\}, \max\{F^{+}(\tau) \mid \tau \in \kappa^{-1}(\varsigma)\}\}\\ &= \min\{(\kappa(F))^{+}(\varrho), (\kappa(F))^{+}(\varsigma)\},\end{aligned}$$

$$\begin{aligned}(\kappa(F))^{+}(\varrho\ddot{\gamma}\varsigma) &= \max\{F^{+}(\ddot{z}) \mid \ddot{z} \in \kappa^{-1}(\varrho\ddot{\gamma}\varsigma)\}\\ &= \max\{F^{+}(\upsilon\ddot{\gamma}\tau) \mid \upsilon \in \kappa^{-1}(\varrho), \tau \in \kappa^{-1}(\varsigma)\}\\ &\geq \max\{F^{+}(\tau) \mid \tau \in \kappa^{-1}(\varsigma)\} \text{ (resp., } \geq \max\{F^{+}(\upsilon) \mid \upsilon \in \kappa^{-1}(\varrho)\})\\ &= (\kappa(F))^{+}(\varsigma).\end{aligned}$$

Hence, $\kappa(F)$ is a BFI of N_S. □

Theorem 3.10. *Let F be a non-empty subset of M_S. Then the BF characteristic set of F, δ_F is a BF left (resp., right) ideal of M_S if and only if F is a left (resp., right) ideal of M_S.*

Proof. Suppose δ_F is a BF left ideal of M_S. Let $\varrho, \varsigma \in M_S$ and $\ddot{\gamma} \in \Gamma$. Then $\delta_F^+(\varrho+\varsigma) \geq \min\{\delta_F^+(\varrho), \delta_F^+(\varsigma)\} = 1$ and $\delta_F^+(\varrho\ddot{\gamma}\varsigma) \geq \delta_F^+(\varsigma) = 1$, so $\varrho+\varsigma \in F$ and $\varrho\ddot{\gamma}\varsigma \in F$. Hence, F is a left ideal of M_S.

Conversely, suppose that F is a left ideal of M_S. Let $\varrho, \varsigma \in F$ and $\ddot{\gamma} \in \Gamma$.

If $\varrho, \varsigma \in F$, then $\varrho+\varsigma \in F$ and $\varrho\ddot{\gamma}\varsigma \in F$. Now,

(i) $\delta_F^+(\varrho+\varsigma) = 1 = \min\{\delta_F^+(\varrho), \delta_F^+(\varsigma)\}$

(ii) $\delta_F^+(\varrho\ddot{\gamma}\varsigma) = 1 = \delta_F^+(\varsigma)$

(iii) $\delta_F^-(\varrho+\varsigma) = -1 = \max\{\delta_F^-(\varrho), \delta_F^-(\varsigma)\}$

(iv) $\delta_F^-(\varrho\ddot{\gamma}\varsigma) = -1 = \delta_F^-(\varsigma)$.

If $\varrho, \varsigma \notin F$, then $\delta_F^+(\varrho) = 0 = \delta_F^-(\varrho)$ and $\delta_F^+(\varsigma) = 0 = \delta_F^=(\varsigma)$. Now,

(i) $\delta_F^+(\varrho+\varsigma) = 0 \geq \min\{\delta_F^+(\varrho), \delta_F^+(\varsigma)\}$

(ii) $\delta_F^+(\varrho\ddot{\gamma}\varsigma) = 0 = \delta_F^+(\varsigma)$

(iii) $\delta_F^-(\varrho+\varsigma) = 0 \leq \max\{\delta_F^-(\varrho), \delta_F^-(\varsigma)\}$

(iv) $\delta_F^-(\varrho\ddot{\gamma}\varsigma) = 0 = \delta_F^-(\varsigma)$.

If $\varrho \notin F$ and $\varsigma \in F$, then $\delta_F^+(\varrho) = 0 = \delta_F^-(\varrho), \delta_F^+ + (\varsigma) = 1$ and $\delta_F^-(\varsigma) = -1$. Now,

(i) $\delta_F^+(\varrho+\varsigma) \geq \min\{\delta_F^+(\varrho), \delta_F^+(\varsigma)\}$

(ii) $\delta_F^+(\varrho\ddot{\gamma}\varsigma) \geq \delta_F^+(\varsigma)$

(iii) $\delta_F^-(\varrho+\varsigma) \leq \max\{\delta_F^-(\varrho), \delta_F^-(\varsigma)\}$

(iv) $\delta_F^+(\varrho\ddot{\gamma}\varsigma) \leq \delta_B^-(\varsigma)$.

A similar argument holds for $\varrho \in F$ and $\varsigma \notin F$.

Hence, δ_F is a BF left ideal of M_S. In a similar pattern, we can prove the case of a BF right ideal of M_S. □

Corollary 3.11. *Let F be a non-empty subset of M_S. Then the BF characteristic set of F, δ_F is a BFI of M_S if and only if F is an ideal of M_S.*

4. Conclusion

This paper introduces the concept of BFIs of Γ-semirings, and we established a one-to-one correspondence between the BFI of Γ-semirings and its level set. Further, we proved that the intersection of BFIs of a Γ-semiring is also a BFI. Also, we investigated that homomorphic and pre-image of a BFI of a Γ-semiring is also a BFI. We expect these structures to be useful in developing bipolar fuzzy normal ideals and maximal ideals of Γ-semirings.

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