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FIXED POINTS OF COUNTABLY CONDENSING MULTIMAPS

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Abstract. In this work, it is proved a fixed point theorem for a Chandrabhan type multimap having a weakly closed graph, and taking convex values only on some subset of its domain. After showing that countably condensing multimaps are Chandrabhan type multimaps, the above result was applied to countably condensing multimaps. These include the fixed point theorems of Mönch, Sadovskii, Darbo, Dhage, and Agarwal and O'Regan as special cases.

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1. Introduction and Preliminaries

Introduced by Sadovskii [15] in 1967, condensing operators have been an important topic in nonlinear functional analysis. He defined the condensing operator for single-valued functions and proved that a condensing function from a closed bounded convex subset of a Banach space into itself has a fixed point.

The concept of the condensing function was generalized by Daher [5] as that of a countable condensing function, which is condensing only in countable sets. And it was extended to multimaps (or maps) by Himmelberg, Porter and Van Vleck [9].

On the other hand, Mönch [11] introduced a new class of single-valued functions to generalize the fixed point theorem of Sadovskii. It is generalized as a class of multimaps called Mönch type maps by O'Regan and Precup [12]. And the maps were relaxed to Chandrabhan type maps by Dhage [7].

Cardinali and Rubbioni [4] tried to prove the fixed point theorem for a countably condensing Mönch type map having a weakly closed graph, and taking convex values only on some subset of its domain. In this paper, we propose a new theorem that correct the result in [4] and extend it for Chandrabhan type

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maps. After showing that countably condensing multimaps are Chandrabhan type multimaps under suitable assumptions, the fixed point theorem for Chandrabhan type maps was applied to countably condensing multimaps.

Throughout this paper, we assume that maps have nonempty values otherwise explicitly stated or obvious from the context.

Definition 1.1. A nonempty subset Y of a locally convex Hausdorff topological linear space E is said to be *quasi-convex* (or *almost convex*) if for any $V \in \mathcal{V}$, where $\mathcal{V}$ is a neighborhood system of the origin 0 in E, and for any finite set $\{y_1, y_2, ..., y_n\} \subset Y$, there exists a finite set $\{z_1, z_2, ..., z_n\} \subset Y$ such that $z_i - y_i \in \mathcal{V}$ for each $i = 1, 2, ..., n$ and $\text{co}\{z_1, z_2, ..., z_n\} \subset Y$.

For example, deleting a certain subset of the boundary of a closed convex set, we get a quasi-convex set. For details, see [8,13].

Definition 1.2. [3,4] Let X be a nonempty subset of a locally convex Hausdorff topological linear space E. It is said that a map $G : X \multimap E$ has a *weakly closed graph* in $X \times E$ if for every net$(x_\delta)_\delta$ in X, $x_\delta \to x$, $x \in X$, and for every net $(y_\delta)_\delta$, $y_\delta \in G(x_\delta)$, $y_\delta \to y$, then $S(x, y) \cap G(x) \neq \emptyset$, where $S(x, y) = \{x + \lambda(y - x) : \lambda \in [0, 1]\}$.

Theorem 1.3 ([3], Teorema I)**.** *Let E be a locally convex Hausdorff topological linear space, K be a nonempty compact subset of E and $G : K \multimap K$ be a map taking closed values and with the properties*

(1) *there exists a quasi-convex subset A of K such that $\overline{A} = K$ and $G(x)$ is convex for every $x \in A$; and*
(2) *G has a weakly closed graph.*

Under these conditions, there exists an $x \in K$ such that $x \in G(x)$.

2. Fixed Point Theorems for Chandrabhan Type Maps

The following theorem is a main result of this paper:

Theorem 2.1. *Let X be a closed convex subset of a Banach space E, B be a relatively compact subset of X and Y be a subset of X such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any relatively compact convex subset K of X. Assume that $F : X \multimap X$ is a map with compact values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*
(2) *F has a weakly closed graph;*
(3) *F maps compact sets into relatively compact sets; and*
(C) *For $A \subset X$, $\overline{A}$ is compact if $A = \text{co}(B \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset C of A.*

Then F has a fixed point.

Proof. Put $K_0 = \text{co}(B)$, $K_{n+1} = \text{co}(B \cup F(K_n))$ for $n = 0, 1, 2, \cdots$ and $K = \bigcup_{n=0}^{\infty} K_n$. The Mazur Theorem implies that K_0 is relatively compact. By induction, $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n \subseteq K_{n+1} \cdots$ and K_n is relatively compact. Note that K is convex, since K_n is convex for $n = 0, 1, 2, \cdots$.

Furthermore we can show that $K = \text{co}(B \cup F(K))$. For each n, $\text{co}(B \cup F(K_n)) \subseteq \text{co}(B \cup F(K))$, so $K = \bigcup_{n=0}^{\infty} \text{co}(B \cup F(K_n)) \subseteq \text{co}(B \cup F(K))$. On the other hand, K is a convex set containing B and $\bigcup_{n=0}^{\infty} F(K_n) = F(K)$, hence $\text{co}(B \cup F(K)) \subseteq K$.

For every $n = 0, 1, 2, \cdots$, consider the space (K_n, d), where d is the metric induced on K_n by the metric generated by $||\cdot||$. The compactness of $\overline{K_n}$ implies that $(\overline{K_n}, d)$ is a separable space. Since every subspace of a separable metric space is separable, there exists a countable set C_n of K_n with $\overline{C_n}^{(K_n,d)} = K_n$. Put $C = \bigcup_{n=0}^{\infty} C_n$, then $\overline{C} = \overline{K}$, since $\overline{K} = \overline{\bigcup_{n=0}^{\infty} K_n} = \bigcup_{n=0}^{\infty} \overline{K_n} = \bigcup_{n=0}^{\infty} \overline{C_n} = \overline{\bigcup_{n=0}^{\infty} C_n} = \overline{C}$. Condition (C) implies that $\overline{K}$ is compact.

Now, we consider the map $T : \overline{K} \multimap \overline{K}$ defined by $T(x) = F(x) \cap \overline{K}$ for all $x \in \overline{K}$. Then the map T has nonempty values. In fact, fixed $x \in \overline{K}$, there exists a sequence $(x_n)_{n\in\mathbb{N}}$ in K such that $x_n \to x$. Let us consider a sequence $(y_n)_{n\in\mathbb{N}}$ such that $y_n \in F(x_n)$, $n \in \mathbb{N}$. Since $F(K) \subset K$ and $\overline{K}$ is compact, there is an $y \in \overline{K}$ such that $y_n \to y$. By (2), $S(x, y) \cap F(x) \neq \emptyset$. As the convexity of $\overline{K}$ implies $S(x, y) \subset \overline{K}$, $T(x) = F(x) \cap \overline{K} \neq \emptyset$.

The above discussion also shows that $\emptyset \neq S(x, y) \cap F(x) = S(x, y) \cap F(x) \cap \overline{K} = S(x, y) \cap T(x)$, so T has a weakly closed graph in $\overline{K} \times \overline{K}$.

Furthermore $Y \cap \overline{K}$ is dense in $\overline{K}$. As F takes closed values and satisfies hypothesis (1), T satisfies all the assumptions of Theorem 1.3. Therefore, there exists $x \in \overline{K}$ such that $x \in T(x) \subset F(x)$. □

Remark 2.2. The proof of Theorem 2.1 was followed a systematic basic idea of Theorem 3.1 in [4]. Theorem 1.3 was also used in the proof of [4], but the condition of Y was different as follows:

$$Y \text{ is a quasi-convex subset of} X \text{ and } \overline{Y} = X. \tag{2.1}$$

The example below shows that condition (2.1) is insufficient to prove Theorem 2.1 using Theorem 1.3. If $X = [0, 1] \times [0, 1]$ has a usual topology and $Y = (0, 1) \times (0, 1) \cup \{(0, 0), (0, 1)\}$, then (2.1) holds but if $K = \{0\} \times [0, 1]$, then $Y \cap K = \{(0, 0), (0, 1)\}$ is neither a dense subset of K nor a quasi-convex subset of K.

Also, as the corollary below, F is a Mönch type maps (that is, B is a point) in [4]:

Corollary 2.3. *Let X be a closed convex subset of a Banach space E and Y be a subset of X such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any compact convex subset K of X. Assume that $F : X \multimap X$ is a map with compact values satisfying the followings:*

(1) *$F(x)$ is convex for every $x \in Y$;*

(2) *F has a weakly closed graph;*

(3) *F maps compact sets into relatively compact sets;*

(M) *there exists $x_0 \in X$ such that $A \subset X$, $\overline{A}$ is compact if $A = \mathrm{co}(\{x_0\} \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset C of A.*

Then F has a fixed point.

Corollary 2.4. [7] *Let X be a closed convex subset of a Banach space E and B be a relatively compact subset of X. Assume that $F : X \multimap X$ is a upper semicontinuous map with convex, compact values satisfying the following:*

(C) *For $A \subset X$, $\overline{A}$ is compact if $A = \mathrm{co}(B \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset C of A.*

Then F has a fixed point.

Proof. The upper semicontinuity of F implies that F has a closed graph and maps compact sets into compact sets. □

Corollary 2.5. *Let X be a closed convex subset of a Banach space E. Assume that $F : X \multimap X$ is a upper semicontinuous map with convex, compact values satisfying the following:*

(M) *there exists $x_0 \in X$ such that $A \subset X$, $\overline{A}$ is compact if $A = \mathrm{co}(\{x_0\} \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset C of A.*

Then F has a fixed point.

Corollary 2.3 includes the following known results in the fixed point theory for single-valued functions in Banach spaces. See [7]:

Corollary 2.6. *Let X be a closed convex subset of a Banach space E and $f : X \to X$ a single-valued function satisfying any one of the following conditions:*

(1) *f is continuous and α-condensing (Sadovskii [15]).*

(2) *f is continuous and a set-contraction (Darbo [6]).*

(3) *f is a compact and continuous map (Schauder).*

Then f has a fixed point.

3. Fixed Point Theorems for Countably Condensing Maps

Definition 3.1. Let E be a Banach space with the norm $\|\cdot\|$, $\mathcal{P}_b(E) = \{H \subset E : H \neq \emptyset, H \text{ bounded}\}$. A function $\beta : \mathcal{P}_b(E) \multimap \mathbb{R}_0^+$ is called a *measure of noncompactness* (MNC, for short) on E provided that the following conditions hold for any $A, B \in \mathcal{P}_b(E)$:

(1) $\beta(\overline{\mathrm{co}} A) = \beta(A)$;

(2) $\overline{A}$ is compact iff $\beta(A) = 0$; and

(3) $\beta(A \cup B) = \max\{\beta(A), \beta(B)\}$.

In [2,10], a MNC was defined as satisfying only (1). However, we follow the definition of [14] such that a MNC also satisfies (2) and (3). Note that a MNC β satisfies the property

(4) $A \subset B$ implies $\beta(A) \leq \beta(B)$.

Definition 3.2. Let X be a nonempty subset of a Banach space E and let β be a MNC. A map $F : X \multimap E$ is said to be *countably condensing* if

(I) $F(X)$ is bounded; and

(II) $\beta(F(A)) < \beta(A)$ for all countable bounded subsets A of X with $\beta(A) > 0$.

The condition (II) can be equivalently formulated as

(II′) for all countable bounded subsets A of X, the relation $\beta(A) \leq \beta(F(A))$ implies that $\overline{A}$ is compact.

Lemma 3.3. *Let X be a closed convex subset of a Banach space E. Suppose that a countably condensing map $F : X \multimap X$ maps compact sets into relatively compact sets. Then F is a Chandrabhan type maps, that is, there exists a relatively compact subset B of X such that $A \subset X$, $\overline{A}$ is compact if $A = \mathrm{co}(B \cup F(A))$ and $\overline{A} = \overline{C}$ with a countable subset C of A.*

Proof. Suppose that for $A = \mathrm{co}(B \cup F(A))$, whose existence was shown in the proof of Theorem 2.1, there exists a countable subset C of A such that $\overline{A} = \overline{C}$. Every point of C can be written as a finite combination of points belonging to the set $B \cup F(A)$, so there exists a countable set $M \subset A$ such that $C \subset \mathrm{co}(B \cup F(M))$. By the definition of a countably condensing map, $F(X)$ is bounded, and the sets A, C and M are also bounded. Since $\beta(B) = 0$,

$$\beta(C) \leq \beta(\mathrm{co}(B \cup F(M))) = \beta(B \cup F(M)) = \beta(F(M)). \tag{*}$$

First, we show that $\beta(M) = 0$. If not, then $\beta(F(M)) < \beta(M)$, because F is countably condensing. Combining above argument, we obtain

$$\beta(C) < \beta(M) \leq \beta(A) = \beta(\overline{A}) = \beta(\overline{C}) = \beta(C),$$

a contradiction. Therefore $\overline{M}$ is compact.

Now, we prove $\beta(\overline{A}) = 0$. As F maps compact sets into relatively compact sets, $\beta(\overline{F(\overline{M})}) = 0$. Hence $\beta(F(M)) = 0$ and $\beta(C) = 0$ by (*), which implies that $\beta(\overline{A}) = \beta(C) = 0$, that is, $\overline{A}$ is compact. □

By Lemma 3.3 and Theorem 2.1, we obtain the following theorem:

Theorem 3.4. *Let X be a closed convex subset of a Banach space E, and Y be a subset of X such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any compact convex subset K of X. Assume that $F : X \multimap X$ is a countably condensing map with compact values satisfying hypotheses (1), (2), (3) of Theorem 2.1. Then F has a fixed point.*

From Theorem 3.4, we also obtain the following corollary in [1]:

Corollary 3.5. *Let X be a closed convex subset of a Banach space E. Assume that $F : X \multimap X$ is a upper semicontinuous, countably condensing map with convex, compact values. Then F has a fixed point.*

Definition 3.6. Let X be a nonempty subset of a Banach space E and let β be a MNC. For $k \in [0, 1)$, a map $F : X \multimap E$ is said to be *countably k-condensing* if

(I) $F(X)$ is bounded; and

(II) $\beta(F(A)) \leq k\beta(A)$ for all countable bounded subsets A of X.

Since a countably k-condensing map is countably condensing, we obtain the following theorem:

Theorem 3.7. *Let X be a closed convex subset of a Banach space E, and Y be a subset of X such that $K \cap Y$ is quasi-convex and $\overline{K \cap Y} = \overline{K}$ for any compact convex subset K of X. Assume that $F : X \multimap X$ is a countably k-condensing map with compact values satisfying hypotheses (1), (2), (3) of Theorem 2.1. Then F has a fixed point.*

The following corollary is in [1]:

Corollary 3.8. *Let X be a closed convex subset of a Banach space E. Assume that $F : X \multimap X$ is a upper semicontinuous, countably k-condensing map with convex, compact values. Then F has a fixed point.*

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