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A SHARP INCLUSION FOR λ-PSEUDO-STAR MAPS

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Abstract. In this short paper, we determine the largest real number ρ such that if $f(z)$, normalized by $f(0) = f'(0) - 1 = 0$, satisfies certain not-linear sums of geometric expression, then $f(z)$ is a λ-pseudo star map of order ρ.

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1. Introduction

Let A denote the class of functions

$$f(z) = z + a_2 z^2 + \cdots$$

which are holomorphic in the unit disk $|z| < 1$.

In [1] Babalola introduced and studied the class $L_\lambda(\beta)$ of λ-pseudo star maps of order β as

$$\text{Re}\,\frac{z(f'(z))^\lambda}{f(z)} > \beta,\ z \in E$$

where $0 \leq \beta < 1$ and $\lambda \geq 1$ are real numbers. Among other interesting results, he proved that the class $L_\lambda(\beta)$ consists only of univalent functions in the unit disk. He also gave examples of such functions, which include and indicate the univalence of certain transcendental functions under the geometry defining the λ-pseudo star maps in the unit disk.

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In this short paper we determine a real number ρ such that if $f(z)$, normalized by $f(0) = f'(0)-1 = 0$, satisfies certain not-linear sums of geometric expressions, then $f(z)$ is a λ-pseudo star map of order ρ, where ρ is the largest possible such real number.

The result shows and strengthen the notion that certain not-linear sums of geometric expressions also guarantees univalence in the unit disk as discussed in [2]. The result also includes the well known result of MacGregor [4] that every convex function is starlike of a best possible order $1/2$.

Furthermore, it is well known that any Caratheodory function $p(z) = 1 + c_1(z) + ...$ is subordinate to the Mobius function $L_o(z) = (1 + z)/(1 - z)$.

In this paper, we determine the largest real number ρ such that

$$Re\left(\frac{z(f'(z))^{\lambda}}{f(z)}\right) > \rho, z \in E$$

given that $f \in L_{\lambda}(\beta)$ for $0 \leq \beta < 1$ and $\lambda \geq 1$ are real numbers.

We shall employ Briot-Bouquet differential subordination technique to achieve this.

2. Preliminary Lemmas

The Preliminary Lemmas which will be useful in proving our main result are stated as follows.

Lemma 1. [3] *Let $p(z)$ be analytic in E and satisfy Briot-Bouquet differential subordination if*

$$p(z) + \frac{zp'(z)}{\gamma p(z) + \delta} \prec h(z), \quad z \in E \tag{1}$$

for complex constants γ and δ and a complex function $h(z)$ with $h(0) = 1$ such that $Re[\gamma h(z) + \delta)] > 0$ in E. If the differential equation

$$q(z) + \frac{zq'(z)}{\gamma q(z) + \delta} = h(z), \quad q(0) = 1 \tag{2}$$

has a solution $q(z)$ which is univalent, then $p(z) \prec q(z) \prec h(z)$ and $q(z)$ is the best dominant.

For more on the technique of differential subordination, see [5,6,7,8]

Lemma 2. [3] *For complex constants λ, γ and a convex univalent function $h(z)$ in E satisfying $h(0) = 1$ and $Re[\gamma h(z) + \delta] > 0$. Suppose the differential subordination (1) is satisfied by $p \in P$. If the differential equation (2) has univalent solution $q(z)$ in E and $q(z)$ is the best dominant of (1), then $p(z) \prec q(z) \prec h(z)$. Moreover, the formal solution of (2) is given as*

$$q(z) = \frac{zF'(z)}{F(z)} = \frac{\gamma + \delta}{\gamma}\left(\frac{H(z)}{F(z)}\right)^{\gamma} - \frac{\delta}{\gamma}. \tag{3}$$

where

$$F(z)^{\gamma} = \frac{\gamma + \delta}{z^{\delta}} \int_0^z t^{\delta-1} H(t)^{\gamma} dt. \tag{4}$$

and

$$H(z) = z\, exp\left(\int_0^z \frac{h(t) - 1}{t} dt\right). \tag{5}$$

Lemma 3. [7] *For a positive measure v on [0,1] and a complex valued function h defined on $E \times [0,1]$ with $h(.,t)$ which is analytic in E for each $t \in [0,1]$ for all $z \in E$. Also suppose Re $[h(z,t)] \geq 0$, $h(-r,t)$ is real and Re $\left[\frac{1}{h(z,t)}\right] \geq \frac{1}{h(-r,t)}$ for $|z| \leq r < 1$ and $t \in [0,1]$ if*

$$h(z) = \int_0^1 h(z,t)dv(t)$$

then

$$Re\left[\frac{1}{h(z)}\right] \geq \frac{1}{h(-r)}.$$

Let a, b and c be real or complex numbers numbers with $(c \neq 0, -1, -2, ...)$, the hypergeometric function is defined by

$$_2F_1(a,b,c,z) = 1 + \frac{a.b}{c}.\frac{z}{1!} + \frac{a(a+1).b(b+1)}{c(c+1)}.\frac{z^2}{2!} + \frac{a(a+1)(a+2).b(b+1)(b+2)}{c(c+1)(c+2)}.\frac{z^3}{3!} + ... \quad (6)$$

The above series converges absolutely for $z \in E$ and thus, represents an analytic function in E. The identities below are associated with the hypergeometric series. Let a, b and c be real numbers and $(c \neq 0, -1, -2, ...)$, then

$$\int_0^1 t^{b-1}(1-t)^{c-b-1}(1-tz)^{(-a)}d(t) = \frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)} {}_2F_1(a,b,c;z) \quad (c > b > 0)$$

$$_2F_1(a,b,c;z) = {}_2F_1(b,a,c;z)$$

and

$$_2F_1(a,b,c;z) = (1-z)^{-a} {}_2F_1\left(a, c-b, c; \frac{z}{z-1}\right)$$

3. Main Result

Here we present the main result.

Theorem 1. *Let $f \in L_\lambda(\beta)$, suppose $0 \leq \beta < 1$ and $\lambda \geq 1$*

Then

$$\frac{z(f'(z))^\lambda}{f(z)} \prec q(z) \prec \frac{1 + (1-2\beta)z}{1-z}, \quad (7)$$

where

$$q(z) = \frac{(1-z)^{2\lambda(\beta-1)}}{\int_0^1 (1-sz)^{2\lambda(\beta-1)}ds}.$$

Furthermore

$$Re\left(\frac{z(f'(z))^\lambda}{f(z)}\right) > \rho,$$

where

$$\rho = \left[{}_2F_1\left(1, 2\lambda(1-\beta); 2; \frac{1}{2}\right)\right]^{-1}$$

and the bound ρ is the best possible.

Proof. Since $f \in L_\lambda(\beta)$, we see that

$$\left\{1 + \frac{zf''(z)}{f'(z)} + \frac{1}{\lambda}\frac{zf'(z)}{f(z)}\left((f'(z))^{\lambda-1} - 1\right)\right\} = F_\lambda(\beta) \prec \frac{1+(1-2\beta)z}{1-z}.$$

Let

$$\frac{z(f'(z))^\lambda}{f(z)} = p(z). \tag{8}$$

Then $p(z)$ is analytic in E with $p(0) = 1$. Thus, for of both sides of (8) if we take the logarithmic differentiations , it yields

$$p(z) + \frac{zp'(z)}{p(z)} = \frac{z(f'(z))^\lambda}{f(z)} + 1 + \frac{\lambda z f''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}$$

$$p(z) + \frac{zp'(z)}{p(z)} = \lambda\left[\frac{1-\lambda}{\lambda} + 1 + \frac{zf''(z)}{f'(z)} + \frac{1}{\lambda}\frac{zf'(z)}{f(z)}\left(f'^{\lambda-1}(z) - 1\right)\right]$$

Thus

$$\frac{1}{\lambda}\left[p(z) + \frac{zp'(z)}{p(z)}\right] + \frac{\lambda-1}{\lambda} = F_\lambda(\beta) \prec \frac{1+(1-2\beta)z}{1-z}$$

which implies

$$\frac{1}{\lambda}\left[q(z) + \frac{zq'(z)}{q(z)}\right] + \frac{\lambda-1}{\lambda} = \frac{1+(1-2\beta)z}{1-z},$$

Hence,

$$q(z) + \frac{zq'(z)}{q(z)} = \frac{\lambda + (1-2\beta)\lambda z}{1-z} - (\lambda - 1) = \frac{1 + [2\lambda(1-\beta) - 1]z}{1-z} = h(z) \tag{9}$$

Then $h(0) = 1$ and we have $\gamma = 1$ and $\delta = 0$, it can be easily verified that $\gamma h(z) + \delta$ has positive real part for $0 \le \beta < 1$. By Lemma 1, $p(z)$ satisfies the differential subordination (1). Thus

$$\frac{z(f'(z))^\lambda}{f(z)} \prec q(z) \prec h(z),$$

where q(z) is the solution of the differential equation (9) obtained as follows.

$$H(z) = z(1-z)^{2\lambda(\beta-1)}$$

and

$$F(z) = \int_0^z (1-t)^{2\lambda(\beta-1)} dt.$$

Now from (3), we have

$$q(z) = \left(\frac{H(z)}{F(z)}\right)^\gamma = \frac{1}{Q(z)},$$

where

$$Q(z) = \int_0^1 \left(\frac{1-sz}{1-z}\right)^{2\lambda(\beta-1)} ds.$$

Next we show that

$$\inf_{|z|<1}\{\mathrm{Re}(q(z))\} = q(-1), z \in E. \tag{10}$$

To prove (10), we need to show that

$$\text{Re}\left[\frac{1}{Q(z)}\right] \geq \frac{1}{Q(-1)}.$$

with some simplications and by Lemma 3, we have

$$Q(z) = \frac{\Gamma(b)}{\Gamma(c)}{}_2F_1\left(1, a, c, \frac{z}{z-1}\right),$$

where $a = 2\lambda(1-\beta)$, $b = 1$ and $c = 2$. Hence,

$$Q(z) = \int_0^1 h(z,s)dv(s),$$

with

$$h(z,s) = \frac{1-z}{1-(1-s)z}(0 \leq s \leq 1)$$

and

$$dv(s) = \frac{\Gamma(b)}{\Gamma(a)\Gamma(c-a)}s^{a-1}(1-s)^{c-a-1}ds$$

which is a positive measure on $[0,1]$. It will be noted that Re $h(z,s) > 0$, $h(-r,s)$ is real for $0 \leq r < 1$

$$\text{Re}\left\{\frac{1}{h(z,s)}\right\} = Re\left\{\frac{1-(1-s)z}{1-z}\right\} \geq \frac{1+(1-s)r}{1+r} = \frac{1}{h(-r,s)}$$

for $|z| \leq r < 1$ and $s \in [0,1]$. Hence by Lemma 3, we have

$$\text{Re}\,\frac{1}{Q(z)} \geq \frac{1}{Q(-r)}$$

and by letting $r \to 1^-$, we have

$$\text{Re}\,\frac{1}{Q(z)} \geq \frac{1}{Q(-1)}.$$

Hence

$$\text{Re}\left\{\frac{z(f'(z))^\lambda}{f(z)}\right\} > q(-1).$$

that is

$$\text{Re}\left\{\frac{z(f'(z))^\lambda}{f(z)}\right\} > \rho,$$

where

$$\rho = (1-z)^{2\lambda(\beta-1)}\left[\int_0^1 (1-sz)^{2\lambda(\beta-1)}ds\right]^{-1}.$$

By Lemma 3, we get

$$\rho = \left[\frac{\Gamma(b)\Gamma(c-b)}{\Gamma(c)}{}_2F_1\left(1, a; c; \frac{z}{z-1}\right)\right]^{-1},$$

which by some simplification, yields

$$\rho = \left[{}_2F_1\left(1, 2\lambda(1-\beta); 2; \frac{1}{2}\right)\right]^{-1}.$$

The bound ρ is the best possible.

□

Next, we give some corollaries. Taking $\lambda = 1$ in Theorem 1, it yields

Corollary 1. *Let* $f \in L_1(\beta)$. *Suppose* $0 \le \beta \le 1$, *then*

$$\frac{zf^{'}(z)}{f(z)} \prec q_1(z) \prec \frac{1+(1-2\beta)z}{1-z}$$

where

$$q_1 = \frac{(1-z)^{2(\beta-1)}}{\int_0^1 (1-sz)^{2(\beta-1)} ds}$$

Furthermore,

$$Re\left\{\frac{z(f'(z))}{f(z)}\right\} > \rho_1$$

where

$$\rho_1 = \left[{}_2F_1\left(1, 2(1-\beta); 2; \frac{1}{2}\right)\right]^{-1}$$

The bound ρ_1 is the best possible.

Taking $\lambda = 2$ in Theorem 1, we have

Corollary 2. *Let* $f \in L_2(\beta)$. *Suppose* $0 \le \beta \le 1$, *then*

$$f^{'}(z)\frac{zf^{'}(z)}{f(z)} \prec q_2(z) \prec \frac{1+(1-2\beta)z}{1-z}$$

where

$$q_2 = \frac{(1-z)^{4(\beta-1)}}{\int_0^1 (1-sz)^{4(\beta-1)} ds}$$

Furthermore,

$$Re\left\{f^{'}(z)\frac{zf^{'}(z)}{f(z)}\right\} > \rho_2$$

where

$$\rho_2 = \left[{}_2F_1\left(1, 4(1-\beta); 2; \frac{1}{2}\right)\right]^{-1}$$

The bound ρ_2 is the best possible. This is a new representation of a product combination for bounded turning and starlike functions.

4. Remark

The Theorem above shows that the geometric expression $\frac{z(f'(z))^\lambda}{f(z)}$ is univalent of order ρ in the unit disk which improves the result of Babalola [1]. By (6), ρ can be written in the series form

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \frac{2\lambda(1-\beta)+j}{j+2}$$

The above series can be rewritten as

$$\frac{1}{\rho} = 1 + \sum_{k=1}^{\infty} \frac{1}{2^k} \prod_{j=1}^{k} \frac{\lambda(1-\beta)+(j+2)}{j+2} + \frac{\lambda(1-\beta)-2}{j+2}$$

5. Conclusion

The above results are new and the corresponding values of ρ which is the best possible for different values of λ improve existing results in geometric function theory.

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References

[1] K.O. Babalola, On λ-pseudo-starlike functions, J. Class. Anal. 3 (2013), 137–147. https://doi.org/10.7153/jca-03-12.

[2] K.O. Babalola, Combinations of geometric expressions implying schlichtness, An. Univ. Oradea, Fasc. Mat. Tom XXI (2014), 91–94.

[3] P. Eenigenburg, S.S. Miller, P.T. Mocanu, M.O. Reade, On a Briot-Bouquet differential surbordination, Rev. Roum. Math. Pures Appl. 29 (1984), 567–573.

[4] T.H. MacGregor, The radius of univalence of certain analytic functions, Proc. Amer. Math. Soc. 14 (1963), 514–520.

[5] S.S. Miller, P.T. Mocanu, Univalent solutions of Briot-Bouquet differential equations, in: C.A. Cazacu, N. Boboc, M. Jurchescu, I. Suciu (Eds.), Complex Analysis – Fifth Romanian-Finnish Seminar, Springer Berlin Heidelberg, Berlin, Heidelberg, 1983: pp. 292-310. https://doi.org/10.1007/BFb0066537.

[6] J. Patel, S. Rout, An application of differential subordinations, Rend. Mat. Ser. VII. 14 (1994), 367–384.

[7] J. Patel, On certain subclass of p-valently Bazilevic functions, J. Ineq. Pure Appl. Math. 6 (2005), 16.

[8] H.M. Srivastava, A.Y. Lashin, Some applications of the Briot-Bouquet differential subordination, J. Ineq. Pure Appl. Math. 6 (2005), 41.