Asia Pac. J. Math. 2023 **10**:46

BIPOLAR FUZZY WEAK BI-IDEALS OF GAMMA-NEAR RINGS

V. P. VINEELA KORADA[1], S. RAGAMAYI[1], AIYARED IAMPAN[2,*]

[1]Department of Engineering Mathematics, Koneru Lakshmaiah Education Foundation, Guntur-522502, Andhra Pradesh, India

[2]Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

*Corresponding author: aiyared.ia@up.ac.th

Received Oct. 10, 2023

Abstract. This article explores the notion of bipolar fuzzy weak bi-ideals of Γ-near rings and studies the algebraic properties intersection and union of bipolar fuzzy weak bi-ideals of Γ-near rings. It also investigates the characterization of bipolar fuzzy weak bi-ideals of Γ-near rings in terms of level cut sets. Further, this article extends to the study of homomorphic image and pre-image of bipolar fuzzy weak bi-ideals of Γ-near rings.

2020 Mathematics Subject Classification. 03E72, 16Y30, 16Y80.

Key words and phrases. bipolar fuzzy weak bi-ideal; Γ-near ring; level cut set; homomorphic image and pre-image.

1. Introduction

The near-ring theory was introduced by Pilz [8]. The concept of Γ-rings, a generalization of a ring, was introduced by Nobusawa [7]. Γ-near rings (GNRs) were defined by Satyanarayana [16], and the ideal theory in GNRs was studied by Satyanarayana [16] and Booth [1]. Further, several authors studied various algebraic structures on GNRs, like ideals, weak ideals, bi-ideals, quasi-ideals, and normal ideals on GNRs. The idea of bipolar-valued fuzzy sets (BFSs) was given by Zhang [20], which is the extension of the theory of Zadeh's fuzzy sets (FSs) [19] to BFSs. Later, taking into consideration, many authors applied fuzzification on crisp sets, like Satyanarayana studied and invented the idea of fuzzy ideals, prime ideals of GNRs. Some results and properties on fuzzy ideals of GNRs are discussed by Jun [4]. In order to study uncertainty, the application of bipolar fuzzification, which is a generalization of FSs, has been developed by Jun and Lee [5]. Several researchers like Ragamayi [9–15, 17, 18] done their research on the development of the BFS theory on different algebraic structures like semigroups, groups, semirings, rings etc.

DOI: 10.28924/APJM/10-46

As a continuity of all these, we introduced bipolar fuzzy ideals and bi-ideals on GNRs in 2023. Now, we are studying bipolar fuzzy weak bi-ideals of GNRs.

2. Preliminaries

Definition 2.1. [8] A *near ring* is a nonempty set R equipped with two binary operations $+$ and $\cdot$ such that
(i) $(R,+)$ is a group,
(ii) $(R,\cdot)$ is a semigroup,
(iii) $(a+b)c = ac+bc, \forall a,b,c \in R$ obeying only right distributive law over addition.

Definition 2.2. [16] A Γ-*near ring* (GNR) is a triple $(M_R,+,\Gamma)$ where
(i) $(M_R,+)$ is a group,
(ii) Γ is a nonempty set of binary operators on M_R such that for each $\alpha \in \Gamma$, $(M_R,+,\alpha)$ is a near ring,
(iii) $\psi\alpha(\omega\beta\kappa) = (\psi\alpha\omega)\beta\kappa, \forall \psi,\omega,\kappa \in M_R, \alpha,\beta \in \Gamma$.

Definition 2.3. [6] A GNR M_R is said to be *zero-symmetric* if $\psi\alpha 0 = 0, \forall \psi \in M_R, \alpha \in \Gamma$.

Definition 2.4. [4] An FS ξ in a GNR M_R is a *fuzzy sub* Γ-*near ring* of M_R if
(i) $\xi(\psi-\omega) \geq \min\{\xi(\psi),\xi(\omega)\}, \forall \psi,\omega \in M_R$,
(ii) $\xi(\psi\alpha\omega) \geq \min\{\xi(\psi),\xi(\omega)\}, \forall \psi,\omega \in M_R, \alpha \in \Gamma$.

Definition 2.5. [5,20] Let M_R be a GNR and B_R be a BFS of M_R. We say that $B_R = (\xi^+_{B_R},\xi^-_{B_R})$ is a *bipolar fuzzy sub* Γ-*near ring* (BFSGNR) of M_R if
(i) $\xi^+_{B_R}(\psi-\omega) \geq \min\{\xi^+_{B_R}(\psi),\xi^+_{B_R}(\omega)\}, \forall \psi,\omega \in M_R$,
(ii) $\xi^-_{B_R}(\psi-\omega) \leq \max\{\xi^-_{B_R}(\psi),\xi^-_{B_R}(\omega)\}, \forall \psi,\omega \in M_R$,
(iii) $\xi^+_{B_R}(\psi\alpha\omega) \geq \min\{\xi^+_{B_R}(\psi),\xi^+_{B_R}(\omega)\}, \forall \psi,\omega \in M_R, \alpha \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi\alpha\omega) \leq \max\{\xi^-_{B_R}(\psi),\xi^-_{B_R}(\omega)\}, \forall \psi,\omega \in M_R, \alpha \in \Gamma$.
If $B = (\xi^+_{B_R},\xi^-_{B_R})$ satisfies the conditions (i) and (ii), then it is called a *bipolar fuzzy subgroup* (BFSG) of M_R.

Definition 2.6. [16] Let M_R be a GNR and A_R be a nonempty subset of M_R. Then A_R is said to be *left* (*resp., right*) *ideal* of M_R if
(i) $\psi-\omega \in A_R, \forall \psi,\omega \in A_R$,
(ii) $\omega+\psi-\omega \in A_R, \forall \psi \in I_R, \omega \in M_R$.
(iii) $a\alpha(\psi+b)-a\alpha b \in A_R$ (resp., $\psi\alpha a \in A_R$), $\forall \psi \in A_R, a,b \in M_R, \alpha \in \Gamma$.

Definition 2.7. [4] An FS ξ in a GNR M_R is called a *fuzzy left* (*resp., right*) *ideal* of M_R if
(i) $\xi(\psi-\omega) \geq \min\{\xi(\psi),\xi(\omega)\}, \forall \psi,\omega \in M_R$,
(ii) $\xi(\omega+\psi-\omega) \geq \xi(\psi), \forall \psi,\omega \in M_R$,

(iii) $\xi(a\alpha(\psi+b)-a\alpha b) \geq \xi(\psi)$ (resp., $\xi(\psi\alpha a) \geq \xi(\psi)$), $\forall \psi, a, b \in M_R, \alpha \in \Gamma$.

Definition 2.8. [4] A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a GNR M_R is called a *bipolar fuzzy ideal* (BFI) of M_R if
(i) $\xi^+_{B_R}(\psi-\omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^+_{B_R}(\omega+\psi-\omega) \geq \xi^+_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(a\alpha(\psi+b)-a\alpha b) \geq \xi^+_{B_R}(\psi)$ and $\xi^+_{B_R}(\psi\alpha a) \geq \xi^+_{B_R}(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi-\omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(v) $\xi^-_{B_R}(\omega+\psi-\omega) \leq \xi^-_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(vi) $\xi^-_{B_R}(a\alpha(\psi+b)-a\alpha b) \leq \xi^-_{B_R}(\psi)$ and $\xi^-_{B_R}(\psi\alpha a) \leq \xi^-_{B_R}(\psi), \forall \psi, a, b \in M_R, \alpha \in \Gamma$.

Definition 2.9. [2] A subgroup B_R of a GNR $(M_R, +, \Gamma)$ is said to be a *bi-ideal* of M_R if $B_R\Gamma M_R\Gamma B_R \subseteq B_R$.

Definition 2.10. [6,18] An FS ξ in a GNR M_R is called a *fuzzy bi-ideal* of M_R if
(i) $\xi(\psi-\omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi(\omega+\psi-\omega) \geq \xi(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi(\min\{(\psi\alpha\omega\beta\kappa), \psi\alpha(\omega+\kappa)-\psi\alpha\omega\} \geq \min\{\xi(\psi), \xi(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$.

Definition 2.11. [3] A subgroup W_R of a GNR $(M_R, +, \Gamma)$ is said to be a *weak bi-ideal* (WBI) of M_R if $W_R\Gamma W_R\Gamma W_R \subseteq W_R$.

Definition 2.12. [3] An FS ξ of a GNR M_R is called a *fuzzy weak bi-ideal* (FWBI) of M_R if
(i) $\xi(\psi-\omega) \geq \min\{\xi(\psi), \xi(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi(\psi\alpha\omega\beta\kappa) \geq \min\{\xi(\psi), \xi(\omega), \xi(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$.

Remark 2.13. [20] Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFSG of a GNR $(M_R, +, \Gamma)$. Let $\psi, \omega, \kappa \in M_R, \alpha \in \Gamma$ be such that $\psi = \omega\alpha\kappa$. Let

$$(\xi^+_{B_R} * \xi^+_{B_R})(\psi) = \sup_{\psi=\omega\alpha\kappa}\{\min\{\xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}\}$$
$$(\xi^-_{B_R} * \xi^-_{B_R})(\psi) = \inf_{\psi=\omega\alpha\kappa}\{\max\{\xi^-_{B_R}(\omega), \xi^-_{B_R}(\kappa)\}\}.$$

Definition 2.14. [18] A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of a zero symmetric GNR M_R is said to be *bipolar fuzzy bi-ideal* (BFBI) of M_R if
(i) $\xi^+_{B_R}(\psi-\omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^+_{B_R}(\omega+\psi-\omega) \geq \xi^+_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(\min\{\psi\alpha\omega\beta\kappa, \psi\alpha(\omega+\kappa)-\psi\alpha\omega\}) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi-\omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(v) $\xi^-_{B_R}(\omega+\psi-\omega) \leq \xi^-_{B_R}(\psi), \forall \psi, \omega \in M_R$,
(vi) $\xi^-_{B_R}(\min\{\psi\alpha\omega\beta\kappa, \psi\alpha(\omega+\kappa)-\psi\alpha\omega\}) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$.

3. Bipolar Fuzzy Weak Bi-Ideals of Γ-Near Rings

This section introduces and studies the notion of bipolar fuzzy weak bi-ideals of GNRs and their properties.

This paper's M_R denotes a zero-symmetric GNR with at least two elements.

Definition 3.1. A BFS $B = (\xi^+_{B_R}, \xi^-_{B_R})$ of M_R is called a *bipolar fuzzy weak bi-ideal* (BFWBI) of M_R if
(i) $\xi^+_{B_R}(\psi - \omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(ii) $\xi^-_{B_R}(\psi - \omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}, \forall \psi, \omega \in M_R$,
(iii) $\xi^+_{B_R}(\psi\alpha\omega\beta\kappa) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$,
(iv) $\xi^-_{B_R}(\psi\alpha\omega\beta\kappa) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega), \xi^-_{B_R}(\kappa)\}, \forall \psi, \omega, \kappa \in M_R, \alpha, \beta \in \Gamma$.

Example 3.2. Let $M_R = \mathbb{R}$ be the set of real numbers, which is clearly a GNR. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$, where $\xi^+_{B_R}(\psi) : M_R \to [0,1], \xi^-_{B_R}(\psi) : M_R \to [-1,0]$ defined by

$$\xi^+_{B_R}(\psi) = \begin{cases} 0.21, & \text{if } \psi = 0 \\ 0.63, & \text{if } \psi > 0 \\ 0.72, & \text{if } \psi < 0 \end{cases}$$

$$\xi^-_{B_R}(\psi) = \begin{cases} -0.42, & \text{if } \psi = 0 \\ -0.51, & \text{if } \psi > 0 \\ -0.64, & \text{if } \psi < 0. \end{cases}$$

Then B_R is a BFWBI of M_R.

Theorem 3.3. *Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFSG of M_R. Then $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a BFWBI of M_R if and only if $\xi^+_{B_R} * \xi^+_{B_R} * \xi^+_{B_R} \subseteq \xi^+_{B_R}$ and $\xi^-_{B_R} * \xi^-_{B_R} * \xi^-_{B_R} \supseteq \xi^-_{B_R}$.*

Proof. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFWBI of M_R. Let $\psi, \omega, \kappa, \omega_1, \omega_2 \in M_R, \alpha, \beta \in \Gamma$ be such that $\psi = \omega\alpha\kappa, \omega = \omega_1\beta\omega_2$. Then

$$\begin{aligned}
&(\xi^+_{B_R} * \xi^+_{B_R} * \xi^+_{B_R})(\psi) \\
&= \sup_{\psi=\omega\alpha\kappa} \{\min\{(\xi^+_{B_R} * \xi^+_{B_R})(\omega), (\xi^+_{B_R})(\kappa)\}\} \\
&= \sup_{\psi=\omega\alpha\kappa} \{\min\{\sup_{\omega=\omega_l\beta\omega_m} \{\min\{\xi^+_{B_R}(\omega_l), \xi^+_{B_R}(\omega_m)\}\}, \xi^+_{B_R}(\kappa)\}\} \\
&= \sup_{\psi=\omega\alpha\kappa} \{\sup_{\omega=\omega_l\beta\omega_m} \{\min\{\min\{\xi^+_{B_R}(\omega_l), \xi^+_{B_R}(\omega_m)\}, \xi^+_{B_R}(\kappa)\}\}\} \\
&= \sup_{\psi=\omega_l\beta\omega_m\alpha\kappa} \{\min\{\xi^+_{B_R}(\omega_l), \xi^+_{B_R}(\omega_m), \xi^+_{B_R}(\kappa)\}\}.
\end{aligned}$$

Since $\xi^{+}_{B_R}$is a BFWBI of M_R, we have

$$\begin{aligned}\xi^{+}_{B_R}(\omega_l\beta\omega_m\alpha\kappa) &\geq \min\{\xi^{+}_{B_R}(\omega_l), \xi^{+}_{B_R}(\omega_m), \xi^{+}_{B_R}(\kappa)\}\\ &\leq \sup_{\psi=\omega_l\beta\omega_m\alpha\kappa}\{\xi^{+}_{B_R}(\omega_l\beta\omega_m\alpha\kappa)\}\\ &= \xi^{+}_{B_R}(\psi).\end{aligned}$$

Now,

$$\begin{aligned}(\xi^{-}_{B_R} * \xi^{-}_{B_R} * \xi^{-}_{B_R})(\psi) &= \inf_{\psi=\omega\alpha\kappa}\{\max\{(\xi^{-}_{B_R} * \xi^{-}_{B_R})(\omega), \xi^{-}_{B_R}(\kappa)\}\}\\ &= \inf_{\psi=\omega\alpha\kappa}\{\max\{\inf_{\omega=\omega_m\beta\omega_m}\{\max\{\xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\omega_m)\}\}, \xi^{-}_{B_R}(\kappa)\}\}\\ &= \inf_{\psi=\omega\alpha\kappa}\{\inf_{\omega=\omega_m\beta\omega_m}\{\max\{\max\{\xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\omega_m)\}, \xi^{-}_{B_R}(\kappa)\}\}\}\\ &= \inf_{\psi=\omega_m\beta\omega_m\alpha\kappa}\{\max\{\xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\kappa)\}\}.\end{aligned}$$

Since $\xi^{-}_{B_R}$is a BFWBI of M_R, we have

$$\begin{aligned}\xi^{-}_{B_R}(\omega_m\beta\omega_m\alpha\kappa) &\leq \max\{\xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\omega_m), \xi^{-}_{B_R}(\kappa)\}\\ &\geq \inf_{x=\omega_m\beta\omega_m\alpha\kappa}\{\xi^{-}_{B_R}(\omega_m\beta\omega_m\alpha\kappa)\}\\ &= \xi^{-}_{B_R}(\psi).\end{aligned}$$

If ψ cannot be expressed as $\psi = \omega\alpha\kappa$, then

$$(\xi^{+}_{B_R} * \xi^{+}_{B_R} * \xi^{+}_{B_R})(\psi) = 0 \leq \xi^{+}_{B_R}(\psi),$$

$$(\xi^{-}_{B_R} * \xi^{-}_{B_R} * \xi^{-}_{B_R})(\psi) = 0 \geq \xi^{-}_{B_R}(\psi).$$

In either case,

$$\xi^{+}_{B_R} * \xi^{+}_{B_R} * \xi^{+}_{B_R} \subseteq \xi^{+}_{B_R},$$

$$\xi^{-}_{B_R} * \xi^{-}_{B_R} * \xi^{-}_{B_R} \supseteq \xi^{-}_{B_R}.$$

Conversely, assuming that $\xi^{+}_{B_R} * \xi^{+}_{B_R} * \xi^{+}_{B_R} \subseteq \xi^{+}_{B_R}$ and $\xi^{-}_{B_R} * \xi^{-}_{B_R} * \xi^{-}_{B_R} \supseteq \xi^{-}_{B_R}$. Let $\psi', \psi, \omega, \kappa \in M_R$ and $\alpha, \beta, \alpha_1, \beta_1 \in \Gamma$ be such that $\psi' = \psi\alpha\omega\beta\kappa$. Then

$$\begin{aligned}\xi^{+}_{\beta}(\psi\alpha\omega\beta\kappa) &= \xi^{+}_{\beta}(\psi')\\ &\geq (\xi^{+}_{B_R} * \xi^{+}_{B_R} * \xi^{+}_{B_R})(\psi')\\ &= \sup_{\psi'=p\alpha_1 q}\{\min\{(\xi^{+}_{B_R} * \xi^{+}_{B_R})(p), \xi^{+}_{B_R}(q)\}\}\\ &= \sup_{\psi'=p\alpha_1 q}\{\min\{\sup_{p=p_l\beta_1 p_m}\{\min\{\xi^{+}_{B_R}(p_l), \xi^{+}_{B_R}(p_m)\}\}, \xi^{+}_{B_R}(q)\}\}\end{aligned}$$

$$
\begin{aligned}
&= \sup_{\psi'=p_l\beta_1 p_m\alpha_1 q}\{\min\{\xi^+_{B_R}(p_l), \xi^+_{B_R}(p_m), \xi^+_{B_R}(q)\}\} \\
&\geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\},
\end{aligned}
$$

$$
\begin{aligned}
\xi^-_{\beta}(\psi\alpha\omega\beta z) &= \xi^-_{\beta}(\psi') \\
&\leq (\xi^-_{B_R} * \xi^-_{B_R} * \xi^-_{B_R})(\psi') \\
&= \inf_{\psi'=p\alpha_1 q}\{\max\{(\xi^-_{B_R} * \xi^-_{B_R})(p), \xi^-_{B_R}(q)\}\} \\
&= \inf_{\psi'=p\alpha_1 q}\{\max\{\inf_{p=p_l\beta_1 p_m}\{\max\{\xi^-_{B_R}(p_l), \xi^-_{B_R}(p_m)\}\}, \xi^-_{B_R}(q)\}\} \\
&= \inf_{\psi'=p_l\beta_1 p_m\alpha_1 q}\{\max\{\xi^-_{B_R}(p_l), \xi^-_{B_R}(p_m), \xi^-_{B_R}(q)\}\} \\
&\leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega), \xi^-_{B_R}(\kappa)\}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\xi^+_{\beta}(\psi\alpha\omega\beta\kappa) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}, \\
\xi^-_{\beta}(\psi\alpha\omega\beta\kappa) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega), \xi^-_{B_R}(\kappa)\}.
\end{aligned}
$$

Hence, $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ is a BFWBI of M_R. □

Theorem 3.4. *If A_R and B_R are BFWBIs of M_R, then the product $A_R * B_R$ and $B_R * A_R$ are also BFWBIs of M_R.*

Proof. Let $A_R = (\xi^+_{A_R}, \xi^-_{A_R})$ and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be BFWBIs of M_R and let $\psi, \omega \in M_R$. Then

$$
\begin{aligned}
&\xi^+_{A_R*B_R}(\psi-\omega) \\
&= \sup_{\psi-\omega=a\alpha b}\{\min\{\xi^+_{A_R}(a_\gamma), \xi^+_{B_R}(b_\gamma)\}\} \\
&\geq \sup_{\psi-\omega=a_l\alpha_1 b_l - a_m\alpha_2 b_m < (a_l-a_m)(b_l-b_m)}\{\min\{\xi^+_{A_R}(a_l-a_m), \xi^+_{B_R}(b_l-b_m)\}\} \\
&\geq \sup\{\min\{\min\{\xi^+_{A_R}(a_l), \xi^+_{A_R}(a_m)\}, \min\{\xi^+_{B_R}(b_l), \xi^+_{B_R}(b_m)\}\}\} \\
&= \sup\{\min\{\min\{\xi^+_{A_R}(a_l), \xi^+_{B_R}(b_l)\}, \min\{\xi^+_{A_R}(a_m), \xi^+_{B_R}(b_m)\}\}\} \\
&\geq \min\{\sup_{\psi=a_l\alpha_1 b_l}\{\min\{\xi^+_{A_R}(a_l), \xi^+_{B_R}(b_l)\}\}, \sup_{y=a_m\alpha_2 b_m}\{\min\{\xi^+_{A_R}(a_m), \xi^+_{B_R}(b_m)\}\}\} \\
&= \min\{\xi^+_{A_R*B_R}(\psi), \xi^+_{A_R*B_R}(\omega)\},
\end{aligned}
$$

$$
\begin{aligned}
&\xi^-_{A_R*B_R}(\psi-\omega) \\
&= \inf_{\psi-\omega=a\alpha b}\{\max\{\xi^-_{A_R}(a_\gamma), \xi^-_{B_R}(b_\gamma)\}\} \\
&\leq \inf_{\psi-\omega=a_l\alpha_1 b_l - a_m\alpha_2 b_m < (a_l-a_m)(b_l-b_m)}\{\max\{\xi^-_{A_R}(a_l-a_m), \xi^-_{B_R}(b_l-b_m)\}\}
\end{aligned}
$$

$$\le \inf\{\max\{\max\{\xi^-_{A_R}(a_l), \xi^-_{A_R}(a_m)\}, \max\{\xi^-_{B_R}(b_l), \xi^-_{B_R}(b_m)\}\}\}$$

$$= \inf\{\max\{\max\{\xi^-_{A_R}(a_l), \xi^-_{B_R}(b_l)\}, \min\{\xi^-_{A_R}(a_m), \xi^-_{B_R}(b_m)\}\}\}$$

$$\le \max\{\inf_{\psi=a_l\alpha_1 b_l}\{\max\{\xi^-_{A_R}(a_l), \xi^-_{B_R}(b_l)\}\}, \inf_{\omega=a_m\alpha_2 b_m}\{\max\{\xi^-_{A_R}(a_m), \xi^-_{B_R}(b_m)\}\}\}$$

$$= \max\{\xi^-_{A_R*B_R}(\psi), \xi^-_{A_R*B_R}(\omega)\}.$$

In the remaining two cases, we got it right away. Hence, $A_R * B_R$ is a BFWBI of M_R. □

Theorem 3.5. *Every BFI of M_R is a BFBI of M_R.*

Proof. Let $B_R = (\mu^+_{B_R}, \mu^-_{B_R})$ be a BFI of M_R. Then

$$\mu^+_{B_R} * M_R * \mu^+_{B_R} \subseteq \mu^+_{B_R} * M_R * M_R \subseteq \mu^+_{B_R} * M_R \subseteq \mu^+_{B_R},$$

$$\mu^-_{B_R} * M_R * \mu^-_{B_R} \supseteq \mu^-_{B_R} * M_R * M_R \supseteq \mu^-_{B_R} * M_R \supseteq \mu^-_{B_R}.$$

Therefore, B_R is a BFBI of M_R. □

Theorem 3.6. *Every BFBI of M_R is a BFWBI of M_R.*

Proof. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFBI of M_R. Then $\xi^+_{B_R} * M_R * \xi^+_{B_R} \subseteq \xi^+_{B_R}$, we have $\xi^+_{B_R} * \xi^+_{B_R} * \xi^+_{B_R} \subseteq \xi^+_{B_R} * M_R * \xi^+_{B_R}$. This implies that $\xi^+_{B_R} * \xi^+_{B_R} * \xi^+_{B_R} \subseteq \xi^+_{B_R} * M_R * \xi^+_{B_R} \subseteq \xi^+_{B_R}$. Similarly, since $\xi^-_{B_R} * M_R * \xi^-_{B_R} \supseteq \xi^-_{B_R}$, we have $\xi^-_{B_R} * \xi^-_{B_R} * \xi^-_{B_R} \supseteq \xi^-_{B_R} * M_R * \xi^-_{B_R}$. This implies that $\xi^-_{B_R} * \xi^-_{B_R} * \xi^-_{B_R} \supseteq \xi^-_{B_R} * M_R * \xi^-_{B_R} \supseteq \xi^-_{B_R}$. Therefore, B_R is a BFWBI of M_R. □

Theorem 3.7. *Every BFI of M_R is a BFWBI of M_R.*

Proof. It is a straightforward result from Theorems 3.5 and 3.6. □

Theorem 3.8. *If A_R and B_R are BFWBIs of M_R, then $A_R \cap B_R$ is also a BFWBI of M_R.*

Proof. Let A_R and B_R be BFWBIs of M_R. Let $\psi, \omega, \kappa \in M_R$ and $\alpha \in \Gamma$. Then

$$(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi - \omega)$$

$$= \min\{\xi^+_{A_R}(\psi - \omega), \xi^+_{B_R}(\psi - \omega)\}$$

$$\ge \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega)\}, \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}\}$$

$$= \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \min\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}\}$$

$$\ge \min\{(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega)\},$$

$$\begin{aligned}
&(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi\alpha\omega\beta\kappa)\\
&= \min\{\xi^+_{A_R}(\psi\alpha\omega\beta\kappa), \xi^+_{B_R}(\psi\alpha\omega\beta\kappa)\}\\
&\geq \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega), \xi^+_{A_R}(\kappa)\}, \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}\}\\
&= \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega), \xi^+_{A_R}(\kappa), \xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}\}\\
&= \min\{\min\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \min\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}, \min\{\xi^+_{A_R}(\kappa), \xi^+_{B_R}(\kappa)\}\}\\
&\geq \min\{(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega), (\xi^+_{A_R} \cap \xi^+_{B_R})(\kappa)\}.
\end{aligned}$$

Similarly, we can show that

$$(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi - \omega) \leq \max\{(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi), (\xi^-_{A_R} \cap \xi^-_{B_R})(\omega)\},$$

$$(\xi^-_{A_R} \cap \xi^-_{B_R})(\psi\alpha\omega\alpha\kappa) \leq \max\{(\xi^+_{A_R} \cap \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cap \xi^+_{B_R})(\omega), (\xi^+_{A_R} \cap \xi^+_{B_R})(\kappa)\}.$$

Hence, $A_R \cap B_R$ is a BFWBI of M_R. □

Theorem 3.9. *If A_R and B_R are BFWBIs of M_R, then $A_R \cup B_R$ is also a BFWBI of M_R if $A_R \subseteq B_R$ or $B_R \subseteq A_R$.*

Proof. Let A_R and B_R be BFWBIs of M_R such that $A_R \subseteq B_R$. Let $\psi, \omega, \kappa \in M_R$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}
&(\xi^+_{A_R} \cup \xi^+_{B_R})(\psi - \omega)\\
&= \max\{\xi^+_{A_R}(\psi - \omega), \xi^+_{B_R}(\psi - \omega)\}\\
&= \xi^+_{B_R}(\psi - \omega)\\
&\geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}\\
&= \min\{\max\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \max\{\xi^+_{A_R}(\omega)\xi^+_{B_R}(\omega)\}\}\\
&= \min\{(\xi^+_{A_R} \cup \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cup \xi^+_{B_R})(\omega)\},
\end{aligned}$$

$$\begin{aligned}
&(\xi^+_{A_R} \cup \xi^+_{B_R})(\psi\alpha\omega\beta\kappa)\\
&= \max\{\xi^+_{A_R}(\psi\alpha\omega\beta\kappa), \xi^+_{B_R}(\psi\alpha\omega\beta\kappa)\}\\
&= \xi^+_{B_R}(\psi\alpha\omega\beta\kappa)\\
&\geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}\\
&= \min\{\max\{\xi^+_{A_R}(\psi), \xi^+_{B_R}(\psi)\}, \max\{\xi^+_{A_R}(\omega), \xi^+_{B_R}(\omega)\}, \max\{\xi^+_{A_R}(\kappa), \xi^+_{B_R}(\kappa)\}\}\\
&= \min\{(\xi^+_{A_R} \cup \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cup \xi^+_{B_R})(\omega), (\xi^+_{A_R} \cup \xi^+_{B_R})(\kappa)\}.
\end{aligned}$$

Similarly, we can show that

$$(\xi^-_{A_R} \cup \xi^-_{B_R})(\psi - \omega) \leq \max\{(\xi^-_{A_R} \cup \xi^-_{B_R})(\psi), (\xi^-_{A_R} \cup \xi^-_{B_R})(\omega)\},$$

$$(\xi^-_{A_R} \cup \xi^-_{B_R})(\psi\alpha\omega\beta\kappa) \leq \max\{(\xi^+_{A_R} \cup \xi^+_{B_R})(\psi), (\xi^+_{A_R} \cup \xi^+_{B_R})(\omega), (\xi^+_{A_R} \cup \xi^+_{B_R})(\kappa)\}.$$

Hence, $A_R \cup B_R$ is a BFWBI of M_R.

Similarly, if $B_R \subseteq A_R$, then $A_R \cup B_R$ is a BFWBI of M_R. □

Remark 3.10. Let $A_R = (\xi^+_{A_R}, \xi^-_{A_R})$ and $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be BFWBIs of M_R defined by

$$\xi^+_{A_R}(\psi) = \begin{cases} 0.21, & \text{if } \psi = 0 \\ 0.63, & \text{if } \psi > 0 \\ 0.72, & \text{if } \psi < 0 \end{cases}, \xi^-_{A_R}(\psi) = \begin{cases} -0.42, & \text{if } \psi = 0 \\ -0.51, & \text{if } \psi > 0 \\ -0.64, & \text{if } \psi < 0 \end{cases}$$

$$\xi^+_{B_R}(\psi) = \begin{cases} 0.19, & \text{if } \psi = 0 \\ 0.54, & \text{if } \psi > 0 \\ 0.83, & \text{if } \psi < 0 \end{cases}, \xi^-_{B_R}(\psi) = \begin{cases} -0.35, & \text{if } \psi = 0 \\ -0.47, & \text{if } \psi > 0 \\ -0.73, & \text{if } \psi < 0 \end{cases}$$

Then

$$\xi^+_{A_R \cup B_R}(\psi) = \begin{cases} 0.21, & \text{if } \psi = 0 \\ 0.63, & \text{if } \psi > 0 \\ 0.83, & \text{if } \psi < 0 \end{cases}, \xi^-_{A_R \cup B_R}(\psi) = \begin{cases} -0.42, & \text{if } \psi = 0 \\ -0.51, & \text{if } \psi > 0 \\ -0.73, & \text{if } \psi < 0 \end{cases}$$

By routine computation, it is clear that the union of two BFWBIs is not a BFWBI.

Theorem 3.11. *A BFS $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ of M_R is a BFWBI of M_R if and only if for all $\rho \in [0,1], \varrho \in [-1,0]$, the (ρ, ϱ)-cut $B_{R\rho,\varrho}$ is a WBI of M_R.*

Proof. Let $B_R = (\xi^+_{B_R}, \xi^-_{B_R})$ be a BFWBI of M_R. Let $\psi, \omega, \kappa \in B_{R\rho,\varrho}$ for $\rho \in [0,1], \varrho \in [-1,0]$. Then $\xi^+_{B_R}(\psi) \geq \rho, \xi^+_{B_R}(\omega) \geq \rho, \xi^+_{B_R}(\kappa) \geq \rho$ and $\xi^-_{B_R}(\psi) \leq \varrho, \xi^-_{B_R}(\omega) \leq \varrho, \xi^-_{B_R}(\kappa) \leq \varrho$. Let $\alpha, \beta \in \Gamma$. Then

$$\xi^+_{B_R}(\psi - \omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\} = \min\{\rho, \rho\} = \rho,$$

$$\xi^+_{B_R}(\psi\alpha\omega\beta\kappa) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\} = \min\{\rho, \rho, \rho\} = \rho,$$

$$\xi^-_{B_R}(\psi - \omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\} = \max\{\varrho, \varrho\} = \varrho,$$

$$\xi^-_{B_R}(\psi\alpha\omega\beta z) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega), \xi^-_{B_R}(\kappa)\} = \max\{\varrho, \varrho, \varrho\} = \varrho.$$

Therefore, $\psi - \omega, \psi\alpha\omega\beta\kappa \in B_{R\rho,\varrho}$. Hence, $B_{R\rho,\varrho}$ is a WBI of M_R.

Conversely, assume that $B_{R\rho,\varrho}$ is a WBI of M_R for all $\rho \in [0,1], \varrho \in [-1,0]$. Let $\psi, \omega, \kappa \in M_R$ and $\alpha, \beta \in \Gamma$. Suppose

$$\xi^+_{B_R}(\psi - \omega) < \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}.$$

Choose $\rho \in [0,1]$ such that $\xi^+_{B_R}(\psi - \omega) < \rho < \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}$. Then $\xi^+_{B_R}(\psi) > \rho, \xi^+_{B_R}(\omega) > \rho$ and $\xi^+_{B_R}(\psi - \omega) < \rho$. Thus $\xi^+_{B_R}(\psi) > \rho, \xi^+_{B_R}(\omega) > \rho$ and $\xi^+_{B_R}(\psi - \omega) < \rho$. Then $\psi, \omega \in B_{R\rho,0}$ but $\psi - \omega \notin B_{R\rho,0}$, which is a contradiction. Thus $\xi^+_{B_R}(\psi - \omega) \geq \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega)\}$. Similarly, we can prove $\xi^-_{B_R}(\psi - \omega) \leq \max\{\xi^-_{B_R}(\psi), \xi^-_{B_R}(\omega)\}$. Again, suppose

$$\xi^+_{B_R}(\psi\alpha\omega\beta\kappa) < \min\{\xi^+_{B_R}(\psi), \xi^+_{B_R}(\omega), \xi^+_{B_R}(\kappa)\}.$$

Choose $\rho \in [0,1]$ such that $\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) < \rho < \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}$. Then $\xi^{+}_{B_R}(\psi) > t, \xi^{+}_{B_R}(\omega) > \rho, \xi^{+}_{B_R}(\kappa) > \rho$, and $\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) < \rho$. Then $\psi, \omega, \kappa \in B_{R\rho,0}$ but $\psi\alpha\omega\beta\kappa \notin B_{R\rho,0}$, which is a contradiction. Thus $\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}$. Similarly, we can prove $\xi^{-}_{B_R}(\psi\alpha\omega\beta\kappa) \leq \max\{\xi^{-}_{B_R}(\psi), \xi^{-}_{B_R}(\omega), \xi^{-}_{B_R}(\kappa)\}$. Therefore, B_R is a BFWBI of M_R. □

Theorem 3.12. *Let S be a non-empty subset of M_R. Then the characteristic set of S, $B_S = (\xi^{+}_{B_R}, \xi^{-}_{B_R})$ is a BFWBI of M_R if and only if S is a WBI of M_R.*

Proof. Assume that $B_S = (\xi^{+}_{B_R}, \xi^{-}_{B_R})$ is a BFWBI of M_R. Let $\psi, \omega, \kappa \in S$ and $\alpha, \beta \in \Gamma$. Then

$$\xi^{+}_{B_R}(\psi - \omega) \geq \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\} = \min\{1, 1\} = 1,$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\} = \min\{1, 1, 1\} = 1,$$

$$\xi^{-}_{B_R}(\psi - \omega) \leq \max\{\xi^{-}_{B_R}(\psi), \xi^{-}_{B_R}(\omega)\} = \max\{-1, -1\} = -1,$$

$$\xi^{-}_{B_R}(\psi\alpha\omega\beta\kappa) \leq \max\{\xi^{-}_{B_R}(\psi), \xi^{-}_{B_R}(\omega), \xi^{-}_{B_R}(\kappa)\} = \max\{-1, -1, -1\} = -1.$$

Thus $\psi - \omega \in S, \psi\alpha\omega\beta\kappa \in S$. Therefore, S is a WBI of M_R.

Conversely, suppose that S is a WBI of M_R.

(i) If $\psi, \omega, \kappa \in S$ and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\omega) = \xi^{+}_{B_R}(\kappa) = 1$$

and also $\psi - \omega, \psi\alpha\omega\beta\kappa \in S$. Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) = 1 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) = 1 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(ii) If $\psi, \omega, \kappa \notin S$ and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\omega) = \xi^{+}_{B_R}(\kappa) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(iii) If $\psi, \omega \in S, \kappa \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\omega) = 1, \xi^{+}_{B_R}(\kappa) = 0, \psi - \omega \in S.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) = 1 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(iv) If $\psi, \kappa \in S, \omega \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\kappa) = 1, \xi^{+}_{B_R}(\omega) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(v) If $\omega, \kappa \in S, \psi \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\omega) = \xi^{+}_{B_R}(\kappa) = 1, \xi^{+}_{B_R}(\psi) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(vi) If $\psi \in S, \omega, \kappa \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\psi) = 1, \xi^{+}_{B_R}(\omega) = \xi^{+}_{B_R}(\kappa) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(vii) If $\omega \in S, \psi, \kappa \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\omega) = 1, \xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\kappa) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

(viii) If $\kappa \in S, \psi, \omega \notin S$, and $\alpha, \beta \in \Gamma$, then

$$\xi^{+}_{B_R}(\kappa) = 1, \xi^{+}_{B_R}(\psi) = \xi^{+}_{B_R}(\omega) = 0.$$

Therefore,

$$\xi^{+}_{B_R}(\psi - \omega) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega)\},$$

$$\xi^{+}_{B_R}(\psi\alpha\omega\beta\kappa) \geq 0 = \min\{\xi^{+}_{B_R}(\psi), \xi^{+}_{B_R}(\omega), \xi^{+}_{B_R}(\kappa)\}.$$

Similarly, we can show that $\xi^{-}_{B_R}(\psi - \omega) \leq \max\{\xi^{-}_{B_R}(\psi), \xi^{-}_{B_R}(\omega)\}$ and $\xi^{-}_{B_R}(\psi\alpha\omega\beta\kappa) \leq \max\{\xi^{-}_{B_R}(\psi), \xi^{-}_{B_R}(\omega), \xi^{-}_{B_R}(\kappa)\}$ for all $\psi, \omega, \kappa \in M_R$ and $\alpha, \beta \in \Gamma$. Therefore, B_S is a BFWBI of M_R. □

Theorem 3.13. *A GNR homomorphic image of BFWBI possessing both supremum and infimum properties is a BFWBI.*

Proof. Let $\phi : M_{R1} \to M_{R2}$ be a GNR homomorphism. Let $A_R = (M_{R1}, \xi^+_{A_R}, \xi^-_{A_R})$ be a BFWBI of M_{R1} possessing both supremum and infimum properties. Let $\xi^+_{A_R}$ be an FWBI of M_{R1}, possessing supremum property. Let $B_R = (M_{R2}, \xi^+_{B_R}, \xi^-_{B_R})$ be the GNR homomorphic image of A_R in M_{R2}. Let $\xi^+_{B_R}$ be the image of $\xi^+_{A_R}$. Let $\phi(\psi), \phi(\omega), \phi(\kappa) \in M_{R2}$. Then

$$\psi_0 \in \phi^{-1}(\phi(\psi)) \ni \xi^+_{A_R}(x_0) = \sup_{a_\gamma \in \phi^{-1}(\phi(\psi))} \{\xi^+_{A_R}(a_\gamma)\},$$
$$\omega_0 \in \phi^{-1}(\phi(\omega)) \ni \xi^+_{A_R}(\omega_0) = \sup_{a_\gamma \in \phi^{-1}(\phi(\omega))} \{\xi^+_{A_R}(a_\gamma)\},$$
$$\kappa_0 \in \phi^{-1}(\phi(\kappa)) \ni \xi^+_{A_R}(\kappa_0) = \sup_{a_\gamma \in \phi^{-1}(\phi(\kappa))} \{\xi^+_{A_R}(a_\gamma)\}.$$

Consider,

$$\begin{aligned}
&\xi^+_{B_R}(\phi(\psi) - \phi(\omega)) \\
&= \sup_{a_\gamma \in \phi^{-1}(\phi(\psi) - \phi(\omega))} \{\xi^+_{A_R}(a_\gamma)\} \\
&= \xi^+_{A_R}(\psi_0 - \omega_0) \\
&\geq \min\{\xi^+_{A_R}(\psi_0), \xi^+_{A_R}(\omega_0)\} \\
&= \min\{\sup_{a_\gamma \in \phi^{-1}(\phi(\psi))} \{\xi^+_{A_R}(a_\gamma)\}, \sup_{a_\gamma \in \phi^{-1}(\phi(\omega))} \{\xi^+_{A_R}(a_\gamma)\}\} \\
&= \min\{\xi^+_{B_R}(\phi(\psi)), \xi^+_{B_R}(\phi(\omega))\},
\end{aligned}$$

$$\begin{aligned}
&\xi^+_{B_R}(\phi(\psi)\alpha\phi(\omega)\beta\phi(\kappa)) \\
&= \sup_{a_\gamma \in \phi^{-1}(\phi(\psi)\alpha\phi(\omega)\beta\phi(\kappa))} \{\xi^+_{A_R}(a_\gamma)\} \\
&= \xi^+_{A_R}(\psi_0\alpha\omega_0\beta\kappa_0) \\
&\geq \min\{\xi^+_{A_R}(x_0), \xi^+_{A_R}(\omega_0), \xi^+_{A_R}(\kappa_0)\} \\
&= \min\{\sup_{a_\gamma \in \phi^{-1}(\phi(\psi))} \{\xi^+_{A_R}(a_\gamma)\}, \sup_{a_\gamma \in \phi^{-1}(\phi(\omega))} \{\xi^+_{A_R}(a_\gamma)\}, \sup_{a_\gamma \in \phi^{-1}(\phi(\kappa))} \{\xi^+_{A_R}(a_\gamma)\}\} \\
&= \min\{\xi^+_{B_R}(\phi(\psi)), \xi^+_{B_R}(\phi(\omega)), \xi^+_{B_R}(\phi(\kappa))\}.
\end{aligned}$$

Therefore, $\xi^+_{B_R}$ is an FWBI of M_{R2}. Similarly, we can show that
$\xi^-_{B_R}(\phi(\psi) - \phi(\omega)) \leq \max\{\xi^-_{B_R}(\phi(\psi)), \xi^-_{B_R}(\phi(\omega))\}$
and $\xi^-_{B_R}(\phi(\psi)\alpha\phi(\omega)\beta\phi(\kappa)) \leq \min\{\xi^-_{B_R}(\phi(\psi)), \xi^-_{B_R}(\phi(\omega)), \xi^-_{B_R}(\phi(\kappa))\}$.
Hence, the GNR homomorphic image of a BFWBI possessing both supremum and infimum properties is a BFWBI. □

Theorem 3.14. *A GNR homomorphic pre-image of a BFWBI is a BFWBI.*

Proof. Let $\phi : M_{R1} \to M_{R2}$ be a GNR homomorphism. Let $B_R = (M_{R2}, \xi^+_{B_R}, \xi^-_{B_R})$ be a BFWBI of M_{R2}. Let $A_R = (M_{R1}, \xi^+_{A_R}, \xi^-_{A_R})$ be the GNR homomorphic pre-image of B_R in M_{R1}. Let $\psi, \omega, \kappa \in M_{R1}$ and $\alpha, \beta \in \Gamma$. Then

$$\begin{aligned}\xi^+_{A_R}(\psi - \omega) = \xi^+_{B_R}(\phi(\psi - \omega)) &= \xi^+_{B_R}(\phi(\psi) - \phi(\omega))\\ &\geq \min\{\xi^+_{B_R}(\phi(\psi) - \xi^+_{B_R}(\phi(\omega))\}\\ &= \min\{\xi^+_{A_R}(\psi) - \xi^+_{A_R}(\omega)\},\end{aligned}$$

$$\begin{aligned}\xi^+_{A_R}(\psi\alpha\omega\beta\kappa) &= \xi^+_{B_R}(\phi(\psi\alpha\omega\beta\kappa))\\ &= \xi^+_{B_R}(\phi(\psi)\alpha\phi(\omega)\beta\phi(\kappa))\\ &\geq \min\{\xi^+_{B_R}(\phi(\psi))\alpha\xi^+_{B_R}(\phi(\omega))\beta\xi^+_{B_R}(\phi(\kappa))\}\\ &= \min\{\xi^+_{A_R}(\psi), \xi^+_{A_R}(\omega), \xi^+_{A_R}(\kappa)\}.\end{aligned}$$

Similarly,

$$\xi^-_{A_R}(\psi - \omega) \leq \max\{\xi^-_{A_R}(\psi) - \xi^-_{A_R}(\omega)\},$$

$$\xi^-_{A_R}(\psi\alpha\omega\alpha\kappa) \leq \max\{\xi^-_{A_R}(\psi), \xi^-_{A_R}(\omega), \xi^-_{A_R}(\kappa)\}.$$

Hence, the GNR homomorphic pre-image of a BFWBI is a BFWBI. □

4. Conclusion

In this article, we explored the notion of BFWBIs of GNRs and studied the algebraic properties of the intersection and the union of BFWBIs of GNRs. We also investigated the characterization of BFWBIs of GNRs in terms of level cut sets. Further, we extended our study to the homomorphic image and pre-image of BFWBIs of GNRs. In future, our work will be followed by the introduction of bipolar fuzzy prime ideals of GNRs, which play a crucial part in the ideal theory of every algebraic structure.

Acknowledgment

This research project was supported by the Thailand Science Research and Innovation Fund and the University of Phayao (Grant No. FF67-UoE-Aiyared-Iampan).

References

[1] G.L. Booth, A note on Γ-near rings, Stud. Sci. Math. Hung. 23 (1988), 471–475.

[2] T.T. Chelvam, N. Meenakumari, Bi-ideals of gamma near-rings, Southeast Asian Bull. Math. 27 (2004), 983–998.

[3] V. Chinnadurai, K. Arulmozhi, S.M. Kadalarasi, Characterization of fuzzy weak bi-ideals of Γ-near rings, Int. J. Algebra Stat. 6 (2017), 95–104. https://doi.org/10.20454/ijas.2017.1259.

[4] Y.B. Jun, M. Sapanci, M.A. Öztürk, Fuzzy ideals in gamma near-rings, Turk. J. Math. 22 (1998), 449–549.

[5] K.J. Lee, Bipolar fuzzy subalgebras and bipolar fuzzy ideals of BCK/BCI-algebras, Bull. Malays. Math. Sci. Soc. 32 (2009), 361–373.

[6] N. Meenakumari, T.T. Chelvam, Fuzzy bi-ideals in gamma near-rings, J. Algebra Discrete Struct. 9 (2011), 43–52.

[7] N. Nobusawa, On a generalization of the ring theory, Osaka J. Math. 1 (1964), 81–89. https://doi.org/10.18910/12354.

[8] G. Pilz, Near-rings: the theory and its applications, North-Holland Pub. Co., Amsterdam, (1983).

[9] S. Ragamayi, Y. Bhargavi, A Study of vague gamma-nearrings, Int. J. Sci. Technol. Res. 9 (2020), 3960–3963.

[10] S. Ragamayi, Y. Bhargavi, Some results on homomorphism of vague ideal of a gamma-near ring, Int. J. Sci. Technol. Res. 9 (2020), 3972–3975.

[11] S. Ragamayi, Y. Bhargavi, C. Krishnaveni, P. Bindu, Lattice-fuzzy prime-ideal of a gamma-nearring, J. Crit. Rev. 7 (2020), 1–5. https://doi.org/10.31838/jcr.07.13.01.

[12] S. Ragamayi, G. Jaya Lalitha, P. Bindu, On L-vague maximal-ideal of a Γ-near ring, AIP Conf. Proc. 2375 (2021), 020017. https://doi.org/10.1063/5.0066395.

[13] S. Ragamayi, N. Konda Reddy, P. Bindu, On L-vague prime-ideal of a Γ-near ring and its characteristic properties, AIP Conf. Proc. 2375 (2021), 020020. https://doi.org/10.1063/5.0066396.

[14] S. Ragamayi, N. Konda Reddy, G. Jaya Lalitha, P. Bindu, A study of L-fuzzy cosets of a Γ-near ring, AIP Conf. Proc. 2375 (2021), 020010. https://doi.org/10.1063/5.0066393.

[15] S. Ragamayi, N. Konda Reddy, B. Srinivasa Kumar, Results on L-Vague ideal of a Γ-near ring, AIP Conf. Proc. 2375 (2021), 020014. https://doi.org/10.1063/5.0066388.

[16] B. Satyanarayana, Contributions to near-rings theory, Doctoral Thesis, Nagarjuna University, (1984).

[17] V.P. Vineela Korada, S. Ragamayi, Application of bi-polar fuzzy theory to ideals in gamma-near rings and it's characteristics, AIP Conf. Proc. 2707 (2023), 020015. https://doi.org/10.1063/5.0143352.

[18] V.P. Vineela Korada, S. Ragamayi, G. Jaya Lalitha, Some results on bi-polar fuzzy bi-ideals of Γ-near rings, AIP Conf. Proc. 2707 (2023), 020014. https://doi.org/10.1063/5.0143353.

[19] L.A. Zadeh, Fuzzy sets, Inf. Control. 8 (1965), 338–353. https://doi.org/10.1016/s0019-9958(65)90241-x.

[20] W.-R. Zhang, Bipolar fuzzy sets and relations: a computational framework for cognitive modelling and multiagent decision analysis, in: Proceedings of the First International Joint Conference of The North American Fuzzy Information Processing Society Biannual Conference, The Industrial Fuzzy Control and Intelligent, San Antonio (1994), 305-309. https://doi.org/10.1109/IJCF.1994.375115.