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$(\in, \in \vee q_k)$ - FUZZY SOFT BOOLEAN NEAR RINGS

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Abstract. In this paper, We present the ideas of $(\in, \in \vee q_k)$ - FSBN and $(\in, \in \vee q_k)$ - FSI over a BN, which are generalizations of $(\in, \in \vee q_k)$ - FSBN and $(\in, \in \vee q_k)$ - FSI resp., and investigate some of its aspects with examples. We also present the idea of a $(\in, \in \vee q_k)$ - FS-sub-BN of an $(\in, \in \vee q_k)$ - FSBN and achieve some related findings

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1. Introduction

The foundational idea of a fuzzy set was first introduced by Zadeh [12], and it offers a framework for the generalization of many fundamental ideas in algebraic constructions. The idea of a $(\in, \in \vee q_k)$ - fuzzy subgroup, which was first introduced by Sandeep Kumar Bhakat and Pratulanada Das [7]. The concepts of a $(\in, \in \vee q_k)$ - fuzzy subnear-ring and $(\in, \in \vee q_k)$ - fuzzy ideals of near-rings have been introduced by Narayanan and Manikantan [9]. In, Dheena and Coumaressane specified the terms $(\in, \in \vee q_k)$ - fuzzy sub-near-ring and $(\in, \in \vee q_k)$ - fuzzy ideals of near-rings, resp,. We refer readers to [4,5,9,10,13] for more information on $(\in, \in \vee q_k)$ - fuzzy algebraic structures. The idea of a soft set, which can be thought of as an efficient mathematical tool to deal with uncertainty, was first introduced by Russian researcher Molodtsov in 1999. The definitions of fuzzy soft set and fuzzy soft set operations were provided by Maji et al. [7]. The terms fuzzy soft ring and $(\in, \in \vee q_k)$ - fuzzy soft ring over a ring were defined by Inan and Ozturk [6]. Ozturk and Inan have expanded these ideas to near-rings

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in [10]. In this article, we establish the concepts of $(\in, \in \vee q_k)$ - FSBN and $(\in, \in \vee q_k)$ - FSI over BN, respectively, which are generalizations of $(\in, \in \vee q_k)$ - FSBN and $(\in, \in \vee q_k)$ - FSI, and we study some of their properties using examples. Additionally, we present the definitions of a $(\in, \in \vee q_k)$ - FSSBN of an $(\in, \in \vee q_k)$ - FSBN and derive some results that are related to them.

2. Preliminaries

The fundamental ideas that will be used in the following sections are presented in this section.

Definition 2.1. A near-ring (NR) is a set $\mathcal{N}$ that is not empty and has the binary operations "+" and "." that satisfy the axioms that,
(i) $\mathcal{N}$ is a group under the operation $+$,
(ii) $\mathcal{N}$ is a semigroup under the operation $\cdot$,
(iii) (b + e) c = b c + e c for all $b, e, c \in \mathcal{N}$

Definition 2.2. A near-ring $\mathcal{N}$ is said to be Boolean Near ring if $a^2 = a$.

Definition 2.3. [1] A F-subset α of $\mathcal{N}$ is named as F-sub-NR of $\mathcal{N}$ if $\forall g, s \in \mathcal{N}$,
(i) $\alpha(g - s) \geq M\{\alpha(g), \alpha(s)\}$, [M is min, in short]
(ii) $\alpha(gs) \geq M\{\alpha(g), \alpha(s)\}$

Definition 2.4. [1] A F-subset α of $\mathcal{N}$ is named as FI of $\mathcal{N}$ if $\forall g, s, r \in \mathcal{N}$,
(i) $\alpha(g - s) \geq M\{\alpha(g), \alpha(s)\}$
(ii) $\alpha(s + g - s) \geq \alpha(g)$
(iii) $\alpha(gs) \geq \alpha(g)$
(iv) $\alpha(g(s + r) - gs) \geq \alpha(r)$

Definition 2.5. [18] A F-subset α of $\mathcal{N}$ is named as $(\in, \in \vee q_k)$-F-sub-NR of $\mathcal{N}$ if
(i) $g_a \in \alpha, s_b \in \alpha \Rightarrow (g + s)_{M\{a,b\}} \in \vee q_k \alpha$
(ii) $g_a \in \alpha \Rightarrow (-g)_a \in \vee q_k \alpha$
(iii) $g_a \in \alpha, s_b \in \alpha \Rightarrow (gs)_{M\{a,b\}} \in \vee q_k \alpha, \forall g, s \in N$ and $a, b \in (0, 1]$.
An $(\in, \in \vee q_k)$-F-sub-NR of $\mathcal{N}$ with K=0 is an $(\in, \in \vee q)$-F-sub-NR of $\mathcal{N}$ (see [9]).

Definition 2.6. [18] A F-subset α of $\mathcal{N}$ is named as $(\in, \in \vee q_k)$-FI of $\mathcal{N}$ if
(i) $g_a \in \alpha, s_b \in \alpha \Rightarrow (g - s)_{M\{a,b\}} \in \vee q_k \alpha$,
(ii) $g_a \in \alpha, s \in \mathcal{N} \Rightarrow (s + g - s)_a \in \vee q_k \alpha$,
(iii) $g_a \in \alpha, s \in \mathcal{N} \Rightarrow (gs)_a \in \vee q_k \alpha$,
(iv) $r_a \in \alpha, g, s \in \mathcal{N} \Rightarrow (g(s + r) - gs)_a \in \vee q_k \alpha$, for all $g, s, r \in \mathcal{N}$ and $a, b \in (0, 1]$.
An $(\in, \in \vee q_k)$-FI of $\mathcal{N}$ with K = 0 is an $(\in, \in \vee q)$-FI of $\mathcal{N}$ (see [9]).

Unless otherwise stated, $\mathcal{P}$(M) is the power set of M and $R \subseteq E$ is a set of parameters, the universe set M.

Definition 2.7. [8] A pair (V, R) is named as soft set M, where $V : R \to \mathcal{P}$(M) is a mapping and $R \subseteq E$.

Definition 2.8. [7] Let $\mathcal{P}$(M) be the collection of all F-subsets of M and $R \subseteq E$. Then the pair (∇, R) is named as FSS over M, where $\nabla : R \to \mathcal{P}(M)$ is a mapping. i.e., for each $\varrho \in R$, $\nabla(\varrho) = \nabla_\varrho : M \to$ [0,1] is a F-subset of M.

Definition 2.9. [11] Let (∇, R) is a FSS M. Then the set $Supp(\nabla, R) = \{\varrho \in R | \nabla(\varrho) \neq \emptyset)\}$ is named as Support of (∇, R).
A FSS (∇, R) is named as notnull, if Supp$(\nabla, R) \neq \emptyset$. It's represented as $(\nabla, R) \neq \emptyset$

Definition 2.10. [7] Let (∇, R_1) and (Ψ, R_2) are a FSS's M. Then the set
(i) "(∇, R_1) AND (Ψ, R_2)", then $(\nabla, R_1) \wedge (\Psi, R_2)$ is described as $(\nabla, R_1) \wedge (\Psi, R_2) = (\Omega, R)$, where $R = R_1 \times R_2$ and $\Omega(\varrho, \phi) = \nabla(\varrho) \cap \Psi(\phi)$ forall $(\varrho, \phi) \in R_1 \times R_2$.
(ii) "(∇, R_1) OR (Ψ, R_2)", then $(\nabla, R_1) \vee (\Psi, R_2)$ is described as $(\nabla, R_1) \vee (\Psi, R_2) = (\Omega, R)$, where $R = R_1 \times R_2$ and $\Omega(\varrho, \phi) = \nabla(\varrho) \cup \Psi(\phi)$ forall $(\varrho, \phi) \in R_1 \times R_2$.

Definition 2.11. [7] Then the union of two FSS's "(∇, R_1) and (Ψ, R_2)" is the FSS (Ω, R), Where $R = R_1 \cup R_2$ and $\forall \varrho \in R$,

$$\Omega(\varrho) = \begin{cases} \nabla(\varrho), if\, \varrho \in R_1 - R_2 \\ \Psi(\varrho), if\, \varrho \in R_2 - R_1 \\ \nabla(\varrho) \cup \Psi(\varrho), if\, \varrho \in R_1 \cap R_2 \end{cases}$$

Then it's represented as $(\Omega, R) = (\nabla, R_1) \sqcup (\Psi, R_2)$.

Definition 2.12. [3] Then the restricted intersection (R-int, in short) of two FSS's "(∇, R_1) and (Ψ, R_2)", then $(\nabla, R_1) \sqcap_r (\Psi, R_2) = (\Omega, R)$, Where $R = R_1 \cap R_2 \neq \emptyset$ and $\Omega(\varrho) = \nabla(\varrho) \cap \Psi(\varrho), \forall \varrho \in R$.

Definition 2.13. [3] Then the extended intersection (E-int, in short) of two FSS's "(∇, R_1) and (Ψ, R_2)", is the FSS (Ω, R), Where $R = R_1 \cup R_2$ and $\forall \varrho \in R$,

$$\Omega(\varrho) = \begin{cases} \nabla(\varrho), if\, \varrho \in R_1 - R_2 \\ \Psi(\varrho), if\, \varrho \in R_2 - R_1 \\ \nabla(\varrho) \cup \Psi(\varrho), if\, \varrho \in R_1 \cap R_2 \end{cases}$$

Then it's represented by $(\Omega, R) = (\nabla, R_1) \sqcap_e (\Psi, R_2)$.

Definition 2.14. [2] The Family of FSS's $\{(\nabla_i, R_i) | i \in I\}$ over M.
(i) Then the R-int of these FSS's is the FSS $\{\sqcap_r(\nabla_i, R_i) | i \in I\} = (\nabla, R)$, where $R = \cap_{i \in I} R_i \neq \emptyset$ and $\nabla(\varrho) = \cap_{i \in I} \nabla_i(\varrho) \forall \varrho \in R$.

(ii) Then the union of these FSS's is the FSS $\{\sqcup(\nabla_i, R_i)|i \in I\} = (\nabla, R)$, where $R = \cup_{i\in I} R_i$ and $\nabla(\varrho) = \cup_{i\in I}\nabla_i(\varrho)\forall\varrho \in R$.
(iii) Then the AND of these FSS's is the FSS $\{\wedge(\nabla_i, R_i)|i \in I\} = (\nabla, R)$, where $R = \prod_{i\in I} R_i$ and $\nabla(\varrho) = \cap_{i\in I}\nabla_i(\varrho_i)\forall\varrho \in R$.

Definition 2.15. [10] Let (∇, R) is the FSS $\mathcal{N}$. Then (∇, R) is named as FSN $\mathcal{N}$, if $\nabla(\varrho)$ be an F-sub-NR of $\mathcal{N}$, $\forall\varrho \in R$.

Definition 2.16. [10] Let ϑ and ξ be F-sub-NR $\mathcal{N}$. Then ϑ is named as F-sub-NR of ξ. If $\vartheta(g) \leq \xi(g), \forall g \in \mathcal{N}$.

Definition 2.17. [10] Let ϑ be a F-subset $\mathcal{N}$, and ξ be a F-sub-NR, $\mathcal{N}$. Then ϑ is an FI of ξ, if $\vartheta(g) \leq \xi(g), \forall g \in \mathcal{N}$ is a FI of $\mathcal{N}$.

Definition 2.18. [10] Let "(∇, R_1) and (Ψ, R_2)" are two FSN $\mathcal{N}$. Then (∇, R_1) is named as FS-sub-BN of (Ψ, R_2)", if
(i) $R_1 \subseteq R_2$ and (ii) $\nabla(\varrho)$ is a F-sub-BN of $\Psi(\varrho), \forall\varrho \in R_1$

3. $(\in, \in \vee q_k)$-Fuzzy Soft Boolean Near-Rings and $(\in, \in \vee q_k)$-Fuzzy Soft idels

The concepts of $(\in, \in \vee q_k)$-FSBN and $(\in, \in \vee q_k)$-FSI over $\mathcal{N}$ are introduced in this section. The concept of a $(\in, \in \vee q_k)$-FS-sub-BN of a $(\in, \in \vee q_k)$-FSBN is further defined, and some of its attributes are examined.

Definition 3.1. Let (∇, R) is a FSS $\mathcal{N}$. Then (∇, R) is named as $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$, if for each $\varrho \in R, \nabla(\varrho) = \nabla_\varrho$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}$. i.e.,
(i) $g_a, s_b \in \nabla_\varrho \Rightarrow (g+s)_{M\{a,b\}} \in \vee q_k \nabla_\varrho$,
(ii) $g_a \in \nabla_\varrho \Rightarrow (-g)_a \in \vee q_k \nabla_\varrho$,
(iii) $g_a, s_b \in \nabla_\varrho \Rightarrow (gs)_{M\{a,b\}} \in \vee q_k \nabla_\varrho, \forall g, s \in \mathcal{N}$ and $a, b \in (0, 1]$.
A F-subset ∇_ϱ of $\mathcal{N}$ is known to satisfy requirements (i) and (ii) iff the following conditions are met:
(iv) $g_a, s_b \in \nabla_\varrho \Rightarrow (g-s)_{M\{a,b\}} \in \vee q_k \nabla_\varrho, \forall g, s \in \mathcal{N}$ and $a, b \in (0, 1]$.
An $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$, with k=0 is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$.

Lemma 3.2. *The definitions 3.1 conditions (i) and (iv) are identical to*
(i) $\nabla_\varrho(g+s) \geq M\{\nabla_\varrho(g), \nabla_\varrho(s), (1-k)/2)\}$,
(ii) $\nabla_\varrho(-g) \geq M\{\nabla_\varrho(g), (1-k)/2)\}$,
(iii) $\nabla_\varrho(gs) \geq M\{\nabla_\varrho(g), \nabla_\varrho(s), (1-k)/2)\}$,
(iv) $\nabla_\varrho(g-s) \geq M\{\nabla_\varrho(g), \nabla_\varrho(s), (1-k)/2)\}, \forall g, s \in \mathcal{N}$.

Proof. The proof can be made from [18] Lemma 3.2 and Lemma 3.3. □

Remark 3.3. Lemma 3.1 condition (i) and (ii) is equivalent to the condition (iv).

Theorem 3.4. *Let* (∇, R) *is a FSS* $\mathcal{N}$. *The* (∇, R) *is named as* $(\in, \in \vee q_k)$*-FSBN* $\mathcal{N}$, *iff* $\varrho \in R$ *and* $g, s \in \mathcal{N}$,
(i) $\nabla_\varrho(g+s) \geq M\{\nabla_\varrho(g), \nabla_\varrho(s), (1-k)/2\}$.
(ii) $\nabla_\varrho(gs) \geq M\{\nabla_\varrho(g), \nabla_\varrho(s), (1-k)/2\}$.

Proof. The evidence is provided by Lemma 3.2. □

Example 3.5. Let the nonempty set $\mathcal{N} = \{0, s, v, l\}$ have the binary operations "+" and ".", in the following terms:

+	0	s	v	1
0	0	s	v	1
s	s	0	1	0
v	v	1	0	s
1	1	0	s	0

$*$	0	s	v	1
0	0	0	0	0
s	0	s	0	s
v	0	0	v	v
1	0	s	v	1

The set of parameters is $R = \{\alpha, \beta, \gamma\}$ and now define FSS (∇, R) on a BN $\mathcal{N}$ by

R	α	β	γ
0	0.7	0.6	0.4
s	0.5	0.5	0.1
v	0.6	0.2	0.3
1	0.5	0.2	0.1

It is simple to verify that (∇, R) is an $(\in, \in \vee q_k) - FSBN, \mathcal{N}$ with k=0.4.

Definition 3.6. Let (ϑ, S) is a FSS $\mathcal{N}$. Then (ϑ, S) is named as $(\in, \in \vee q_k)$-FSI $\mathcal{N}$, if for each $\varrho \in S, \vartheta(\varrho) = \vartheta_\varrho$ is an $(\in, \in \vee q_k)$-FI of $\mathcal{N}$, i.e.,
(i) $g_a, s_b \in \vartheta_\varrho \Rightarrow (g+s)_{M\{a,b\}} \in \vee q_k \vartheta_\varrho$,
(ii) $g_a \in \vartheta_\varrho, s \in \mathcal{N} \Rightarrow (s+g-s)_a \in \vee q_k \vartheta_\varrho$,
(iii) $g_a \in \vartheta_\varrho, s \in \mathcal{N} \Rightarrow (gs)_a \in \vee q_k \vartheta_\varrho$,
(iv) $g_a \in \vartheta_\varrho, s, r \in \mathcal{N} \Rightarrow (s(r+g) - sr)_r \in \vee q_k \vartheta_\varrho, \forall g, s, r \in \mathcal{N}$ and $a, b \in (0, 1]$.
An $(\in, \in \vee q_k)$-FSI $\mathcal{N}$, with k=0 is named as $(\in, \in \vee q_k)$-FSI $\mathcal{N}$.

Lemma 3.7. *The preceding definition's conditions* (i) *through* (iv) *are identical to,*
(i) $\vartheta_\varrho(g+s) \geq M\{\vartheta_\varrho(g), \vartheta_\varrho(s), (1-k)/2\}$,
(ii) $\vartheta_\varrho(s+g-s) \geq M\{\vartheta_\varrho(g), (1-k)/2\}$,
(iii) $\vartheta_\varrho(gs) \geq M\{\vartheta_\varrho(g), (1-k)/2\}$,
(iv) $\vartheta_\varrho(g(s+r) - gs) \geq M\{\vartheta_\varrho(r), (1-k)/2\}, \forall g, s, r \in \mathcal{N}$.

Proof. Lemma 3.11 of [18] can be used to achieve the proof. □

Theorem 3.8. *Let (ϑ, S) is a FSS $\mathcal{N}$. Then (ϑ, S) is an $(\in, \in \vee q_k)$-FSI $\mathcal{N}$, iff $\forall \varrho \in R$ and $g, s, r \in \mathcal{N}$.*
(i) $\vartheta_\varrho(g+s) \geq M\{\vartheta_\varrho(g), \vartheta_\varrho(s), (1-k)/2\}$,
(ii) $\vartheta_\varrho(s+g-s) \geq M\{\vartheta_\varrho(g), (1-k)/2\}$,
(iii) $\vartheta_\varrho(gs) \geq M\{\vartheta_\varrho(g), (1-k)/2\}$,
(iv) $\vartheta_\varrho(g(s+r)-gs) \geq M\{\vartheta_\varrho(r), (1-k)/2\}$.

Proof. The evidence is provided by lemma 3.2. □

Example 3.9. Let $\mathcal{N} = \{0, s, v, l\}$ be a BN in example 3.1. Define a FSS (ϑ, S), where $S = \{g, s\}$, as depicted below
$\vartheta_g = \{(0, 0.7), (s, 0.5), (v, 0.2), (l, 0.2)\}$ and
$\vartheta_s = \{(0, 0.6), (s, 0.3), (v, 0.3), (l, 0.5)\}$.
We can quickly confirm that (ϑ, S) is an $(\in, \in \vee q_k)$-FSBN and an $(\in, \in \vee q_k)$-FSI $\mathcal{N}$, with k=0.4.

Theorem 3.10. *The R-int of two $(\in, \in \vee q_k)$-FSBN's $\mathcal{N}$ is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$,if it's notnull.*

Proof. Let (∇, R_1) and (Ψ, R_2) are $(\in, \in \vee q_k)$ - FSBN's of $\mathcal{N}$ and let $(\nabla, R_1) \sqcap_r (\Psi, R_2) = (\Omega, R)$. Where $R = R_1 \cap R_2$ and $\Omega(\varrho) = \nabla(\varrho) \cap \Psi(\varrho), \forall \varrho \in Supp(\Omega, R)$. Since $\nabla(\varrho)$ and $\Psi(\varrho)$ are$(\in, \in \vee q_k)$ - F-sub-BN's of $\mathcal{N}$, by theorem 3.16 in [18], $\nabla(\varrho) \cap \Psi(\varrho) = \Omega(\varrho)$ is $(\in, \in \vee q_k)$ - F-sub-BN's of $\mathcal{N}$, $\forall \varrho \in Supp(\Omega, R)$. Hence (Ω, R) is an $(\in, \in \vee q_k)$ - FSBN $\mathcal{N}$.
In a similar manner, we can demonstrate for $(\in, \in \vee q_k)$ - FSI. □

Theorem 3.11. *The E-Int of two $(\in, \in \vee q_k)$ - FSBN's $\mathcal{N}$ is an $(\in, \in \vee q_k)$ - FSBN $\mathcal{N}$, if it's notnull.*

Proof. Let (∇, R_1) and (Ψ, R_2) are $(\in, \in \vee q_k)$ - FSBN's of $\mathcal{N}$ and let $(\nabla, R_1) \sqcap_e (\Psi, R_2) = (\Omega, R)$ be the E-Int of (∇, R_1) and (Ψ, R_2). Then $R = R_1 \cup R_2$ and for any $\varrho \in Supp(\Omega, R)$,
if $\varrho \in R_1 - R_2$ then $\Omega(\varrho) = \nabla(\varrho)$ is an $(\in, \in \vee q_k)$ - F-sub-BN $\mathcal{N}$,
if $\varrho \in R_2 - R_1$ then $\Omega(\varrho) = \Psi(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN $\mathcal{N}$, and if $\varrho \in R_1 \cap R_2$,
then $\Omega(\varrho) = \nabla(\varrho) \cap \Psi(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}$, Since the intersection of two $(\in, \in \vee q_k)$-F-sub-BN's $\mathcal{N}$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}$. Therefore $\Omega(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}$, $\forall \varrho \in supp(\Omega, R)$. Hence (Ω, R) is an $(\in, \in \vee q_k)$ - FSBN $\mathcal{N}$.
In a similar manner, we can demonstrate for $(\in, \in \vee q_k)$ - FSI. □

Theorem 3.12. *If (∇, R_1) and (Ψ, R_2) are $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$ such that $R_1 \cap R_2 = \emptyset$, then their union is an $(\in, \in \vee q_k)$ - FSBN $\mathcal{N}$.*

Proof. by applying definition (union), Now we can write $(\nabla, R_1) \sqcup (\Psi, R_2) = (\Omega, R)$, where $R = R_1 \cup R_2$. Since $R_1 \cap R_2 = \emptyset$, we have either $\varrho \in R_1 - R_2$ or $\varrho \in R_2 - R_1, \forall \varrho \in Supp(\Omega, R)$. For any $\varrho \in Supp(\Omega, R)$,

if $\varrho \in R_1 - R_2$ then $\Omega(\varrho) = \nabla(\varrho)$ is an $(\in, \in \vee q_k)$ -F-sub-BN $\mathcal{N}$, and
if $\varrho \in R_2 - R_1$ then $\Omega(\varrho) = \Psi(\varrho)$ is an $(\in, \in \vee q_k)$ -F-sub-BN $\mathcal{N}$.
Thus $\Omega(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN $\mathcal{N}, \forall \varrho \in Supp(\Omega, R)$. Hence (Ω, R) is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$.
In a similar manner, we can demonstrate for $(\in, \in \vee q_k)$-FSI. □

Theorem 3.13. *if (∇, R_1) and (Ψ, R_2) are $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$. Then $(\nabla, R_1) \wedge (\Psi, R_2)$ is an $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$, if it's notnull.*

Proof. by applying Definition (AND), we can write $(\nabla, R_1) \wedge (\Psi, R_2) = (\Omega, R)$, where $R = R_1 \times R_2$ and $\Omega(\varrho, \phi) = \nabla(\varrho) \cap \Psi(\phi), \forall (\varrho, \phi) \in R_1 \times R_2$. Since $\nabla(\varrho)$ and $\Psi(\phi)$ are $(\in, \in \vee q_k)$-F-sub-BN's of $\mathcal{N}, \forall \varrho \in R_1$ and $\psi \in R_2, \nabla(\varrho) \cap \Psi(\phi) = \Omega(\varrho, \phi)$ is also an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}, \forall (\varrho, \phi) \in R_1 \times R_2$. Therefore (Ω, R) is an $(\in, \in \vee q_k)$ - FSBN N.
In a similar manner, we can demonstrate for $(\in, \in \vee q_k)$-FSI's. □

Theorem 3.14. *If (∇, R_1) and (Ψ, R_2) is $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$, and let $\nabla(\varrho) \subseteq \Psi(\phi)$ or $\Psi(\phi) \subseteq \nabla(\varrho), \forall (\varrho, \phi) \in R_1 \times R_2$. Then $(\nabla, R_1) \vee (\Psi, R_2)$ is an $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$.*

Proof. By applying definition (OR), we can write $(\nabla, R_1) \vee (\Psi, R_2) = (\Omega, R)$, where $R = R_1 \times R_2$ and $\Omega(\varrho, \phi) = \nabla(\varrho) \cup \Psi(\phi), \forall (\varrho, \phi) \in R_1 \times R_2$. For any $(\varrho, \phi) \in R_1 \times R_2$, if $\nabla(\varrho) \subseteq \Psi(\phi)$, then $\Omega(\varrho, \phi) = \nabla(\varrho) \cup \Psi(\phi) = \Psi(\phi)$ is an $(\in, \in \vee q_k)$-F-sub-BN $\mathcal{N}$, and if $\Psi(\phi) \subseteq \nabla(\varrho)$, then $\Omega(\varrho, \phi) = \nabla(\varrho) \cup \Psi(\phi) = \nabla(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN $\mathcal{N}$. Hence (Ω, R) is an $(\in, \in \vee q_k)$ -FSBN $\mathcal{N}$. □

Theorem 3.15. *Let $\{(\nabla_i, R_i) | i \in I\}$ be a family of $(\in, \in \vee q_k)$ -FSBN's $\mathcal{N}$. Then*
(i) $\{\sqcap_r (\nabla_i, R_i) | i \in I\}$ is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$, if $\{\sqcap_r (\nabla_i, R_i)\} \neq \emptyset$.
(ii) $\{\wedge (\nabla_i, R_i) | i \in I\}$ is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$, if $\{\wedge (\nabla_i, R_i)\} \neq \emptyset$.

Proof. (i) Let $\{\sqcap_r (\nabla_i, R_i) | i \in I\} = (\nabla, R)\}$, where $R = \cap_{i \in I} R_i$ and for each $\varrho \in R, \nabla(\varrho) = \cap_{i \in I} \nabla_i(\varrho)$. Suppose that $(\nabla, R) \neq \emptyset$. If $\varrho \in R$, then $\nabla(\varrho) = \cap_{i \in I} \nabla_{i \in I}(\varrho) \neq \emptyset$. Since (∇_i, R_i) is an $(\in, \in \vee q_k)$-FSBN, $\forall i \in I, \nabla_i(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}, \forall \varrho_i \in Supp(\nabla_i, R_i)$ and $i \in I$ and so by the theorem 3.16 in [18], $\cap_{i \in I} \nabla_i(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}, \forall \varrho \in Supp(\nabla, R)$. Hence $(\nabla, R) = \{\sqcap_r (\nabla_i, R_i) | i \in I\}$ is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$.
(ii) Let $\{\wedge (\nabla_i, R_i) | i \in I) = (\nabla, R)\}$, where $R = \prod_{i \in I} R_i$ and for each $\varrho \in R, \nabla(\varrho) = \cap_{i \in I} \nabla_i(\varrho_i)$. Suppose $(\nabla, R) \neq \emptyset$. If $\varrho \in R$, then $\nabla(\varrho) = \cap_{i \in I} \nabla_i(\varrho_i) \neq \emptyset$. Since (∇_i, R_i) is an $(\in, \in \vee q_k)$-FSBN, $\forall i \in I, \nabla_i(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}, \forall \varrho_i \in Supp(\nabla_i, R_i)$ and $i \in I$ and so by the theorem 3.16 in [18], $\cap_{i \in I} \nabla_i(\varrho_i)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\mathcal{N}, \forall \varrho_i \in Supp(\nabla_i, R_i)$. Hence $(\nabla, R) = \{\wedge (\nabla_i, R_i) | i \in I\}$ is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$. □

Definition 3.16. Let (∇, R_1) and (Ψ, R_2) are $(\in, \in \vee q_k)$-FSBN's $\mathcal{N}$. Then (∇, R_1) is named as $(\in, \in \vee q_k)$-FS-sub-BN of (Ψ, R_2),

(i) if $R_1 \subseteq R_2$,

(ii) $\nabla(\varrho)$ is an $(\in, \in \vee q_k)$-F-sub-BN of $\Psi(\varrho), \forall \varrho \in Supp(\nabla, R_1)$.

Theorem 3.17. *Let* (∇, R_1) *and* (Ψ, R_2) *are* $(\in, \in \vee q_k)$*-FSBN's N, and if* $\nabla(\varrho) \subseteq \Psi(\varrho), \forall \varrho \in Supp(\nabla, R_1)$*, then* (∇, R_1) *is an* $(\in, \in \vee q_k)$ *-FS-sub-BN' of* (Ψ, R_2).

Proof. Proof is straight forward. □

Definition 3.18. Let (∇, R) is an $(\in, \in \vee q_k)$ - FSBN $\mathcal{N}$. Then the FSS (ϑ, S) $\mathcal{N}$ is named as $(\in, \in \vee q_k)$-FSI of (∇, R) expressed as $(\vartheta, S) \trianglelefteq (\nabla, R)$,

(i) if $S \subseteq R$ and

(ii) $\vartheta(\varrho)$ is an $(\in, \in \vee q_k)$-FI of $\nabla(\varrho)$, represented as $\vartheta(\varrho) \triangleleft \nabla(\varrho), \forall \varrho \in Supp(\vartheta, S)$.

Example 3.19. Let (∇, R) is an $(\in, \in \vee q_k)$-FSBN $\mathcal{N}$, as in starting example in this section. Define a FSS (ϑ, S), where S={a,m}, as follows:
$\vartheta_a = \{(0, 0.4), (s, 0.5), (v, 0.6), (l, 0.4)\}$ and $\vartheta_b = \{(0, 0.3), (s, 0.2), (v, 0.1), (l, 0.1)\}$. We can quickly confirm that (ϑ, S) is an $(\in, \in \vee q_k)$ - FSI of (∇, R) $\mathcal{N}$ with k=0.2

Theorem 3.20. *The R-Int of two* $(\in, \in \vee q_k)$*-FSI's* $\mathcal{N}$ *of an* $(\in, \in \vee q_k)$*-FSBN* $\mathcal{N}$ *is an* $(\in, \in \vee q_k)$*-FSI of* (∇, R)*, when it's notnull.*

Proof. Let $(\alpha, S_1) \trianglelefteq (\nabla, R)$ and $(\xi, S_2) \trianglelefteq (\nabla, R)$. By the definition (R-Int), we can write $(\alpha, S_1) \sqcap_r (\xi, S_2) = (\vartheta, S)$, where $S = S_1 \cap S_2 \neq \emptyset$ and $\vartheta(\varrho) = \alpha(\varrho) \cap \xi(\varrho), \forall \varrho \in Supp(\vartheta, S)$. Since $S_1 \subseteq R$ and $S_2 \subseteq R$, we have $S_1 \cap S_2 = S \subseteq R$. Suppose that (ϑ, R) is notnull. Since $(\alpha, S_1) \trianglelefteq (\nabla, R)$ and $(\xi, S_2) \trianglelefteq (\nabla, R)$, we have $\alpha(\varrho) \triangleleft \nabla(\varrho)$ and $\xi(\varrho) \triangleleft \nabla(\varrho), \forall \varrho \in Supp(\vartheta, s)$. So $\alpha(\varrho) \cap \xi(\varrho) = \vartheta(\varrho) \triangleleft \nabla(\varrho), \forall \varrho \in Supp(\vartheta, s)$. Hence $(\alpha, S_1) \sqcap_r (\xi, S_2) = (\vartheta, S)$ is an $(\in, \in \vee q_k)$-FSI of (∇, R). □

Theorem 3.21. *The union of two* $(\in, \in \vee q_k)$*-FSI's* $\mathcal{N}$ *of an* $(\in, \in \vee q_k)$*-FSBN* (∇, R) *is an* $(\in, \in \vee q_k)$*-FSI of* (∇, R).

Proof. Let $(\alpha, S_1) \trianglelefteq (\nabla, R)$ and $(\xi, S_2) \trianglelefteq (\nabla, R)$. By the definition (Union), we can write $(\alpha, S_1) \sqcup (\xi, S_2) = (\vartheta, S)$, where $S = S_1 \cup S_2$ and for all $\varrho \in Supp(\vartheta, S)$ $\vartheta(\varrho) = \begin{cases} \alpha(\varrho), if\, \varrho \in S_1 - S_2 \\ \xi(\varrho), if\, \varrho \in S_2 - S_1 \\ \alpha(\varrho) \cup \xi(\varrho), if\, \varrho \in S_1 \cap S_2 \end{cases}$

Obviously $S_1 \cup S_2 = S \subseteq R$. Since $S_1 \cap S_2 = \emptyset$, either $\varrho \in S_1 - S_2$ or $\varrho \in S_2 - S_1, \forall \in Supp(\vartheta, S)$.

If $\varrho \in S_1 - S_2$, then $\vartheta(\varrho) = \alpha(\varrho) \triangleleft \nabla(\varrho)$ and

if $\varrho \in S_2 - S_1$, then $\vartheta(\varrho) = \xi(\varrho) \triangleleft \nabla(\varrho)$.

Thus $\vartheta(\varrho) \triangleleft \nabla(\varrho), \forall \varrho \in Supp(\vartheta, S)$. Therefore (ϑ, S) is an $(\in, \in \vee q_k)$ -FSI of (∇, R). □

4. Conclusion

In this paper, we've discussed the ideas of $(\in, \in \vee q_k)$-FSBN and $(\in, \in \vee q_k)$ - FSI over BN resp., Additionally, we have added the idea of $(\in, \in \vee q_k)$-FS-sub-BN of an $(\in, \in \vee q_k)$-FSBN and examined some of their characteristics using illustrative instances. The FSS's operations have been transferred to the $(\in, \in \vee q_k)$-FSBN's and $(\in, \in \vee q_k)$-FSI's.

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