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ON THE COMPLETE PRODUCT OF INTUITIONISTIC FUZZY GRAPHS

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Abstract. An Intuitionistic fuzzy graph (IFG) is a kind of fuzzy graph (FG) which explains the degree of membership (MS) and non-membership (NMS) of vertices and edges. We present the novel concept, the Complete product of a pair of IFGs. We explore the idea of the Complete product of a pair of strong IFGs and develop certain essential theorems based on strong IFGs. Also, we discuss about the Complete product of a pair of complete IFGs.

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Key words and phrases. IFG; strong IFG; complete IFG; complete product.

1. Introduction

Euler pioneered the notion of graph theory. A graph is a convenient approach to express relationships between objects. Vertices of the graph represent objects, while edges describe relations. Designing a fuzzy network model is necessary when there is ambiguity in the description of the objects and their relations. Zadeh [10] put forth the concept of fuzzy set and this concept brought about revolutionary changes in the area of interdisciplinary research. Rosenfeld [8] developed fuzzy graph (FG) theory in 1975. Another innovative study on fuzzy set was made by Atanassov [2] who introduced intuitionistic fuzzy sets. Next breakthrough came when R. Parvathi [7] developed intuitionistic fuzzy graph (IFG). Ch. Chaitanya and T.V. Pradeep Kumar [3] introduced complete product of FGs. Many perspectives on intuitionistic fuzzy sets and IFGs are discussed in [1,5,9].

An IFG explains the degree of MS and NMS of vertices and edges. An IFG is strong if it is λ-strong, δ-strong and ρ-strong. We introduce the idea of complete product of a pair of IFGs and prove certain results based on this product. We prove that the complete product of a pair of strong IFGs is a strong IFG. For a pair of complete IFGs, their complete product is also a complete IFG. Also, we prove that if

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the complete product of a pair of IFGs is strong, then at least one of the IFG will be strong. We refer to Harary [4] for fundamental graph theoretic terms.

2. Preliminaries

Definition 2.1. [4] A graph $G = (V, E)$, V is the vertex set and E is the edge set. Each edge has either one or two vertices connected to it, which are referred to as its end points.

Definition 2.2. [6] A FG is $G = (V, \sigma, \mu)$, V is the vertex set, $\sigma : V \to [0, 1]$, $\mu : V \times V \to [0, 1]$ with $\mu(u, v) \leq \sigma(u) \wedge \sigma(v), \forall u, v \in V$.

Definition 2.3. [7] An IFG is $G = (V, E, \sigma, \mu)$, V is the vertex set, $\sigma = (\lambda_1, \delta_1), \mu = (\lambda_2, \delta_2)$ and $\lambda_1, \delta_1 : V \to [0, 1]$ stands for the degree of MS, NMS of $v \in V$,

$$0 \leq \lambda_1(v) + \delta_1(v) \leq 1.$$

$\lambda_2, \delta_2 : V \times V \to [0, 1]$ stands for the degree of MS, NMS of the edge $x = (u, v) \in V \times V$,

$$\lambda_2(x) \leq \lambda_1(u) \wedge \lambda_1(v)$$

$$\delta_2(x) \leq \delta_1(u) \vee \delta_1(v)$$

$$0 \leq \lambda_2(x) + \delta_2(x) \leq 1, \forall x$$

Definition 2.4. An IFG G is

$$\lambda\text{-strong if} \quad \lambda_2(x) = \lambda_1(u) \wedge \lambda_1(v),$$

$$\delta\text{-strong if} \quad \delta_2(x) = \delta_1(u) \vee \delta_1(v), \forall x = (u, v) \in E$$

An IFG G is a strong IFG if G is λ-strong and δ-strong.

Example 2.5. Consider the IFG G with vertices u_1, u_2, u_3 in figure 1.
$\lambda_2(u_1, u_2) = 0.5, \quad \lambda_1(u_1) \wedge \lambda_1(u_2) = 0.6 \wedge 0.5 = 0.5$
$\lambda_2(u_2, u_3) = 0.4, \quad \lambda_1(u_2) \wedge \lambda_1(u_3) = 0.5 \wedge 0.4 = 0.4$
$\lambda_2(u_1, u_2) = \lambda_1(u_1) \wedge \lambda_1(u_2)$
$\lambda_2(u_2, u_3) = \lambda_1(u_2) \wedge \lambda_1(u_3)$.
i.e., G is a λ-strong IFG.
$\delta_2(u_1, u_2) = 0.4, \quad \delta_1(u_1) \vee \delta_1(u_2) = 0.3 \vee 0.4 = 0.4$
$\delta_2(u_2, u_3) = 0.6, \quad \delta_1(u_2) \vee \delta_1(u_3) = 0.4 \vee 0.6 = 0.6$
$\delta_2(u_1, u_2) = \delta_1(u_1) \vee \delta_1(u_2)$
$\delta_2(u_2, u_3) = \delta_1(u_2) \vee \delta_1(u_3)$.
i.e., G is δ-strong .
Since, G is λ-strong and δ-strong, G is a strong IFG.

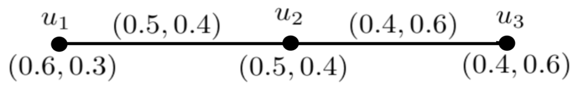

FIGURE 1. Strong IFG G

Definition 2.6. An IFG G is a complete IFG if

$$\lambda_2(x) = \lambda_1(u) \wedge \lambda_1(v)$$

$$\delta_2(x) = \delta_1(u) \vee \delta_1(v), \forall u, v \in V$$

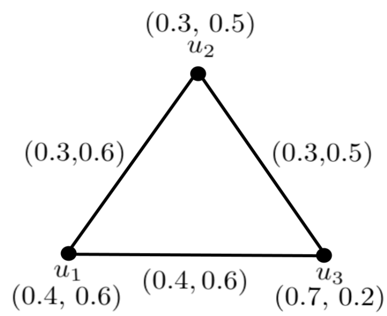

FIGURE 2. Complete IFG G

3. MAIN RESULTS

We can construct different products in IFGs like tensor product, normal product etc. But these products are defined on specific domains (proper subsets of $U \times V$) and not on the whole cartesian product $U \times V$ of the two vertex sets U and V of the two IFGs. Now we discuss the complete product of IFG which is defined on the whole cartesian product.

Definition 3.1. Complete product of the IFGs, $G_1 = (U, E_U, \sigma, \mu)$, $G_2 = (V, E_V, \sigma', \mu')$ where $\sigma = (\lambda_1, \delta_1)$, $\mu = (\lambda_2, \delta_2)$, $\sigma' = (\lambda_1', \delta_1')$ and $\mu' = (\lambda_2', \delta_2')$ is the IFG
$G = G_1 \circledast G_2 = (U \times V, E, \sigma \circledast \sigma', \mu \circledast \mu')$,
$E = E_1 \cup E_2 \cup ...... \cup E_8$ such that

$$E_1 = \{w : u_1 = u_2, w_2 \in E_V\}$$

$$E_2 = \{w : u_1 = u_2, w_2 \notin E_V\}$$

$$E_3 = \{w : v_1 = v_2, w_1 \in E_U\}$$

$$E_4 = \{w : v_1 = v_2, w_1 \notin E_U\}$$

$$E_5 = \{w : w_1 \in E_U, w_2 \notin E_V\}$$

$$E_6 = \{w : w_1 \notin E_U, w_2 \in E_V\}$$

$$E_7 = \{w : w_1 \in E_U, w_2 \in E_V\}$$

$$E_8 = \{w : w_1 \notin E_U, w_2 \notin E_V\},$$

$$\text{where,} \quad w = ((u_1, v_1), (u_2, v_2)),$$

$$w_1 = (u_1, u_2),$$

$$w_2 = (v_1, v_2).$$

$$(\lambda_1 \circledast \lambda_1')(x) = \lambda_1(u) \wedge \lambda_1'(v)$$

$$(\delta_1 \circledast \delta_1')(x) = \delta_1(u) \vee \delta_1'(v), \quad x = (u, v)$$

$$(\lambda_2 \circledast \lambda_2')(w) = \begin{cases} \lambda_1(u_1) \wedge \lambda_2'(w_2), \text{if } w \in E_1 \\ \lambda_1(u_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{if } w \in E_2 \\ \lambda_1'(v_1) \wedge \lambda_2(w_1), \text{if } w \in E_3 \\ \lambda_1'(v_1) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), \text{if } w \in E_4 \\ \lambda_2(w_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{if } w \in E_5 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_2'(w_2), \text{if } w \in E_6 \\ \lambda_2(w_1) \wedge \lambda_2'(w_2), \text{if } w \in E_7 \\ \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{if } w \in E_8 \end{cases}$$

$$(\delta_2 \circledast \delta_2')(w) = \begin{cases} \delta_1(u_1) \vee \delta_2'(w_2), \text{if } w \in E_1 \\ \delta_1(u_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{if } w \in E_2 \\ \delta_1'(v_1) \vee \delta_2(w_1), \text{if } w \in E_3 \\ \delta_1'(v_1) \vee \delta_1(u_1) \vee \delta_1(u_2), \text{if } w \in E_4 \\ \delta_2(w_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{if } w \in E_5 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_2'(w_2), \text{if } w \in E_6 \\ \delta_2(w_1) \vee \delta_2'(w_2), \text{if } w \in E_7 \\ \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{if } w \in E_8 \end{cases}$$

Theorem 3.2. For a pair of strong IFGs G_1, G_2, their complete product is also a strong IFG .

Proof. Consider the strong IFGs G_1, G_2.

For $w_1 \in E_U, w_2 \in E_V$,

$$\lambda_2(w_1) = \lambda_1(u_1) \wedge \lambda_1(u_2), \lambda'_2(w_2) = \lambda'_1(v_1) \wedge \lambda'_1(v_2),$$
$$\delta_2(w_1) = \delta_1(u_1) \vee \delta_1(u_2), \delta'_2(w_2) = \delta'_1(v_1) \vee \delta'_1(v_2).$$

Case(i) When $w \in E_1$

$$\begin{aligned}(\lambda_2 \circledast \lambda'_2)(w) &= \lambda_1(u_1) \wedge \lambda'_2(w_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2), \text{ since } u_1 = u_2\\ &= (\lambda_1 \circledast \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \circledast \lambda'_1)(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta'_2)(w) &= \delta_1(u_1) \vee \delta'_2(w_2)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2), \text{ since } u_1 = u_2\\ &= (\delta_1 \circledast \delta'_1)(u_1, v_1) \vee (\delta_1 \circledast \delta'_1)(u_2, v_2)\end{aligned}$$

Case(ii) When $w \in E_2$

$$\begin{aligned}(\lambda_2 \circledast \lambda'_2)(w) &= \lambda_1(u_1) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda'_1(v_1) \wedge \lambda'_1(v_2), \text{ since } u_1 = u_2\\ &= (\lambda_1 \circledast \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \circledast \lambda'_1)(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta'_2)(w) &= \delta_1(u_1) \vee \delta'_1(v_1) \vee \delta'_1(v_2)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta'_1(v_1) \vee \delta'_1(v_2), \text{ since } u_1 = u_2\\ &= (\delta_1 \circledast \delta'_1)(u_1, v_1) \vee (\delta_1 \circledast \delta'_1)(u_2, v_2)\end{aligned}$$

Case(iii) When $w \in E_3$

$$\begin{aligned}(\lambda_2 \circledast \lambda'_2)(w) &= \lambda'_1(v_1) \wedge \lambda_2(w_1)\\ &= \lambda'_1(v_1) \wedge \lambda'_1(v_2) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), \text{since } v_1 = v_2\\ &= (\lambda_1 \circledast \lambda'_1)(u_1, v_1) \wedge (\lambda_1 \circledast \lambda'_1)(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta'_2)(w) &= \delta'_1(v_1) \vee \delta_2(w_1)\\ &= \delta'_1(v_1) \vee \delta'_1(v_2) \vee \delta_1(u_1) \vee \delta_1(u_2), \text{ since } v_1 = v_2\\ &= (\delta_1 \circledast \delta'_1)(u_1, v_1) \vee (\delta_1 \circledast \delta'_1)(u_2, v_2)\end{aligned}$$

Case(iv) When $w \in E_4$

$$\begin{aligned}(\lambda_2 \circledast \lambda_2')(w) &= \lambda_1'(v_1) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2)\\ &= \lambda_1'(v_1) \wedge \lambda_1'(v_2) \wedge \lambda_1(u_1) \wedge \lambda_1(u_2), \text{since } v_1 = v_2\\ &= (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta_2')(w) &= \delta_1'(v_1) \vee \delta_1(u_1) \vee \delta_1(u_2)\\ &= \delta_1'(v_1) \vee \delta_1'(v_2) \vee \delta_1(u_1) \vee \delta_1(u_2), \text{ since } v_1 = v_2\\ &= (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2)\end{aligned}$$

Case(v) When $w \in E_5$

$$\begin{aligned}(\lambda_2 \circledast \lambda_2')(w) &= \lambda_2(w_1) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta_2')(w) &= \delta_2(w_1) \vee \delta_1'(v_1) \vee \delta_1'(v_2)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)\\ &= (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2)\end{aligned}$$

Case(vi) When $w \in E_6$

$$\begin{aligned}(\lambda_2 \circledast \lambda_2')(w) &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_2'(w_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2)\end{aligned}$$

$$\begin{aligned}(\delta_2 \circledast \delta_2')(w) &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_2'(w_2)\\ &= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)\\ &= (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2)\end{aligned}$$

Case(vii) When $w \in E_7$

$$\begin{aligned}(\lambda_2 \circledast \lambda_2')(w) &= \lambda_2(w_1) \wedge \lambda_2'(w_2)\\ &= \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)\\ &= (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2)\end{aligned}$$

$$(\delta_2 \circledast \delta_2')(w) = \delta_2(w_1) \vee \delta_2'(w_2)$$
$$= \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2)$$

Case(viii) When $w \in E_8$

$$(\lambda_2 \circledast \lambda_2')(w) = \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2)$$
$$= (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2)$$

$$(\delta_2 \circledast \delta_2')(w) = \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2)$$
$$= (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2)$$

Thus, $G = G_1 \circledast G_2$ is a strong IFG. □

Example 3.3. In figure 3, G_1 and G_2 are two strong IFGs.

Their complete product $G_1 \circledast G_2$ shown in figure 4 is a strong IFG since all the edges are strong edges.

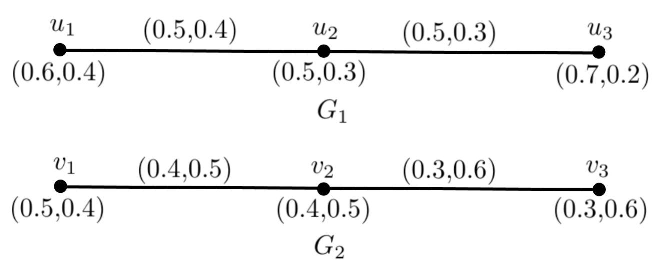

FIGURE 3. Strong IFGs G_1 and G_2

Theorem 3.4. For two complete IFGs G_1, G_2 their complete product is also a complete IFG .

Proof. Similar to 3.2. □

Theorem 3.5. If G_1,G_2 are two IFGs such that their complete product is strong, then at least one of the IFG will be strong.

Proof. Suppose G_1, G_2 are not strong. Then $\exists$ at least one $w_1 \in E_U, w_2 \in E_V$, with

$$\lambda_2(w_1) < \lambda_1(u_1) \wedge \lambda_1(u_2), \quad \lambda_2'(w_2) < \lambda_1'(v_1) \wedge \lambda_1'(v_2),$$
$$\delta_2(w_1) < \delta_1(u_1) \vee \delta_1(u_2), \quad \delta_2'(w_2) < \delta_1'(v_1) \vee \delta_1'(v_2),$$

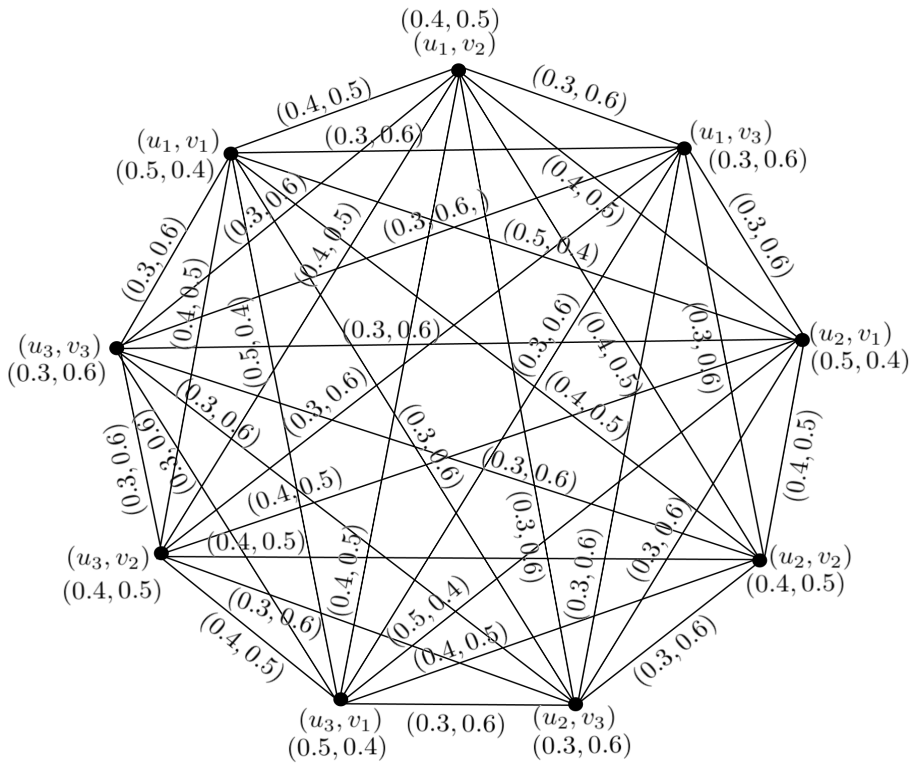

FIGURE 4. Complete product of strong IFGs G_1 and G_2

Let $w \in E_1$. Then,

$$(\lambda_2 \circledast \lambda_2')(w) = \lambda_1(u_1) \wedge \lambda_2'(w_2)$$
$$< \lambda_1(u_1) \wedge \lambda_1(u_2) \wedge \lambda_1'(v_1) \wedge \lambda_1'(v_2), \text{since } u_1 = u_2$$
$$i.e., (\lambda_2 \circledast \lambda_2')(w) < (\lambda_1 \circledast \lambda_1')(u_1, v_1) \wedge (\lambda_1 \circledast \lambda_1')(u_2, v_2).$$

$$(\delta_2 \circledast \delta_2')(w) = \delta_1(u_1) \vee \delta_2'(w_2)$$
$$< \delta_1(u_1) \vee \delta_1(u_2) \vee \delta_1'(v_1) \vee \delta_1'(v_2), \text{ since } u_1 = u_2$$
$$i.e., (\delta_2 \circledast \delta_2')(w) < (\delta_1 \circledast \delta_1')(u_1, v_1) \vee (\delta_1 \circledast \delta_1')(u_2, v_2).$$

Hence the complete product is not strong, a contradiction. So at least one of the IFG will be strong. □

4. APPLICATIONS

IFGs can be suitably used in real life problems. It can work as a good aid in solving companies' merger problems. Consider two strong networks G_1 and G_2 as in figure 3, with vertices indicating distinct companies. The MS degree of the vertices indicates the market worth of the companies and the MS degree of the edges indicates the market worth of the joint ventures of the companies. Since G_1, G_2 are strong, all the edges in G_1, G_2 are strong and all the edges in the complete product $G_1 \circledast G_2$ are also strong as in figure 4. As the complete product is defined on the whole cartesian product, it includes all the possible edges between every pair of vertices. Thus, this product is stronger and more reliable than other products and the decision on merger problems based on this result will be more accurate.

For example, consider two strong companies, one which is successful in the production of scooters and another company which is expert in the production of batteries. A joint venture, if initiated, will benefit both the companies and expertise of both the companies in their respective production, will be a strong foundation to introduce a new production unit for manufacturing electric scooters and thus produce a new brand of electric scooters. Thus, the production carried out by the joint venture will be surely successful and may result in making both the companies involved in the joint venture more stronger.

5. Conclusions

IFGs has many uses in the fields of robotics, artificial intelligence and medical diagnosis. We investigated a novel product known as the complete product of two IFGs, which accounts for all potential edges. We proved that complete product of a pair of strong IFGs is a strong IFG and complete product of two complete IFGs is a complete IFG. Also we proved that if the complete product of a pair of IFGs is strong, then at least one of the IFG will be strong. IFG models provide exact and accurate outcomes for making decisions and resolving merger related problems. Our future work is to broaden the scope of our investigation to study the complement of the complete product of IFG.

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