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A NOTE ON TWO NEW THREE-PARAMETER LOGARITHMIC AND EXPONENTIAL COPULAS

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Abstract. The copula approach was developed for describing the connection between two or more quantitative variables. Despite the numerous existing copulas, there is still a need for new copulas or generalized versions of well-established copulas. In this study, we suggest two brand-new copulas with three parameters derived from the construction of the Farlie-Gumbel-Morgenstern copula and unique non-separable logarithmic and exponential perturbation functions. In particular, the exponential-based copula has the merit of naturally extending the well-known Celebioglu-Cuadras copula. Widely acceptable domains are established for the involved parameters. The related features and capabilities of the proposed copulas are examined in detail. It is demonstrated that they possess various kinds of shapes, are diagonally symmetric, have manageable series expansions, satisfy interesting first-order copula orders, have a versatile quadrant dependence, have no tail dependence, are not Archimedean, are typically not radially symmetric, and can model a weak or moderate dependence with the rho of Spearman as a benchmark. Numerical tables and figures are used to illustrate some findings. As a final remark, some new two-dimensional inequalities are established, and they might be interesting for reasons unrelated to those of this study.

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1. Introduction

In many practical scenarios, modeling the association (or dependence) between two or more quantitative variables is crucial. To this end, in the quantitative case, the copulas offer efficient

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solutions. Due to the varied complexity of dependence that manifests in heterogeneous current issues, they have recently attracted a lot of attention. Significant applications are in a variety of applied fields (see [28], [25], [33], and [29], among others). When only two quantitative variables are of interest, two-dimensional copulas are required. A definition of such a copula in the absolutely continuous case is provided below as a mathematical foundation (see [24]).

Definition 1. *A two-dimensional function $C(x, y)$, $(x, y) \in [0, 1]^2$, is a (absolutely continuous two-dimensional) copula if and only if, for any $(x, y) \in [0, 1]^2$,*

(I): *we have $C(x, 0) = 0$ and $C(0, y) = 0$,*

(II): *we have $C(x, 1) = x$ and $C(1, y) = y$,*

(III): *we have (in the absolutely continuous case)*

$$\partial_{x,y} C(x, y) \geq 0,$$

where $\partial_{x,y} = \partial^2/(\partial x \partial y)$ denotes the mixed second order partial derivatives according to x and y.

The basics on the copula theory can be found in [24], [15], [19], [10], and [23], and recent developments include [1], [26], [3], [17], [6], [7], [30], [31], [16], [22], [13], [34], and [27], among others. In order to motivate our study, a retrospective on the Farlie-Gumbel-Morgenstern (FGM) copula is necessary. To begin, it is defined as

$$C(x, y; \theta) = xy + \theta xy(1 - x)(1 - y), \quad (x, y) \in [0, 1]^2,$$

where $\theta \in [-1, 1]$. Thus, we can write this copula as $C(x, y; \theta) = \Pi(x, y) + P(x, y; \theta)$, where $\Pi(x, y)$ is the independence copula, i.e., $\Pi(x, y) = xy$, and $P(x, y; \theta)$ is the separable perturbation function defined by $P(x, y; \theta) = \theta\phi(x)\phi(y)$, with $\phi(u) = u(1 - u)$. We have $P(x, y; \theta) \in [-1/16, 1/16]$, and the parameter θ is known as the dependence parameter; when $\theta = 0$, we have $C(x, y; \theta) = \Pi(x, y)$. The main informations about the FGM copula can be found in [24], [15], [19], [10], and [23]. We may also refer to [12], where the numerous qualities of FGM copula are developed. However, it is limited in terms of possible shapes and correlation range, which are both crucial in a data fitting scenario. Because of its simple algebraic features, the FGM copula has attracted the attention of researchers to a large extent. For these reasons, new types of FGM copula have been suggested by researchers. We may refer to [17], [20], [2], [14], and [5], among others.

Following the spirit of the above references, we introduce two new copulas of the general form $C(x, y; \alpha) = \Pi(x, y) + Q(x, y; \alpha)$, where $Q(x, y; \alpha)$ is a perturbation function defined with original logarithmic or exponential transformations, and which share the characteristics of (i) being not separable with respect to x and y (unlike $P(x, y; \theta)$), and of (ii) being dependent on three parameters represented by α. In particular, the established exponential-based copula has the merit of extending in a natural way the well-known Celebioglu-Cuadras copula (see [4], [11], and [8]). Thus, we aim to present a new way of constructing such copulas beyond the standard forms and with the use of three parameters. In 2023, with computer developments having broken down the barriers of mathematical complexity, the practical management of three parameters in a function is not a problem anymore, as long as we know their admissible values. Hence, for each copula, we determine the admissible values of their parameters. The limits, two-dimensional differentiations, factorizations, and mathematical inequalities, are the main foundations for the underlying proofs. The characteristics of the proposed copulas are then examined, including their shapes, associated functions (survival copula, copula density, etc.), symmetry (diagonal, radial, etc.), tractable series expansions, quadrant dependence, first-order copula ordering, tail dependence, medial and Spearman correlations, and generation of two-dimensional distributions. In particular, we show that the proposed copulas are suitable for modeling weak or moderate dependence. Analyses in both value tables and graphics are offered when appropriate. This research can be seen as the first step in creating new multi-dimensional copulas, which continue to hold special interest in many practical fields. On the other hand, based on our copula findings, a number of two-dimensional inequalities are found and may be of independent interest.

The rest of the paper is divided into three sections: Section 2 is devoted to the proposed logarithmic-type copula, and Section 3 investigates the proposed exponential-type copula. A short conclusion is given in Section 4.

2. Logarithmic copula

This section is devoted to the first proposed copula involving an original logarithmic perturbed function.

2.1. **Main result.** The main result of this section is presented below.

Proposition 2.1. *Let us consider the following two-dimensional function:*

$$C(x,y;\alpha) = xy + a\log\left[1 + bx^c y^c (1-x)(1-y)\right], \quad (x,y)\in[0,1]^2, \tag{1}$$

where $\alpha=(a,b,c)$, and a, b and c are such that $c\geq 1$, and one of the following condition set is satisfied:

Condition set 1: $b \geq 0$, *and* $|a|bc^2 \leq 1$,

Condition set 2: $b\in(-(c+1)^{2(c+1)}/c^{2c}, 0]$, *and* $(1+bc^{2c}/(c+1)^{2(c+1)})^2 + |a|bc^2 \geq 0$.

Then $C(x,y;\alpha)$ is a two-dimensional copula.

Proof. We aim to show that the proposed function satisfies the items **(I)**, **(II)** and **(III)** of Definition 1.

(I): For any $x\in[0,1]$, since $c\geq 1$, we have

$$C(x,0;\alpha) = x\times 0 + a\log\left[1+bx^c\times 0^c\times(1-x)(1-0)\right] = 0 + a\log(1) = 0.$$

Using a similar development, for any $y\in[0,1]$, we get $C(0,y;\alpha)=0$.

(II): For any $x\in[0,1]$, we have

$$C(x,1;\alpha) = x\times 1 + a\log\left[1+bx^c\times 1^c\times(1-x)(1-1)\right] = x + a\log(1) = x.$$

Similarly, for any $y\in[0,1]$, we have $C(1,y;\alpha)=y$.

(III): After differentiation, several simplifications and factorizations, we get

$$\partial_{x,y}C(x,y;\alpha) = \frac{\begin{aligned} &bx^{c+1}y^c\left\{a(c+1)[c(y-1)+y] - 2(y-1)y\right\} - abcx^c\left[c(y-1)+y\right]y^c \\ &\quad + b^2(y-1)^2x^{2c+1}y^{2c+1} - 2b^2(y-1)^2x^{2c+2}y^{2c+1} \\ &\quad + b^2(y-1)^2x^{2c+3}y^{2c+1} + 2b(y-1)x^{c+2}y^{c+1} + xy\end{aligned}}{xy\left[1+bx^cy^c(1-x)(1-y)\right]^2}$$

$$= 1 + \frac{abx^{c-1}y^{c-1}\left[c(x-1)+x\right]\left[c(y-1)+y\right]}{\left[1+bx^cy^c(1-x)(1-y)\right]^2},$$

which is a quite manageable condensed expression.

For any $(x,y)\in[0,1]^2$, with the manipulation of absolute values, the following inequality holds:

$$\partial_{x,y}C(x,y;\alpha) \geq 1 - \frac{|a||b|x^{c-1}y^{c-1}\left|c(x-1)+x\right|\left|c(y-1)+y\right|}{\left[1+bx^cy^c(1-x)(1-y)\right]^2}.$$

Clearly, since $c \geq 1$, for $(x, y) \in [0, 1]^2$, we have $c(x-1)+x \in [-c, 1]$ and $c(y-1)+y \in [-c, 1]$, implying that $|c(x-1)+x| \leq \max(|c|, 1) = c$ and $|c(y-1)+y| \leq \max(|c|, 1) = c$. Furthermore, under the same conditions, we have $x^{c-1}y^{c-1} \leq 1$. We deduce that

$$\partial_{x,y} C(x, y; \alpha) \geq 1 - \frac{|a||b|c^2}{[1 + bx^c y^c (1-x)(1-y)]^2}.$$

In order to prove the positivity of $\partial_{x,y} C(x, y; \alpha)$, let us work on the right term by distinguishing the two condition sets on the parameters.

Condition set 1: We recall that $b \geq 0$, and $|a|bc^2 \leq 1$.

Since $b \geq 0$, and $(x, y) \in [0, 1]^2$, we have $bx^c y^c (1-x)(1-y) \geq 0$, implying that $[1 + bx^c y^c (1-x)(1-y)]^2 \geq 1$. This inequality and the assumption $|a|bc^2 \leq 1$ imply that

$$\partial_{x,y} C(x, y; \alpha) \geq 1 - |a|bc^2 \geq 0.$$

Condition set 2: We recall that $b \in (-(c+1)^{2(c+1)}/c^{2c}, 0]$, and $(1 + bc^{2c}/(c+1)^{2(c+1)})^2 + |a|bc^2 \geq 0$.

Since $b \in (-(c+1)^{2(c+1)}/c^{2c}, 0]$ with $c \geq 1$, a function study gives $m = \sup_{x \in [0,1]} x^c(1-x) = c^c/(c+1)^{c+1}$. Therefore we have

$$1 + bx^c y^c (1-x)(1-y) \geq 1 + bm^2 = 1 + b\frac{c^{2c}}{(c+1)^{2(c+1)}},$$

which is strictly positive, implying that $[1 + bx^c y^c (1-x)(1-y)]^2 \geq (1 + bc^{2c}/(c+1)^{2(c+1)})^2$. By virtue of this inequality, $|b| = -b$ and the assumption $(1 + bc^{2c}/(c+1)^{2(c+1)})^2 + |a|bc^2 \geq 0$, we get

$$\partial_{x,y} C(x, y; \alpha) \geq 1 + \frac{|a|bc^2}{(1 + bc^{2c}/(c+1)^{2(c+1)})^2} \geq 0.$$

Thus, under the two considered sets, we have

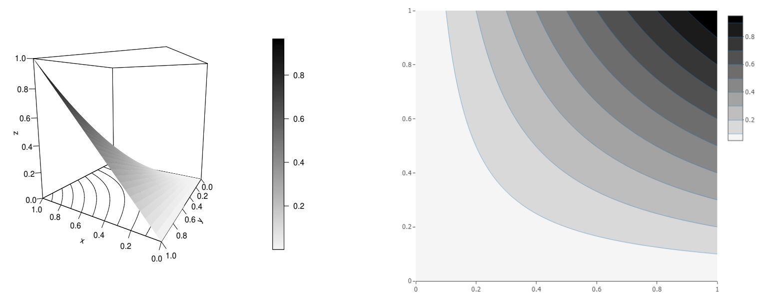
$$\partial_{x,y} C(x, y; \alpha) \geq 0.$$

The item **(III)** is proved.

The proof of the proposition ends. □

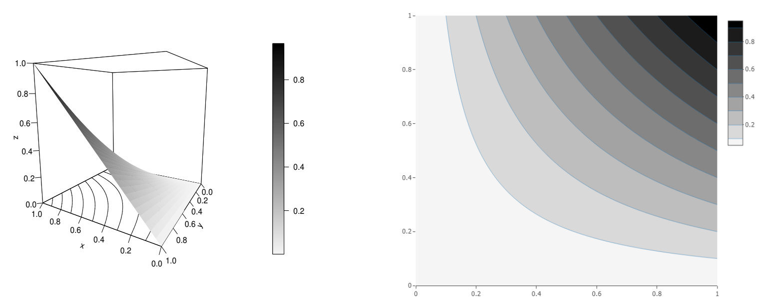
For the purposes of this study, the copula presented in Equation (1) is called the logarithmic (L) copula. The L copula is then presumably taken into account under Configurations sets 1 or 2 from Proposition 2.1. The independence copula is a special case of the L copula; it is obtained by taking $a = 0$ or $b = 0$. To the best of our knowledge, the other parameter values produce new two-dimensional copulas. Furthermore, there is some connection between the L copula and an extended version of the FGM copula; this will be discussed later.

Plots of the L copula are presented in Figures 1 and 2 for arbitrary parameters that belong to Configuration sets 1 and 2, respectively.



(a) (b)

FIGURE 1. Display of the (a) perspective plot and (b) contour plot of the L copula for $a = 1/2$, $b = 1/4$ and $c = 2$, belonging to Configuration set 1



(a) (b)

FIGURE 2. Display of the (a) perspective plot and (b) contour plot of the L copula for $a = 3/2$, $b = -1/4$ and $c = 3/2$, belonging to Configuration set 2

Different form morphologies for the L copula are seen in these figures. These forms are undoubtedly impacted by a, b and c.

Remark 2.2. *Based on the definition of the L copula, it is natural to think of the following two-dimensional function as an alternative:*

$$C(x, y; \alpha) = xy + ax^c y^c \log\left[1 + b(1-x)(1-y)\right], \quad (x, y) \in [0, 1]^2,$$

*where $\alpha = (a, b, c)$. However, the item **(III)** of Definition 1 is difficult to deal with, particularly in determining the admissible values for a, b, and c that satisfy it. This is thus a mathematical challenge that we postponed for a future study.*

Remark 2.3. *The expression of the L copula can inspire the construction of higher-dimensional copulas. For instance, one can consider the following three-dimensional function:*

$$C(x, y, z; \nu) = xyz + a \log \left[1 + bxyz(1 - x)(1 - y)(1 - z)\right],$$
$$(x, y, z) \in [0, 1]^3,$$

with $\nu = (a, b)$; to reduce the functional complexity, we have put $c = 1$. Then one can prove that it is a three-dimensional copula under the following configuration sets:

Condition set 1: $b \in [0, 1]$, *and* $|a|b \leq 1$,
Condition set 2: $b > 1$, *and* $|a|b(1 + b/4^3) \leq 1$,
Condition set 3: $b \in (-4^3, 0]$, *and* $(1 + b/4^3)^2 + |a|b(1 - b/4^3) \geq 0$.

These statements are made possible primarily by a manageable mixed third order partial derivative of $C(x, y, z; \nu)$ with respect to x, y, and z; we have

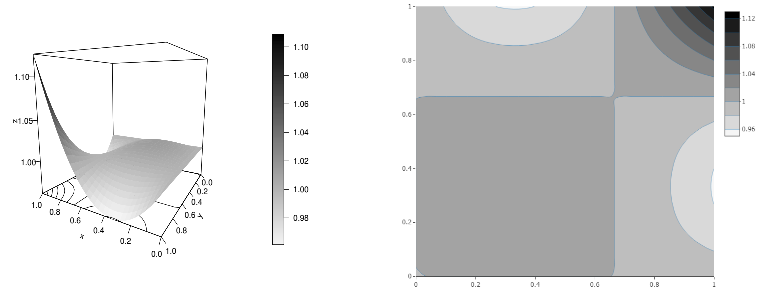
$$\partial_{x,y,z} C(x, y, z; \nu) =$$
$$1 - ab \frac{(2x - 1)(2y - 1)(2z - 1)[1 - bxyz(1 - x)(1 - y)(1 - z)]}{[1 + bxyz(1 - x)(1 - y)(1 - z)]^3},$$
$$(x, y, z) \in [0, 1]^3.$$

For more information about the potential applicability of such an original three-dimensional copula; we may refer the reader to [18] and [9], and the references therein.

2.2. **Related functions.** To begin, based on Equation (1), the L copula density is calculated as

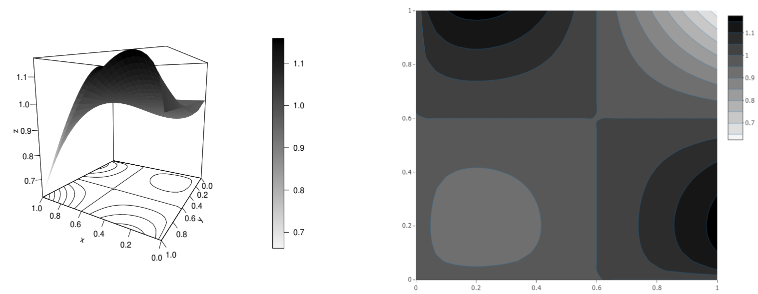
$$c(x, y; \alpha) = \partial_{x,y} C(x, y; \alpha)$$
$$= 1 + \frac{abx^{c-1}y^{c-1}\left[c(x - 1) + x\right]\left[c(y - 1) + y\right]}{\left[1 + bx^c y^c (1 - x)(1 - y)\right]^2}, \quad (x, y) \in [0, 1]^2.$$

We can examine the modeling potential of the L copula as well as the effects of the parameters a, b, and c on its shapes by looking at the forms of this function. Figures 3 and 4 display the L copula density plots for arbitrary parameters from Configuration sets 1 and 2, respectively.



(a) (b)

FIGURE 3. Display of the (a) perspective plot and (b) contour plot of the L copula density for $a = 1/2$, $b = 1/4$ and $c = 2$, belonging to Configuration set 1



(a) (b)

FIGURE 4. Display of the (a) perspective plot and (b) contour plot of the L copula density for $a = 3/2$, $b = -1/4$ and $c = 3/2$, belonging to Configuration set 2

These figures demonstrate a type of dependence flexibility by showing completely different shapes of the L copula density. It is important to consider how a, b and c affect these shapes, particularly in the neighborhood of the extrema $(0, 0)$, $(1, 1)$, $(0, 1)$, and $(1, 0)$.

The L survival copula is given by

$$\begin{aligned}\hat{C}(x, y; \alpha) &= x + y - 1 + C(1 - x, 1 - y; \alpha)\\ &= xy + a \log\left[1 + bxy(1 - x)^c(1 - y)^c\right], \quad (x, y) \in [0, 1]^2.\end{aligned}$$

The main difference between the L copula and the L survival copula is the effect of c which is not the same. The L survival copula is a brand-new three-parameter copula as well, which is added to the body of current research.

2.3. **Properties.** In this part, some fundamental properties of the L copula are established in order to comprehend its modeling capabilities. The book of [24] contains all of the details of the future mentioned notions.

To begin, since $C(x, y; \alpha) = C(y, x; \alpha)$ for any $(x, y) \in [0, 1]^2$, the L copula is diagonally symmetric. It is not Archimedean because, for example, when $a = 1$, $b = 1$, and $c = 1$,

$$C\left[\frac{1}{4}, C\left(\frac{1}{2}, \frac{1}{3}; \alpha\right); \alpha\right] = 0.08692625 \neq 0.0879255$$
$$= C\left[C\left(\frac{1}{4}, \frac{1}{2}; \alpha\right), \frac{1}{3}; \alpha\right],$$

proving that it is not associative. For $a \neq 0$ and $c > 1$, the L copula is not radially symmetric because of the moving of the parameter c in the two expressions; there exists (x, y) such that $\hat{C}(x, y; \alpha) \neq C(x, y; \alpha)$. In the cases $a = 0$, or $a \neq 0$ and $c = 1$, it is radially symmetric.

Of course, as for any copula, the Fréchet-Hoeffding bounds hold: For any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C(x, y; \alpha) \leq \min(x, y)$.

Remark 2.4. *Immediate mathematical consequences of the Fréchet-Hoeffding bounds are the following two-dimensional inequalities: for any $(x, y) \in [0, 1]^2$, we have*

$$\max(x + y - 1, 0) - xy \leq a \log\left[1 + bx^c y^c (1 - x)(1 - y)\right]$$
$$\leq \min(x, y) - xy,$$

where a, b, and c, satisfy either Configuration sets 1 or 2. This three-dimensional logarithmic inequality can be used in various two-dimensional analysis studies beyond the copula's scope.

Let us now investigate the diverse quadrant dependence properties with respect to the parameters.

- For $a \geq 0$ and $b \geq 0$ (which is compatible with Configuration set 1, with some restrictions), or $a \leq 0$ and $b \in [-1, 0]$ (which is compatible with Configuration set 2, with some restrictions), because $a \log\left[1 + bx^c y^c (1 - x)(1 - y)\right] \geq 0$, the L copula is positively quadrant dependent, i.e., $C(x, y; \alpha) \geq xy$ for any $(x, y) \in [0, 1]^2$.
- For $a \geq 0$ and $b \in [-1, 0]$ (which is compatible with Configuration set 2, with some restrictions), or $a \leq 0$ and $b \geq 0$ (which is compatible with Configuration set 1, with some restrictions), because $a \log\left[1 + bx^c y^c (1 - x)(1 - y)\right] \leq 0$, the L copula is negatively quadrant dependent, i.e., $C(x, y; \alpha) \leq xy$ for any $(x, y) \in [0, 1]^2$.

In addition, interesting first-order copula orders are satisfied. Thanks to the following logarithmic inequality: $\log(1 + x) \leq x$ for $x > -1$, the results below are established.

- For $a \geq 0$, we have $C(x,y;\alpha) \leq C_\star(x,y;\alpha)$ for any $(x,y) \in [0,1]^2$, where

$$C_\star(x,y;\alpha) = xy + abx^c y^c (1-x)(1-y), \quad (x,y) \in [0,1]^2, \tag{2}$$

  is an extended version of the FGM copula with dependence parameter ab (see [17]).
- For $a \leq 0$, we have $C(x,y;\alpha) \geq C_\star(x,y;\alpha)$ for any $(x,y) \in [0,1]^2$.

For $b \in (-1,1)$ and any $(x,y) \in [0,1)^2$, the logarithmic series expansion gives

$$C(x,y;\alpha) = xy + a\sum_{i=1}^{\infty} \frac{(-1)^{i+1}}{i} b^i x^{ci} y^{ci} (1-x)^i (1-y)^i. \tag{3}$$

This expansion can express or approximate various crucial correlation measures, which makes it useful in particular situations.

The tail dependence of the L copula is examined below. Using standard limit and equivalence techniques, since $c \geq 1$, we have

$$\begin{aligned}\lambda_{low} &= \lim_{x\to 0} \frac{C(x,x;\alpha)}{x} \\ &= \lim_{x\to 0} \left\{ x + \frac{a\log\left[1+bx^{2c}(1-x)^2\right]}{x} \right\} \\ &= \lim_{x\to 0} \left(x + abx^{2c-1}\right) = 0.\end{aligned}$$

Thus, the L copula has no lower tail dependence. Concerning the upper tail dependence, using similar arguments, we have

$$\begin{aligned}\lambda_{up} &= \lim_{x\to 1} \frac{1-2x+C(x,x;\alpha)}{1-x} \\ &= \lim_{x\to 1} \frac{1-2x+x^2+a\log\left[1+bx^{2c}(1-x)^2\right]}{1-x} \\ &= \lim_{x\to 1} \left(1-x+\frac{a\log\left[1+bx^{2c}(1-x)^2\right]}{1-x}\right) \\ &= \lim_{x\to 1} \left[1-x+ab(1-x)\right] = 0.\end{aligned}$$

Thus, the L copula has no upper tail dependence.

The medial correlation (or coefficient of Blomqvist) of the L copula is expressed as

$$M_{cor} = 4C\left(\frac{1}{2},\frac{1}{2};\alpha\right) - 1 = 4a\log\left(1+b2^{-2(c+1)}\right).$$

The rho of Spearman of the L copula is defined by

$$\rho = 12\int_0^1\int_0^1 [C(x,y;\alpha) - xy]\, dxdy$$
$$= 12a\int_0^1\int_0^1 \log\left[1 + bx^c y^c(1-x)(1-y)\right] dxdy.$$

There is no a simple expression of this measure, but the following expansion is derived from Equation (3):

$$\rho = 12a\sum_{i=1}^{\infty}\frac{(-1)^{i+1}}{i}b^i B(ci+1, i+1)^2,$$

where $B(u,v) = \int_0^1 t^{u-1}(1-t)^{v-1}dt$ with $u > 0$ and $v > 0$, is the standard beta function. Tables 1 and 2 determine numerical values of ρ for arbitrary parameters that belong to Configuration sets 1 and 2, respectively.

TABLE 1. Some values of ρ of the L copula for $b = 1/2$ and $c = \sqrt{2}$, and varying $a \in [-1, 1]$, belonging to Configuration set 1

a	-1.0	-0.7	-0.4	-0.1	0.2	0.5	0.8
ρ	-0.0878	-0.0615	-0.0351	-0.0088	0.0176	0.0439	0.0702

TABLE 2. Some values of ρ of the L copula for $b = -1$ and $c = 1$, and varying $a \in [-(15/16)^2, (15/16)^2]$, belonging to Configuration set 2

a	-0.87890625	-0.57890625	-0.27890625	0.02109375	0.32109375	0.62109375
ρ	0.299	0.197	0.0949	-0.0072	-0.1092	-0.2113

According to the obtained values in these tables, ρ can have a moderate amplitude and can be either positive or negative. The L copula is hence the best choice for modeling a mild dependence.

The L copula has the same ability to define new parametric distributional models as all other two-dimensional copulas. In fact, by merging two uni-dimensional cumulative distribution functions, say $F(x)$ and $G(x)$, we build a new two-dimensional cumulative distribution function as follows:

$$
\begin{aligned}
H(x,y;\xi) &= C(F(x),G(y);\alpha)\\
&= F(x)G(y) + a\log\left[1 + bF(x)^c G(y)^c (1-F(x))(1-G(y))\right],\\
&(x,y)\in\mathbb{R}^2,
\end{aligned}
$$

where ξ represents the vector of the involved parameters, including a, b and c (so α), and those appearing in $F(x)$ and $G(x)$.

Based on this function, novel two-dimensional distributions could be produced. Among other references, the discussion of motivated lifetime cumulative distribution functions is found in [32]. As an example, let us consider two exponential distributions with parameters γ and τ, respectively, then $F(x) = 1 - e^{-\gamma x}$ and $G(x) = 1 - e^{-\tau x}$ for $x \geq 0$, and $F(x) = G(x) = 0$ for $x \leq 0$. Then the related distribution based on the L copula is defined by the following cumulative distribution function:

$$
\begin{aligned}
&H(x,y;\xi) =\\
&(1-e^{-\gamma x})(1-e^{-\tau y}) + a\log\left[1 + be^{-\gamma x}e^{-\tau y}(1-e^{-\gamma x})^c(1-e^{-\tau y})^c\right],\\
&(x,y)\in[0,\infty)^2,
\end{aligned}
$$

and $H(x,y;\xi) = 0$ for $(x,y) \notin [0,\infty)^2$, where $\xi = (a,b,c,\gamma,\tau)$. To the best of our knowledge, such a two-dimensional logarithmic distribution is new in the literature. On the topic of FGM two-dimensional distributions, we may refer to [21], and the references therein.

3. Exponential copula

This section is devoted to the second proposed copula involving an original exponential perturbed function.

3.1. Main result.

The main result of this section is presented below.

Proposition 3.1. *Let us consider the following two-dimensional function:*

$$
C(x,y;\alpha) = xy + ax^c y^c\left(e^{b(1-x)(1-y)} - 1\right), \quad (x,y)\in[0,1]^2, \tag{4}
$$

where $\alpha = (a,b,c)$, and a, b and c are such that $c \geq 1$, and one of the following condition set is satisfied:

Condition set 1: $b \in [0,1]$, $a \geq 0$, *and* $abc \leq 1$,

Condition set 2: $b \in [-1,0]$, $a \geq 0$, *and* $ac^2 \leq 1$,

Condition set 3: $b \in [-1,0]$, $a \leq 0$, *and* $abc \leq 1$.

Then $C(x, y; \alpha)$ is a two-dimensional copula.

Proof. We aim to show that the proposed function satisfies the items **(I)**, **(II)** and **(III)** of Definition 1.

(I): For any $x \in [0, 1]$, since $c \geq 1$, we have

$$C(x, 0; \alpha) = x \times 0 + ax^c \times 0^c \times \left(e^{b(1-x)(1-0)} - 1\right) = 0.$$

Using a similar development, for any $y \in [0, 1]$, we get $C(0, y; \alpha) = 0$.

(II): For any $x \in [0, 1]$, we have

$$C(x, 1; \alpha) = x \times 1 + ax^c \times 1^c \times \left(e^{b(1-x)(1-1)} - 1\right) = x + ax^c \times \left(e^0 - 1\right) = x.$$

Similarly, for any $y \in [0, 1]$, we have $C(1, y; \alpha) = y$.

(III): We have

$$\begin{aligned}
\partial_{x,y} C(x, y; \alpha) &= ax^{c-1}y^{c-1}e^{b(1-x)(1-y)} \times \\
&\left\{bc\left[x(2y-1) - y\right] + bxy\left[1 + b(1-x)(1-y)\right] + c^2\right\} \\
&+ 1 - ac^2x^{c-1}y^{c-1} \\
&= ax^{c-1}y^{c-1}e^{b(1-x)(1-y)} f(x, y; \beta) + 1 - ac^2x^{c-1}y^{c-1}, \qquad (5)
\end{aligned}$$

where

$$f(x, y; \beta) = bc\left[xy + (1-x)(1-y)\right] + bxy\left[1 + b(1-x)(1-y)\right] + c^2 - bc,$$

and $\beta = (b, c)$.

In order to prove the positivity of $\partial_{x,y} C(x, y; \alpha)$, let us work on the right term, and the function $f(x, y; \beta)$ in particular, by distinguishing the three condition sets on the parameters.

Condition set 1: We recall that $b \in [0, 1]$, $a \geq 0$, $c \geq 1$, and $abc \leq 1$.

To begin, let us prove that $f(x, y; \beta) \geq 0$. Since $b \in [0, 1]$, $c \geq 1$, and $(x, y) \in [0, 1]^2$, we have $bc\left[xy + (1-x)(1-y)\right] \geq 0$, $bxy\left[1 + b(1-x)(1-y)\right] \geq 0$ and $c^2 - bc = c(c-b) \geq c(1-b) \geq 0$, implying that $f(x, y; \beta) \geq 0$.

On the other hand, we have $b(1-x)(1-y) \geq 0$, so $e^{b(1-x)(1-y)} \geq 1$. Therefore, since $a \geq 0$, we obtain

$$\partial_{x,y}C(x,y;\alpha) \geq ax^{c-1}y^{c-1}f(x,y;\beta)+1-ac^2x^{c-1}y^{c-1}$$
$$= ax^{c-1}y^{c-1}\times$$
$$\{bc\,[xy+(1-x)(1-y)]+bxy\,[1+b(1-x)(1-y)]\}$$
$$+1-abcx^{c-1}y^{c-1}.$$

The first main term (involving the curly brackets) is clearly positive as a sum of positive terms. Since $c \geq 1$, for any $(x,y) \in [0,1]^2$, have $x^{c-1}y^{c-1} \leq 1$, and owing to the condition $abc \leq 1$ (so $abc \in [0,1]$), we have $1-abcx^{c-1}y^{c-1} \geq 1-abc \geq 0$. It follows from the above results and Equation (5), as a sum of positive terms, we have

$$\partial_{x,y}C(x,y;\alpha) \geq 0.$$

Condition set 2: We recall that $b \in [-1,0]$, $a \geq 0$, $c \geq 1$, and $ac^2 \leq 1$.

Let us now prove that $f(x,y;\beta) \geq 0$, which is more technical than the developments in Condition set 1. After a well-arranged factorization, we can write

$$f(x,y;\beta) = -bc(1-xy)+b^2xy(1-x)(1-y)+c(c+b)$$
$$-bcx(1-y)-by(c-x).$$

Since $b \in [-1,0]$, $c \geq 1$, and $(x,y) \in [0,1]^2$, we have $-bc(1-xy) \geq 0$, $b^2xy(1-x)(1-y) \geq 0$, $-bcx(1-y) \geq 0$, $c(c+b) \geq c(1+b) \geq 0$, and $-by(c-x) \geq -by(1-x) \geq 0$. By putting all these results together, we get $f(x,y;\beta) \geq 0$.

On the other hand, it is clear that $ax^{c-1}y^{c-1}e^{b(1-x)(1-y)} \geq 0$. Since $c \geq 1$ and $ac^2 \leq 1$ (so $ac^2 \in [0,1]$), we have $1-ac^2x^{c-1}y^{c-1} \geq 1-ac^2 \geq 0$. It follows from the above results and Equation (5) that

$$\partial_{x,y}C(x,y;\alpha) \geq 0.$$

Condition set 3: We recall that $b \in [-1,0]$, $a \leq 0$, $c \geq 1$ and $abc \leq 1$.

We still have $f(x,y;\beta) \geq 0$ thanks to the same developments made for Condition set 2 because $f(x,y;\beta)$ is independent of a.

On the the hand, since $b \in [-1,0]$ and $(x,y) \in [0,1]^2$, we have $b(1-x)(1-y) \leq 0$ and $e^{b(1-x)(1-y)} \leq 1$. Therefore, since $a \leq 0$, we obtain

$$\partial_{x,y}C(x,y;\alpha) \geq ax^{c-1}y^{c-1}f(x,y;\beta) + 1 - ac^2x^{c-1}y^{c-1}$$
$$= ax^{c-1}y^{c-1}\times$$
$$\{bc\,[xy + (1-x)(1-y)] + bxy\,[1 + b(1-x)(1-y)]\}$$
$$+ 1 - abcx^{c-1}y^{c-1}.$$

Since $b \in [-1,0]$ and $c \geq 1$ and $(x,y) \in [0,1]^2$, we have $bc\,[xy + (1-x)(1-y)] \leq 0$ and $bxy\,[1 + b(1-x)(1-y)] \leq bxy\,[1 - (1-x)(1-y)] \leq 0$. Therefore, the term in curly brackets is negative. Since $a \leq 0$ and $x^{c-1}y^{c-1} \geq 0$, we have

$$ax^{c-1}y^{c-1}\{bc\,[xy + (1-x)(1-y)] + bxy\,[1 + b(1-x)(1-y)]\} \geq 0.$$

For the remaining term, since $c \geq 1$, $abc \leq 1$ (so $abc \in [0,1]$ since a and b are negative), and $(x,y) \in [0,1]^2$, we have $1 - abcx^{c-1}y^{c-1} \geq 1 - abc \geq 0$. Therefore, as a sum of positive terms, we have

$$\partial_{x,y}C(x,y;\alpha) \geq 0.$$

Thus, under the two considered sets, we have

$$\partial_{x,y}C(x,y;\alpha) \geq 0.$$

The item **(III)** is proved.

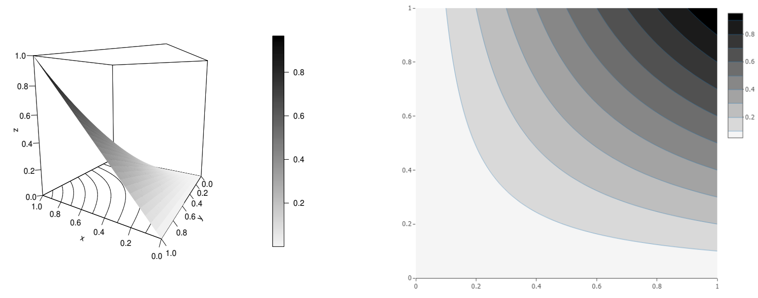
This ends the proof of the proposition. □

For the purposes of this study, the copula presented in Equation (4) is called the exponential (E) copula. Following this, it is assumed that the E copula is considered under Configurations sets 1, 2, or 3 of Proposition 3.1. The independence copula is a special case of the E copula by assuming that $a = 0$. For $a = 1$ and $c = 1$, the E copula is simply expressed as

$$C(x,y;\alpha) = xye^{b(1-x)(1-y)}, \quad (x,y) \in [0,1]^2,$$

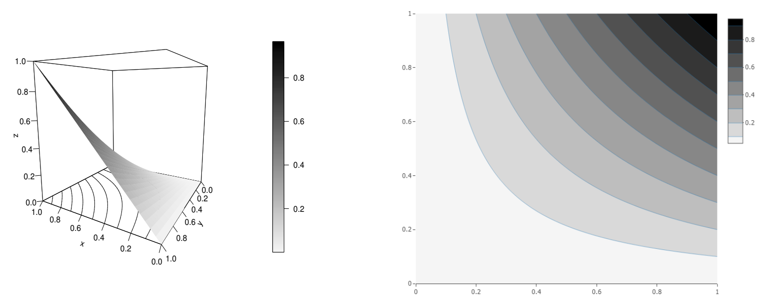
which is exactly the CC copula as described in [4], [11], and [8]. In this sense, the E copula is a generalization of the CC copula; the significance of the parameters b will be emphasized throughout this study. To the best of our knowledge, the other parameter values produce new two-dimensional copulas. There is some relationship with an extended version of the FGM copula, which will be covered later.

Plots of the E copula are presented in Figures 5, 6, and 7 for arbitrary parameters that belong to Configuration sets 1, 2, and 3, respectively.



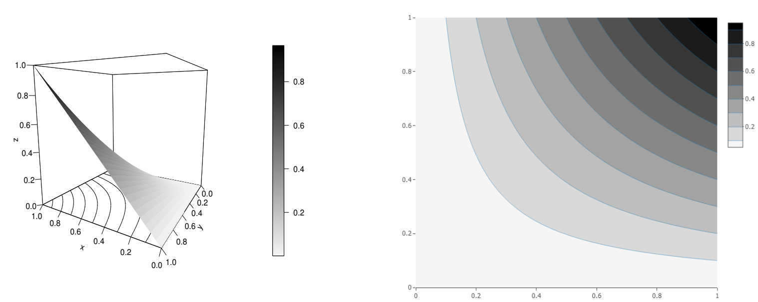
(a) (b)

FIGURE 5. Display of the (a) perspective plot and (b) contour plot of the E copula for $a = 1/2$, $b = 1$ and $c = 2$, belonging to Configuration set 1



(a) (b)

FIGURE 6. Display of the (a) perspective plot and (b) contour plot of the E copula for $a = 1/2$, $b = -1$ and $c = \sqrt{2}$, belonging to Configuration set 2



(a) (b)

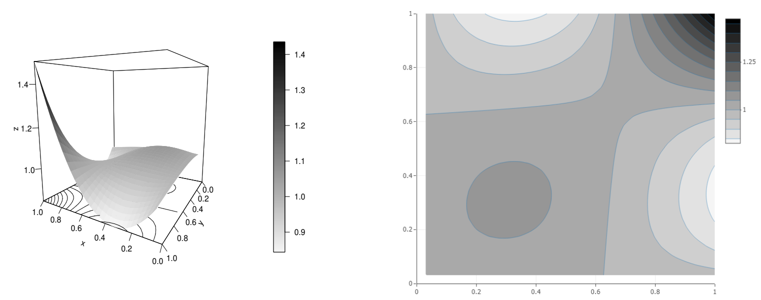
FIGURE 7. Display of the (a) perspective plot and (b) contour plot of the E copula for $a = -1$, $b = -1/2$ and $c = 2$, belonging to Configuration set 3

Different form morphologies for the E copula are seen in these figures. They are undoubtedly impacted by a, b and c.

3.2. **Related functions.** To begin, based on Equation (4), the E copula density is given by

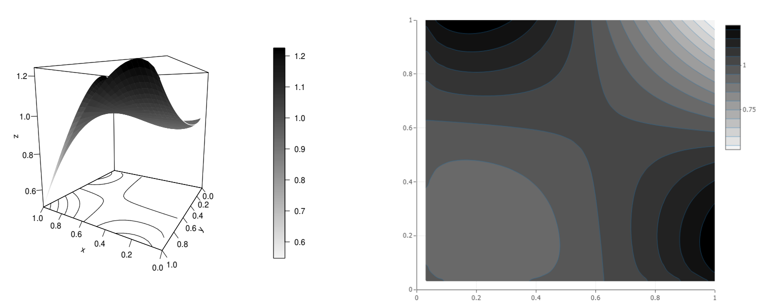
$$c(x,y;\alpha)=\partial_{x,y}C(x,y;\alpha)=ax^{c-1}y^{c-1}e^{b(1-x)(1-y)}\times$$
$$\left\{bc\left[x(2y-1)-y\right]+bxy\left[1+b(1-x)(1-y)\right]+c^2\right\}+1-ac^2x^{c-1}y^{c-1},$$
$$(x,y)\in[0,1]^2.$$

We can examine the modeling potential of the E copula as well as the effects of the parameters a, b, and c on its shapes by looking at the forms of this function. Figures 8, 9 and 10 display the E copula density plots for arbitrary parameters from Configuration sets 1, 2, and 3, respectively.



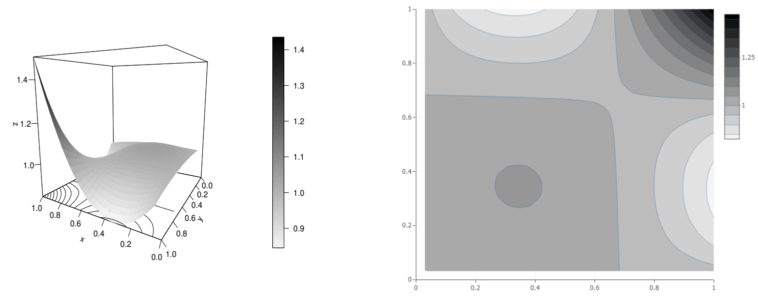
(a) (b)

FIGURE 8. Display of the (a) perspective plot and (b) contour plot of the E copula density for $a=1/2$, $b=1$ and $c=2$, belonging to Configuration set 1



(a) (b)

FIGURE 9. Display of the (a) perspective plot and (b) contour plot of the E copula density for $a=1/2$, $b=-1$ and $c=\sqrt{2}$, belonging to Configuration set 2



(a) (b)

Figure 10. Display of the (a) perspective plot and (b) contour plot of the E copula density for $a = -1$, $b = -1/2$ and $c = 2$, belonging to Configuration set 3

The entirely diverse shapes of the E copula density in these figures exhibit a type of dependence flexibility. The effects of a, b, and c on them are significant.

The E survival copula is defined by

$$\begin{aligned}\hat{C}(x, y; \alpha) &= x + y - 1 + C(1 - x, 1 - y; \alpha)\\ &= xy + a(1 - x)^c(1 - y)^c\left(e^{bxy} - 1\right), \quad (x, y) \in [0, 1]^2.\end{aligned}$$

The effect of c, which differs from that of c in the E copula, is the primary distinction between the two. Furthermore, the E survival copula is a novel three-parameter copula to be included in the body of current literature.

3.3. **Properties.** In order to better understand its modeling capacities, several essential characteristics of the E copula are developed in this part.

To begin, the E copula is diagonally symmetric because $C(x, y; \alpha) = C(y, x; \alpha)$ for any $(x, y) \in [0, 1]^2$. It is not Archimedean because, for example, when $a = 1$, $b = 1$, and $c = 1$, it corresponds to a special case of the CC copula, which is not (the associative property is not satisfied, see [8]).

For $a \neq 0$ and $b \neq 0$, the E copula is not radially symmetric because of the moving of the parameter c in the two expressions; there exists (x, y) such that $\hat{C}(x, y; \alpha) \neq C(x, y; \alpha)$. For $a = 0$ or $b = 0$, it is radially symmetric.

The Fréchet-Hoeffding bounds are satisfied, as for any other two-dimensional copula: For any $(x, y) \in [0, 1]^2$, we have $\max(x + y - 1, 0) \leq C(x, y; \alpha) \leq \min(x, y)$.

Remark 3.2. *Immediate mathematical consequences of the Fréchet-Hoeffding bounds are the following two-dimensional inequalities: for any* $(x, y) \in [0, 1]^2$, *we have*

$$\max(x+y-1,0)-xy \leq ax^c y^c \left(e^{b(1-x)(1-y)}-1\right) \leq \min(x,y)-xy,$$

where a, b, and c, satisfy either Configuration sets 1, 2, or 3.

The following quadrant dependence properties hold:

- For $a \geq 0$ and $b \geq 0$ (which is compatible with Configuration set 1, with some restrictions), or $a \leq 0$ and $b \in [-1,0]$ (which is compatible with Configuration set 3, with some restrictions), because $ax^c y^c \left(e^{b(1-x)(1-y)}-1\right) \geq 0$, the E copula is positively quadrant dependent, i.e., $C(x,y;\alpha) \geq xy$ for any $(x,y) \in [0,1]^2$.
- For $a \geq 0$ and $b \in [-1,0]$ (which is compatible with Configuration set 2, with some restrictions), because $ax^c y^c \left(e^{b(1-x)(1-y)}-1\right) \leq 0$, the E copula is negatively quadrant dependent, i.e., $C(x,y;\alpha) \leq xy$ for any $(x,y) \in [0,1]^2$.

In addition, interesting first-order copula orders are satisfied. Thanks to the following exponential inequality: $e^x - 1 \geq x$ for $x \in \mathbb{R}$, the results below are obtained.

- For $a \geq 0$, we have $C(x,y;\alpha) \geq C_\star(x,y;\alpha)$ for any $(x,y) \in [0,1]^2$, where $C_\star(x,y;\alpha)$ is defined by Equation (2).
- For $a \leq 0$, we have $C(x,y;\alpha) \leq C_\star(x,y;\alpha)$ for any $(x,y) \in [0,1]^2$.

The exponential series expansion gives

$$C(x,y;\alpha) = xy + ax^c y^c \sum_{i=1}^{\infty} \frac{1}{i!} b^i (1-x)^i (1-y)^i. \tag{6}$$

This expansion is helpful in specific circumstances since it may represent or approximate a variety of important correlation measures.

The tail dependence of the E copula is investigated below. Using standard limit and equivalence techniques, since $c \geq 1$, we have

$$\lambda_{low} = \lim_{x\to 0} \frac{C(x,x;\alpha)}{x} = \lim_{x\to 0} \left[x + ax^{2c-1}\left(e^{b(1-x)^2}-1\right)\right]$$
$$= \lim_{x\to 0} \left[x + ax^{2c-1}(e^b-1)\right] = 0.$$

There is no lower tail dependence in the E copula as a result. Regarding the upper tail dependence, we have

$$\lambda_{up} = \lim_{x \to 1} \frac{1 - 2x + C(x, x; \alpha)}{1 - x} = \lim_{x \to 1} \frac{1 - 2x + x^2 + ax^{2c}\left(e^{b(1-x)^2} - 1\right)}{1 - x}$$
$$= \lim_{x \to 1} [1 - x + ab(1 - x)] = 0.$$

Thus, the E copula has no upper tail dependence.

The medial correlation of the E copula is expressed as

$$M_{cor} = 4C\left(\frac{1}{2}, \frac{1}{2}; \alpha\right) - 1 = a2^{2(1-c)}\left(e^{b/4} - 1\right).$$

The rho of Spearman of the E copula is defined by

$$\rho = 12 \int_0^1 \int_0^1 [C(x, y; \alpha) - xy] \, dxdy$$
$$= 12a \int_0^1 \int_0^1 x^c y^c \left(e^{b(1-x)(1-y)} - 1\right) dxdy.$$

There is no a simple expression of this measure, but the following expansion is derived from Equation (6):

$$\rho = 12a \sum_{i=1}^{\infty} \frac{1}{i!} b^i B(c + 1, i + 1)^2.$$

Tables 3, 4, and 5 show numerical values of ρ for arbitrary parameters that belong to Configuration sets 1, 2, and 3, respectively.

TABLE 3. Some values of ρ of the E copula for $b = 1/2$ and $c = 2$, and varying $a \in [0, 1]$, belonging to Configuration set 1

a	0.00	0.12	0.24	0.36	0.48	0.60	0.72	0.84	0.96
ρ	0	0.0052	0.0104	0.0156	0.0208	0.026	0.0313	0.0365	0.0417

TABLE 4. Some values of ρ of the E copula for $b = -1$ and $c = \sqrt{2}$, and varying $a \in [0, 1/2]$, belonging to Configuration set 2

a	0.00	0.06	0.12	0.18	0.24	0.30	0.36	0.42	0.485
ρ	0	-0.0096	-0.0192	-0.0288	-0.0384	-0.0481	-0.0577	-0.0673	-0.0769

Table 5. Some values of ρ of the E copula for $b = -1/2$ and $c = 2$, and varying $a \in [-1, 0]$, belonging to Configuration set 3

a	-1.00	-0.88	-0.76	-0.64	-0.52	-0.40	-0.28	-0.16	-0.04
ρ	0.0401	0.0353	0.0305	0.0256	0.0208	0.016	0.0112	0.0064	0.0016

The values in these tables show that ρ can have a small amplitude and be either negative or positive. As a result, the E copula is ideal for modeling various weak correlations between two random variables. However, this conclusion is formulated in the light of the considered values of the parameters; since the CC copula is a special case of the E copula and has a greater amplitude for ρ, similar properties can be reached for the E copula.

Like all other two-dimensional copulas, the E copula has the capacity to define novel parametric distributional models. In fact, we construct a new two-dimensional cumulative distribution function by combining two uni-dimensional cumulative distribution functions, say $F(x)$ and $G(x)$, as follows:

$$\begin{aligned} H(x, y; \xi) &= C(F(x), G(y); \alpha) \\ &= F(x)G(y) + aF(x)^c G(y)^c \left\{ e^{b[1-F(x))(1-G(y)]} - 1 \right\}, \\ &\quad (x, y) \in \mathbb{R}^2, \end{aligned}$$

where ξ represents the vector of the involved parameters, including a, b and c, and those appearing in $F(x)$ and $G(x)$.

Based on this function, various novel two-dimensional distributions could be produced. As an example derived from the example made in the framework of the L copula, let us consider two exponential distributions with parameters γ and τ, respectively, then $F(x) = 1 - e^{-\gamma x}$ and $G(x) = 1 - e^{-\tau x}$ for $x \geq 0$, and $F(x) = G(x) = 0$ for $x \leq 0$. Then the related distribution based on the E copula is defined by the following cumulative distribution function:

$$\begin{aligned} H(x, y; \xi) =&(1 - e^{-\gamma x})(1 - e^{-\tau y}) + a \left(1 - e^{-\gamma x}\right)^c \left(1 - e^{-\tau y}\right)^c \times \\ &\left\{ e^{be^{-(\gamma x + \tau y)}} - 1 \right\}, \quad (x, y) \in [0, \infty)^2, \end{aligned}$$

and $H(x, y; \xi) = 0$ for $(x, y) \notin [0, \infty)^2$, where $\xi = (a, b, c, \gamma, \tau)$. We believe this two-dimensional exponential distribution to be novel in the literature.

4. Conclusion

In this study, we introduced and examined two copulas of the following forms:

$$C(x,y;\alpha) = xy + a\log\left[1 + bx^c y^c(1-x)(1-y)\right], \quad (x,y) \in [0,1]^2,$$

and

$$C(x,y;\alpha) = xy + ax^c y^c\left(e^{b(1-x)(1-y)} - 1\right), \quad (x,y) \in [0,1]^2,$$

where $\alpha = (a,b,c)$. The logarithmic-type copula is totally new, whereas the exponential-type copula can be viewed as a three-parameter generalization of the Celebioglu-Cuadras copula. The range of admissible values for a, b, and c, were determined. In addition, we emphasized their main features. Among other results, it was demonstrated that they possess interesting shapes, are diagonally symmetric, have manageable series expansions, satisfy precise first-order copula orders, have a versatile quadrant dependence, have no tail dependence, are not Archimedean, are typically not radially symmetric, and can model a weak or moderate dependence with the rho of Spearman as a benchmark. Although the contributions are mostly theoretical, the suggested copulas provide a foundation for cutting-edge dependence models that may be useful in a number of domains. Finally, our results indicate some two-dimensional inequalities that may be of independent interest.

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