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D-PSEUDO SUPPLEMENTED ALMOST DISTRIBUTIVE LATTICES

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Abstract. In this paper, we introduce D-pseudo supplementation and it's dual on (almost distributive lattices) ADLs (with dense elements) which is a generalization of pseudo supplemented ADLs, we obtain some algebraic properties and characterize both using the set $B_D(A) = \{a_1 \in L \mid$ there exists $a_2 \in L$ such that $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is desne$\}$, where A is an ADL with dense elements. Finally, we prove a one to one correspondence between D-pseudo supplemented ADLs (dual D-pseudo supplemented ADLs) and the class of principal ideals (sectionally intervals) in an ADL.

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1. Introduction and Preliminaries

Frink [3] introduced the theory of pseudo-complements for semi lattices in 1962, and then Venkatanarasimhan extended some of the results of [3] to posets in 1972. Later on several generalizations of distributive lattices were came into the picture. In this context, Swamy and Rao introduced a generalization of both lattices and as well as distributive lattices and named as an almost distributive lattice (ADL) [9] in 1981. An ADL is a common abstraction of most existing ring theoretic and lattice theoretic generalization of a distributive lattice which satisfies almost all conditions of a distributive lattice except right distributivity of $\vee$ over $\wedge$ and commutative with respect $\vee$ & $\wedge$. A pseudo complementation [11] $*$ on an ADL A is a unary operation which satisfies

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$$x \wedge y = 0 \Leftrightarrow x^* \wedge y = y.$$
$$(x \vee y)^* = x^* \wedge y^*,$$

for all $x, y \in A$. The authors proved that $\{x^* | x \in A\}$ forms a Boolean algebra. Later, the authors [5] generalized the pseudo complementation on an ADL A (with maximal elements), using Birkhoff center [10] B (That is, given $x \in A$, there exists $b \in B$ such that $x \wedge b = b \Leftrightarrow x^* \wedge c = c$, where $c \in B$ and $x \wedge c = c$) and then they derived a good number of algebraic properties on them. The existence of pseudo complemented almost distributive lattices and pseudo supplemented ADLs always depends upon maximal elements. This leads us to concentrate a class of ADLs (without maximal elements) and introduce a similar concept of pseudo complementation (pseudo supplementation). It is well known that, in an ADL, every maximal element is dense but not the converse. The (dense) class of elements in ADLs helps to introduce D-pseudo supplementation in which is a generalization of pseudo complementation (pseudo supplementation).

In the following, let us recall some of definitions and results of almost distributive lattice, which used in the sequel.

Definition 1.1. [9] An algebra $(A, \wedge, \vee, 0)$ of type (2,2,0) which satisfies the following identities, is called as an *Almost Distributive Lattice* (abbreviated as ADL);

(i) $0 \wedge a_1 = 0$

(ii) $a_1 \vee 0 = a_1$

(iii) $a_1 \wedge (a_2 \vee a_3) = (a_1 \wedge a_2) \vee (a_1 \wedge a_3)$

(iv) $(a_1 \vee a_2) \wedge a_3 = (a_1 \wedge a_3) \vee (a_2 \wedge a_3)$

(v) $a_1 \vee (a_2 \wedge a_3) = (a_1 \vee a_2) \wedge (a_1 \vee a_3)$

(vi) $(a_1 \vee a_2) \wedge a_2 = a_2$

for all $a_1, a_2, a_3 \in A$.

Example 1.2. [9] Let X be a non-empty set. Fix $a_0 \in X$. For any $a_1, a_2 \in X$, define,

$$a_1 \wedge a_2 = \begin{cases} a_0, & \text{if } a_1 = a_0 \\ a_2, & \text{if } a_1 \neq a_0 \end{cases} \quad \text{and} \quad a_1 \vee a_2 = \begin{cases} a_2, & \text{if } a_1 = a_0 \\ a_1, & \text{if } a_1 \neq a_0. \end{cases}$$

Then $(X, \wedge, \vee, a_0)$ is an ADL in which a_0 is the zero element. This ADL (called as discrete) is neither lattice nor distributive.

Given $a_1, a_2 \in A$, we say that a_1 is less than or equal to a_2 ($a_1 \leq a_2$), provided $a_1 \wedge a_2 = a_1$, or equivalently, $a_1 \vee a_2 = a_2$. Then $\leq$ forms a partial ordering on A.

Theorem 1.3. [9] *For any* $a_1, a_2 \in A$,

(i) $a_1 \wedge 0 = 0$ *and* $0 \vee a_1 = a_1$

(ii) $a_1 \wedge a_1 = a_1 \vee a_1 = a_1$

(iii) $a_1 \vee (a_2 \vee a_1) = a_1 \vee a_2 = (a_1 \vee a_2) \vee a_1$

(iv) $a_1 \wedge a_2 = 0 \Leftrightarrow a_2 \wedge a_1 = 0$

(v) $a_1 \wedge a_2 \leq a_2$ *and* $a_1 \leq a_1 \vee a_2$

(vi) $a_1 \vee a_2 = a_2 \vee a_1 \Leftrightarrow a_1 \wedge a_2 = a_2 \wedge a_1$

(vii) $(a_1 \wedge a_2) \vee a_2 = a_2$, $a_1 \vee (a_1 \wedge a_2) = a_1$ *and* $a_1 \wedge (a_1 \vee a_2) = a_1$

(viii) $a_1 \wedge a_2 = a_1 \Leftrightarrow a_1 \vee a_2 = a_2$

(ix) $a_1 \wedge a_2 = a_2 \Leftrightarrow a_1 \vee a_2 = a_1$

(x) $a_1 \wedge a_2 = a_2 \wedge a_1$, *whenever* $a_1 \leq a_2$.

Theorem 1.4. [9] *For any* $a_1, a_2, a_3 \in A$,

(i) $(a_1 \vee a_2) \wedge a_3 = (a_2 \vee a_1) \wedge a_3$

(ii) $\wedge$ *is associative in* A

(iii) $a_1 \wedge a_2 \wedge a_3 = a_2 \wedge a_1 \wedge a_3$.

If for each $a_1, a_2 \in I$ and $x \in A, a_1 \vee a_2$, and $a_1 \wedge x \in I$, then the non-empty subset I of A is said to be an ideal of A. Notably, for any $a_1 \in A,\ (a_1] = \{a_1 \wedge x \mid x \in A\}$ is the principal ideal generated by a_1. The set $\mathcal{I}(A)$ ideals of A forms a bounded distributive lattice, where $I \cap J$ is the infimum and $I \vee J = \{i \vee j \mid i \in I \text{ and } j \in J\}$ is the supremum of I and J in $\mathcal{I}(A)$. The set $\mathcal{PI}(A)$ principal ideals of A form a sublattice of $\mathcal{I}(A)$, where $(a_1] \wedge (a_2] = (a_1 \wedge a_2]$ and $(a_1] \vee (a_2] = (a_1 \vee a_2]$, for any $a_1, a_2 \in A$. A non-empty subset F of A is called a filter of A, if for any $a_1, a_2 \in F$ and $x \in A,\ a_1 \wedge a_2,\ x \vee a_1 \in F$.

Given a non-empty subset S of A, the set $S^* = \{a_1 \in A \mid a_1 \wedge x = 0, \text{ for all } x \in S\}$ is an ideal of A. Especially, for any $a_1 \in A,\ \{a_1\}^* = (a_1)^*$, where $(a_1) = (a_1]$ is the principal ideal generated by a_1.

Lemma 1.5. [6] *For any* $a_1, a_2 \in A$,

(i) $a_1 \leq a_2$ *implies* $(a_2)^* \subseteq (a_1)^*$

(ii) $(a_1 \vee a_2)^* = (a_1)^* \cap (a_2)^*$

(iii) $(a_1 \wedge a_2)^{**} = (a_1)^{**} \cap (a_2)^{**}$.

An element $d \in A$ is called dense [7], if $(d)^* = \{0\}$. The set D denotes the set of dense elements in A. An element $m \in A$ is called maximal [9], if for any $a_1 \in A$, $m \leq a_1$ implies $m = a_1$. It is easy to observe that every maximal element is dense but not the converse.

Theorem 1.6. [9] *For any* $m \in A$, *the following are equivalent;*

(i) m *is maximal*

(ii) $m \wedge x = x$, *for all* $x \in A$

(iii) $m \vee x = m$, *for all* $x \in A$.

Definition 1.7. [9] A non-empty subset S of A is called a subADL of A, if it is closed under $\wedge$, $\vee$ and S contains zero element.

Definition 1.8. [5] Let A be an ADL with maximal element m and Birkhoff center B(A). A is said to be pseudo-supplemented ADL if for each $x \in A$, there exists $a_1 \in B(A)$ such that $x \wedge a_1 = a_1$ and if $a_2 \in B(A)$ and $x \wedge a_2 = a_2$, then $a_1 \wedge a_2 = a_2$.

Theorem 1.9. [4] *Let A be an ADL with dense elements and D is the set of dense elements in A. Then the set $B_D(A) = \{a_1 \in L \mid$ there exists $a_2 \in L$ such that $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is dense$\}$ a weakly relatively complemented sub-ADL of A.*

2. D-Pseudo Supplemented ADLs

In this section, we introduce a D-pseudo-supplemented almost distributive lattice which is a generalization of a pseudo-supplemented almost distributive lattice using the set $B_D(A) = \{a_1 \in L \mid$ there exists $a_2 \in L$ such that $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is maximal$\}$, where A is an almost distributive lattice with dense elements. We prove several algebraic properties on D-pseudo supplemented almost distributive lattices. We obtain a necessary and sufficient conditions for an almost distributive lattice with dense elements to become a D-pseudo- supplemented almost distributive lattice.

Definition 2.1. A is said to be a D-pseudo supplemented ADL, if for any $x \in A$, there exists $a_1 \in B_D(A)$ such that

$S_1 : x \wedge a_1 = a_1,$

$S_2 : If\ a_2 \in B_D(A)$ *and* $x \wedge a_2 = a_2,$ *then* $a_1 \wedge a_2 = a_2.$

Lemma 2.2. *Given $x \in A$, there exist $a_1, a_2 \in B_D(A)$ which satisfies S_1 and S_2 in the Definition 2.1, then $a_1 \wedge a = a_2 \wedge a$, for all $a \in A$.*

Proof. Let $x \in A$ and $a_1, a_2 \in B_D(A)$ (satisfy both S_1 and S_2). Then $x \wedge a_1 = a_1$ and $x \wedge a_2 = a_2$. By S_2, $a_1 \wedge a_2 = a_2$ and $a_2 \wedge a_1 = a_1$. For any $a \in A,\ a_1 \wedge a = a_2 \wedge a_1 \wedge a = a_1 \wedge a_2 \wedge a = a_2 \wedge a$. ☐

In the Lemma 2.2., the existence of elements in $B_D(A)$ may not be unique. But if we fix an element $d \in D$, then it is unique (in $B_D(L)$) and in the form $a_1 \wedge d$. It is denoted by $x^d (= a_1 \wedge d)$ and called as the D-pseudo supplement of x,

Lemma 2.3. *Let A be a D-pseudo supplemented ADL and $x \in A$. Then we have*

(i) $x \wedge x^d = x^d$

(ii) $a_1 \wedge x^d = x^d$ *and* $x^d \wedge a_1 = d \wedge a_1$

(iii) *If $a_2 \in B_D(A)$ and $x \wedge a_2 = a_2$, then $x^d \wedge a_2 = d \wedge a_2$ and $a_2 \wedge x^d = a_2 \wedge d$, for all $d \in D$.*

Proof. Let $x \in A$. Then there exists $a_1 \in B_D(A)$ satisfying $x \wedge a_1 = a_1$ and if $a_2 \in B_D$ such that $x \wedge a_2 = a_2$, then $a_1 \wedge a_2 = a_2$. Therefore $x^d = a_1 \wedge d$, for any $d \in D$. Now,
(i) $x \wedge x^d = x \wedge (a_1 \wedge d) = (x \wedge a_1) \wedge d = a_1 \wedge d = x^d$.
(ii) $a_1 \wedge x^d = a_1 \wedge (a_1 \wedge d) = a_1 \wedge d = x^d$ and $x^d \wedge a_1 = (a_1 \wedge d) \wedge a_1 = (d \wedge a_1) \wedge a_1 = d \wedge a_1$.
(iii) If $a_2 \in B_D(A)$ and $x \wedge a_2 = a_2$, then $a_1 \wedge a_2 = a_2$. Therefore $x^d \wedge a_2 = a_1 \wedge d \wedge a_2 = d \wedge a_1 \wedge a_2 = d \wedge a_2$ and $a_2 \wedge x^d = a_2 \wedge (a_1 \wedge d) = a_1 \wedge a_2 \wedge d = a_2 \wedge d$. □

Lemma 2.4. *Let A be a D-pseudo supplemented ADL and $x, y \in A$. Then, we have*

(i) $x \vee x^d = x$
(ii) $0^d = 0$
(iii) $x \wedge d = d \Leftrightarrow x^d = d$*, for all* $d \in D$
(iv) $x \wedge m = m \Rightarrow x^d = d$*, for any maximal element* m *in* A
(v) $m^d = d$*, for any maximal element* m *in* A
(vi) $x \in B_D(A) \Rightarrow x^d = x \wedge d$
(vii) $x^{dd} = x^d$
(viii) $y \wedge x = x \Rightarrow y^d \wedge x^d = x^d$.

Proof. Let $x \in A$. Then there exists $a_1 \in B_D(A)$ such that $x \wedge a_1 = a_1$ and if $x \wedge a_2 = a_2$, then $a_1 \wedge a_2 = a_2$, for some $a_2 \in A$.
(i) By the above lemma, $x \wedge x^d = x^d$ for all $x \in A$, Therefore $x \vee x^d = x$.
(ii) From(i), $0 \vee 0^d = 0$. Therefore $0^d = 0$.
(iii) Suppose $x \wedge d = d$. Then $a_1 \wedge d = d$ (by Definition 2.1.). Therefore $x^d = d$. Conversely suppose that $x^d = d$. Then $x \wedge d = x \wedge x^d = x^d = d$.
(iv) Suppose $x \wedge m = m$. Then $a_1 \wedge m = m$. Now, $x^d = a_1 \wedge d = (a_1 \vee m) \wedge d = (a_1 \wedge d) \vee (m \wedge d) = (a_1 \wedge d) \vee d = d$. Therefore $x^d = d$.
(v) Let m be a maximal element in A. Then there exists $a_m \in B_D(A)$ such that $m^d = a_m \wedge d$. Since $m \wedge x = x$ for all $x \in A$, $m \wedge d = d$. By Definition 2.1., $a_m \wedge d = d$. Hence $m^d = a_m \wedge d = d$.
(vi) We know that $x \wedge x^d = x^d$ and $x \wedge x = x$, for all $x \in A$. If $x \in B_D(A)$. Then $a_1 \wedge x = x$ (by Definition 2.1.). Now, $x^d = a_1 \wedge d = x \wedge a_1 \wedge d = a_1 \wedge x \wedge d = x \wedge d$. Hence $x^d = a_1 \wedge d = x \wedge d$.
(vii) We know that $x^d \in B_D(A)$ for all $x \in A$. Therefore $x^{dd} = (x^d)^d = x^d \wedge d = a_1 \wedge d \wedge d = a_1 \wedge d = x^d$ (by (vi)).
(viii) If $y \wedge x = x$, then $y \wedge x^d = y \wedge x \wedge x^d = x \wedge x^d = x^d$. Since $x^d \in B_D(A)$ and $y \wedge x^d = x^d$, $y^d \wedge x^d = d \wedge x^d = d \wedge a_1 \wedge d = a_1 \wedge d = x^d$(by Lemma 2.3(iii)). □

Theorem 2.5. *Given elements x, y in a D-pseudo supplemented ADL A, we have*

(i) $x^d \wedge y^d = y^d \wedge x^d$
(ii) $x^d \vee y^d = y^d \vee x^d$

(iii) $(x \wedge y)^d = x^d \wedge y^d$

(iv) $(x \wedge y)^d = (y \wedge x)^d$

(v) $x^d \vee y^d \leq (x \vee y)^d$

(vi) $(x \vee y)^d = (y \vee x)^d$

(vii) $(x \wedge m)^d = x^d$.

Proof. Let $x, y \in A$. Then there exist $a_1, a' \in B_D(A)$ such that $x^d = a_1 \wedge d$ and $y^d = a' \wedge d$.
(i) $x^d \wedge y^d = a_1 \wedge d \wedge a' \wedge d = a' \wedge d \wedge a_1 \wedge d = y^d \wedge x^d$.
(ii) $x^d \vee y^d = (a_1 \wedge d) \vee (a' \wedge d) = (a_1 \vee a') \wedge d = (a' \vee a) \wedge d = (a' \wedge d) \vee (a_1 \wedge d) = y^d \vee x^d$.
(iii) Let $x, y \in A$. Then $(x \wedge y) \wedge (x^d \wedge y^d) = (x \wedge x^d) \wedge (y \wedge y^d) = x^d \wedge y^d$ and $x^d \wedge y^d \in B_D(A)$. By Lemma 2.3.(iii)., $x^d \wedge y^d \wedge (x \wedge y)^d = x^d \wedge y^d \wedge d = x^d \wedge a' \wedge d \wedge d = x^d \wedge a' \wedge d = x^d \wedge y^d$, where $y^d = a' \wedge d$. Therefore $x^d \wedge y^d \leq (x \wedge y)^d$. We know that $x \wedge (x \wedge y) = x \wedge y$. By Lemma 2.4.(viii), $x^d \wedge (x \wedge y)^d = (x \wedge y)^d$. Therefore $(x \wedge y)^d \wedge x^d = (x \wedge y)^d$. Hence $(x \wedge y)^d \leq x^d$. Similarly we can prove $(x \wedge y)^d \leq y^d$. Now, $(x \wedge y)^d \wedge x^d \wedge y^d = (x \wedge y)^d \wedge y^d = (x \wedge y)^d$. Therefore $(x \wedge y)^d \leq x^d \wedge y^d$. Hence $(x \wedge y)^d = x^d \wedge y^d$.
(iv) From (i) and (iii), we can observe this property.
(v) Let $x, y \in A$. Then we have $x^d, y^d \in B_D(A)$. Now, $(x^d \vee y^d) \wedge (x \vee y)^d = [x^d \wedge (x \vee y)^d] \vee [y^d \wedge (x \vee y)^d] = [x \wedge (x \vee y)]^d \vee [y \wedge (x \vee y)]^d = x^d \vee [(x \vee y) \wedge y]^d = x^d \vee y^d$. Therefore $x^d \vee y^d \leq (x \vee y)^d$.
(vi) $(y \vee x)^d \wedge (x \vee y)^d = (y \vee x)^d \wedge (y \vee x)^d = (y \vee x)^d$. Then $(y \vee x)^d \leq (x \vee y)^d$. Similarly we can prove $(x \vee y)^d \leq (y \vee x)^d$. Hence $(x \vee y)^d = (y \vee x)^d$.
(vii) $(x \wedge m)^d = (m \wedge x)^d = x^d$. □

Theorem 2.6. *Let A be a D-pseudo supplemented ADL. Then for any dense element d in A, $[0, d]$ is a D-pseudo supplemented lattice.*

Proof. Suppose that A is D-pseudo supplemented ADL and d be a dense element in A. Let $x \in [0, d]$. Then there exists $a_1 \in B_D(A)$ such that $S_1 : x \wedge a_1 = a_1$ and $S_2 :\ If\ a_2 \in B_D(A)$ such that $x \wedge a_2 = a_2$, then $a_1 \wedge a_2 = a_2$. Define $x^d = a_1 \wedge d$. For $a_1 \in B_D(A)$, there exists $c \in A$ such that $a_1 \wedge c = 0$ and $a_1 \vee c$ is dense. Then $c \wedge d \wedge x^d = (c \wedge d) \wedge (a_1 \wedge d) = (a_1 \wedge c) \wedge d = 0$ and $(c \wedge d) \vee x^d = (c \wedge d) \vee (a_1 \wedge d) = (c \vee a_1) \wedge d = (a_1 \vee c) \wedge d$ is dense in $[0, d]$. Therefore $x^d \in B_D([0, d])$. Now, $x \wedge x^d = x \wedge a_1 \wedge d = a_1 \wedge d = x^d$. Let $y \in B_D([0, d])$ such that $x \wedge y = y$. We have $y \in B_D([0, d]) \subseteq B_D(A)$ and have $x \wedge y = y$. Now, $x^d \wedge y = a_1 \wedge d \wedge y = d \wedge a_1 \wedge y = d \wedge y = y$. Therefore $[0, d]$ is D-pseudo supplemented lattice. □

The converse of above Theorem 2.6. is discussed as follows.

Theorem 2.7. *Let A be an ADL in which every dense element is maximal and d is dense. If $[0, d]$ is D-pseudo supplemented lattice, then A is a D-pseudo supplemented ADL.*

Proof. Suppose that $[0, d]$ is a D-pseudo supplemented lattice in which every D-pseudo supplement of x is denoted by x^d. Let $x \in A$. Then $x \wedge d \in [0, d]$. Take $a_1 = (x \wedge d)^d$. Then $a_1 \in B_D([0, d]) \subseteq B_D(A)$. Now, $x \wedge a_1 = x \wedge d \wedge a_1 = x \wedge d \wedge a_1 \wedge a_1 = (x \wedge d \wedge a_1) \wedge (x \wedge d)^d = a_1 \wedge (x \wedge d) \wedge (x \wedge d)^d = a_1 \wedge (x \wedge d)^d = a_1$ (since $a_1 \in [0, d]$). Let $a_2 \in B_D(A)$ such that $x \wedge a_2 = a_2$. Then $a_2 \wedge d \in B_D([0, d])$ and $x \wedge a_2 \wedge d = a_2 \wedge d$. Therefore $(x \wedge d) \wedge (a_2 \wedge d) = a_2 \wedge d$. Since $a_2 \wedge d \in B_D([0, d])$, $(a_2 \wedge d) \wedge (x \wedge d)^d = (a_2 \wedge d) \wedge d = a_2 \wedge d$ (by Lemma 2.3(iii)). So that $a_2 \wedge d \leq a_1$. Hence $a_1 \wedge a_2 \wedge d = a_2 \wedge d$. Now, $a_1 \wedge a_2 = a_1 \wedge a_2 \wedge a_2 = a_1 \wedge a_2 \wedge d \wedge a_2 = a_2 \wedge d \wedge a_2 = d \wedge a_2 = a_2$ (Since every dense element is maximal). Therefore A is D-pseudo supplemented ADL. □

Remark 2.8. If $a_1 \in B_D([0, m])$, then $a_1 = a_2 \wedge m$, for some $a_2 \in B_D([0, m])$. Since $B_D([0, m]) \subseteq B_D(A)$, $a_1 \in B_D(A)$. By Lemma 2.4(vi), $a_1^m = a_1 \wedge m = a_2 \wedge m \wedge m = a_2 \wedge m = a_1$. Hence $B_D([0, m]) = \{x^m \mid x \in A\}$.

Theorem 2.9. *Let A be an ADL with dense elements and $B_D(A)$ is the dense center. Then A is a D-pseudo supplemented ADL if and only if $PI(A)$ is a D-pseudo supplemented lattice.*

Proof. Suppose that A is D-pseudo supplemented ADL. Then for any $x \in A$, there exists $x^d \in A$ such that $x^d = a_1 \wedge d$ (by Lemma 2.2.).

For this $x \in A$, we can define $(x]^d = (x^d]$. Since $x^d \in B_D(A)$, there exists $y \in A$ such that $x^d \wedge y = 0$ and $x^d \vee y$ is dense. Therefore $(x^d] \wedge (y] = (x^d \wedge y] = (0]$ and $(x^d] \vee (y] = (x^d \vee y]$ is an ideal generated by a dense element $x^d \vee y$. So that $(x^d] \in B_D(PI(A))$. Now, $(x] \wedge (x^d] = (x \wedge x^d] = (x^d]$ and hence $(x^d] = (x]^d \subseteq (x]$. Let $(a_1] \in B_D(PI(A))$ such that $(a_1] \subseteq (x]$. For this $(a_1] \in B_D(PI(A))$, there exists $(a_2] \in PI(A)$ such that $(a_1] \wedge (a_2] = (a_1 \wedge a_2] = (0]$ and $(a_1] \vee (a_2] = (a_1 \vee a_2]$ is a principal ideal generated by a dense element $a_1 \vee a_2$. Therefore $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is dense and hence $a_1 \in B_D(A)$ and $x \wedge a_1 = a_1$. So that $x^d \wedge a_1 = a_1$. Now $(x^d \wedge a_1] = (x^d] \wedge (a_1] = (x]^d \wedge (a_1] = (a_1]$. Hence $(a_1] \subseteq (x]^d$. Thus $PI(A)$ is a D-pseudo supplemented lattice. Conversely suppose that $PI(A)$ is D-pseudo supplemented lattice. For any $x \in A$, $(x]^d \in B(PI(A))$. Take $(x]^d = (a_1]$, for some $a_1 \in A$ and also $a_1 \in B_D(A)$. Then $(a_1] \subseteq (x]$. So that $a_1 \in (x]$ and hence $x \wedge a_1 = a_1$. Let $a_2 \in B_D$ such that $x \wedge a_2 = a_2$. Then $(a_2] \in B_D(PI(A))$ and $(x] \wedge (a_2] = (a_2]$. Therefore $(a_2] \subseteq (x]$. So that $(a_2] \subseteq (x]^d$. That is $(a_2] \subseteq (a_1]$. Hence $a_1 \wedge a_2 = a_2$. Thus A is D-pseudo supplemented ADL. □

3. Dual D-Pseudo Supplemented ADLs

In this section we discuss the dual of a D-pseudo supplementation and then D-pseud-supplemented ADLs. We prove a good number of algebraic properties and necessary conditions on them

Definition 3.1. Let A be an ADL with dense elements and $B_D(A)$ is the dense center. A is said to be a dual D-pseudo supplemented ADL, if for each $x \in A$, there exists an element $a_1 \in B_D(A)$ such that

$PS_1: \ a_1 \wedge x = x,$

$PS_2: \ If \ a_2 \in B_D(A)$ such that $a_2 \wedge x = x$, then $a_2 \wedge a_1 = a_1$.

Lemma 3.2. *Let $x \in A$ and $a_1, a_2 \in B_D(A)$ satisfying PS_1 and PS_2 in the Definition 3.1. Then $a_1 \wedge d = a_2 \wedge d$, for all $d \in D$.*

Lemma 3.3. *For any $x \in A$, there exists a unique $a_1 \in B_D(A)$ such that ${}^d x = a_1 \wedge d \in B_D(A)$, for all $d \in D$.*

Proof. By the Lemma 3.2., for any $x \in A$, there exists a unique $a_1 \in B_D(A)$ such that ${}^d x = a_1 \wedge d$, for all $d \in D$. For this $a_1 \in B_D(A)$, there exists $a_2 \in A$ such that $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is dense. Now $a_2 \wedge {}^d x = a_2 \wedge a_1 \wedge d = 0$ and $a_2 \vee {}^d x = a_2 \vee (a_1 \wedge d) = (a_2 \vee a_1) \wedge (a_2 \vee d)$ is dense in A. Therefore ${}^d x \in B_D(A)$. □

Lemma 3.4. *For any $x \in A$,*

(i) ${}^d x \wedge x = x$

(ii) *If $a_2 \in B_D(A)$ such that $a_2 \wedge x = x$, then $a_2 \wedge {}^d x = {}^d x$*

(iii) *${}^d x \wedge m = {}^d x$, for all maximal elements $m \in A$.*

Theorem 3.5. *For any $x, y \in A$ and $a_1 \in B_D(A)$,*

(i) ${}^d x \vee x = {}^d x$

(ii) *${}^d m = m$, for any maximal element $m \in A$*

(iii) ${}^d 0 = 0$

(iv) *$a_1 \in B_D(A)$, then ${}^d a_1 = a_1 \wedge d$*

(v) ${}^d x = {}^{dd} x$

(vi) *If $y \wedge x = x$, then ${}^d y \wedge {}^d x = {}^d x$. Hence ${}^d x \leq {}^d y$.*

Proof. Let $x, y \in A$ and $a_1 \in B_D(A)$.

(i) By the Lemma 3.2, ${}^d x \wedge x = x$, therefore ${}^d x \vee x = {}^d x$.

(ii) Let m be a maximal element in A. Then ${}^d m = a_1 \wedge d$, for any $d \in D$. By the Definition 3.1., ${}^d m = a_1 \wedge m$ and $a_1 \wedge m = m$(since m is dense). Therefore ${}^d m = m$.

(iii) Since $0 \in B_D(A)$ and $0 \wedge 0 = 0$. By Lemma 4.2, we get $0 \wedge {}^d 0 = {}^d 0$. Therefore ${}^d 0 = 0$.

(iv) For any $a_1 \in A$, we have that ${}^d a_1 = x \wedge d$, where $d \in D$ and $x \wedge a_1 = a_1$. Since $a_1 \wedge a_1 = a_1, a_1 \wedge x = x$. Therefore ${}^d a_1 = x \wedge d = a_1 \wedge x \wedge d = x \wedge a_1 \wedge d = a_1 \wedge d$.

(v) Since ${}^d x \in B_D(A)$, by (iv) ${}^{dd} x = {}^d x \wedge m = {}^d x$.

(vi) Suppose that $y \wedge x = x$. Then ${}^d y \wedge y \wedge x = y \wedge x = x$. Then ${}^d y \wedge x = x$. Since ${}^d y \in B_D(A)$, ${}^d y \wedge {}^d x = {}^d x$. □

Theorem 3.6. *For any $x, y \in A$,*

(i) ${}^d (x \vee y) = {}^d x \vee {}^d y$

(ii) ${}^{d}(x \vee y) = {}^{d}(y \vee x)$

(iii) ${}^{d}x \wedge {}^{d}y \leq {}^{d}(x \wedge y)$

(iv) ${}^{d}(x \wedge y) = {}^{d}(y \wedge x)$

(v) ${}^{d}(x \wedge m) = {}^{d}x$.

Proof. (i) Let $x, y \in A$. Then ${}^{d}x, {}^{d}y \in B_D(A)$. Since $B_D(A)$ is a subADL, ${}^{d}x \vee {}^{d}y \in B_D(A)$. Now, $({}^{d}x \vee {}^{d}y) \wedge (x \vee y) = [({}^{d}x \vee {}^{d}y) \wedge x] \vee [({}^{d}x \vee {}^{d}y) \wedge y] = [({}^{d}x \wedge x) \vee ({}^{d}y \wedge x)] \vee [({}^{d}x \wedge y) \vee ({}^{d}y \wedge y)] = [x \vee ({}^{d}y \wedge x)] \vee [({}^{d}x \wedge y) \vee y] = x \vee y$. Therefore $({}^{d}x \vee {}^{d}y) \wedge {}^{d}(x \vee y) = {}^{d}(x \vee y)$. Since $x = (x \vee y) \wedge x$ and $y = (x \vee y) \wedge y$, by the Theorem 3.5 (vi), ${}^{d}x \leq {}^{d}(x \vee y)$ and ${}^{d}y \leq {}^{d}(x \vee y)$. Therefore ${}^{d}x \vee {}^{d}y \leq {}^{d}(x \vee y)$. Hence ${}^{d}x \vee {}^{d}y = {}^{d}(x \vee y)$.

(ii) Let ${}^{d}(x \vee y) = {}^{d}x \vee {}^{d}y = (a_1 \wedge d) \vee (a_2 \wedge d) = (a_1 \vee a_2) \wedge d = (a_2 \vee a_1) \wedge d = (a_2 \wedge d) \vee (a_1 \wedge d) = {}^{d}y \vee {}^{d}x = {}^{d}(y \vee x)$, for any $d \in D$.

(iii) Since $x \wedge (x \wedge y) = x \wedge y$. By the Theorem 3.5(vi), ${}^{d}(x \wedge y) \leq {}^{d}x$. Similarly we can write $y \wedge (x \wedge y) = x \wedge y$. Therefore ${}^{d}(x \wedge y) \leq {}^{d}y$. Hence ${}^{d}(x \wedge y) \leq {}^{d}x \wedge {}^{d}y$.

(iv) We can write $(x \wedge y) \wedge (y \wedge x) = y \wedge x$. Then we get $(y \wedge x)^d \leq (x \wedge y)^d$. Now, $(y \wedge x) \wedge (x \wedge y) = x \wedge y$. Hence by the Theorem 3.1(vi), we get ${}^{d}(x \wedge y) \leq {}^{d}(y \wedge x)$. Then ${}^{d}(x \wedge y) = {}^{d}(y \wedge x)$.

(v) For any $x \in A,\ (x \wedge m)^d = (m \wedge x)^d = x^d(\text{By(iv)})$. □

Theorem 3.7. *If A is a dual D-pseudo supplemented ADL, then $[0, d]$ is dual D-pseudo supplemented lattice.*

Proof. Suppose that A is a dual D-pseudo supplemented ADL with dense element d. Let $x \in [0, d]$. Then there exists $a_1 \in B_D(A)$ such that $a_1 \wedge x = x$ and if $a_2 \in B_D(A)$ such that $a_2 \wedge x = a_2$, then $a_2 \wedge a_1 = a_1$. Define ${}^{d}x = a_1 \wedge d$. Then ${}^{d}x \in B_D([0, d])$. For any $a_1 \in B_D(A)$, there exists $c \in A$ such that $a_1 \wedge c = 0$ and $a_1 \vee c$ is dense. Then $(c \wedge d) \wedge {}^{d}x = (c \wedge d) \wedge (a_1 \wedge d) = (c \wedge a_1) \wedge d = 0$ and $(c \wedge d) \vee {}^{d}x = (c \wedge d) \vee (a_1 \wedge d) = (c \vee a_1) \wedge d$ is also dense. Since $c \wedge d,\ a_1 \wedge d \in [0, d],\ {}^{d}x \in B_D([0, d])$. Since ${}^{d}x \wedge x = x,\ x \leq {}^{d}x$. Take $a_1 \in B_D([0, d])$ such that $a_1 \wedge x = x$. Since $a_1 \in [0, d],\ a_1 = a_2 \wedge d$, for some $a_2 \in B_D(A)$. Therefore $a_2 \wedge x = a_2 \wedge d \wedge x = a_1 \wedge x = x$ (since $x \in [0, d]$). We get $a_2 \wedge {}^{d}x = {}^{d}x$. Therefore $a_1 \wedge {}^{d}x = a_2 \wedge d \wedge {}^{d}x = a_2 \wedge {}^{d}x = {}^{d}x$(since ${}^{d}x \in [0, d]$). Hence $a \leq {}^{d}x$. Thus $[0, d]$ is a dual D-pseudo supplemented lattice. □

Corollary 3.8. *Let A be an ADL in which every dense element d is maximal. If $[0, d]$ is dual D-pseudo supplemented lattice, then A be a dual D-pseudo supplemented ADL.*

Proof. For any $x \in [0, d]$, dual D-pseudo supplement of x is denoted by ${}^{d}x$. For any $y \in A$. Define $y^* = {}^{d}(y \wedge d)$. We have $y \wedge d < y^*$. Therefore $y \wedge d = y^* \wedge y \wedge d$. Since every dense element is maximal, $y = y^* \wedge y$. Let $a_1 \in B_D(A)$ such that $a_1 \wedge y = y$. Then $a_1 \wedge d \in B_D([0, d])$. Therefore $a_1 \wedge d \wedge y \wedge d = a_1 \wedge y \wedge d = y \wedge d$. So that $a_1 \wedge d \geq y \wedge d$. Hence $a_1 \wedge d > y^*$. Now $a_1 \wedge y^* = a_1 \wedge d \wedge y^* = y^*$(since every dense is maximal). Hence A is dual pseudo supplemented ADL. The converse of the above theorem is true when ever every dense element is maximal. □

Theorem 3.9. *A is a dual D-pseudo supplemented ADL if and only if $PI(A)$ is a dual D-pseudo supplemented lattice.*

Proof. Suppose that A is dual D-pseudo supplemented ADL. Let $x \in A$. Define $^d(x] = (^dx]$. Since $^dx \in B_D(A)$ there exists $y \in A$ such that $^dx \wedge y = 0$ and $^dx \vee y$ is dense. Therefore $(^dx] \wedge (y] = (^dx \wedge y] = (0]$ and $(^dx \vee y] = (^dx] \vee (y]$ is a principal ideal generated by a dense element $^dx \vee y$. So that $(^dx] \in B_D PI(A)$. Now, $(^dx] \wedge (x] = (^dx \wedge x] = (x]$. Therefore $(x] \leq (^dx]$. Let $(a_1] \in B_D PI(A)$ such that $(x] \subseteq (a_1]$. For $(a_1] \in B_D PI(A)$, there exists $(a_2] \in PI(A)$ such that $(a_1] \wedge (a_2] = (0]$ and $(a_1] \vee (a_2]$ is a principal ideal generated by a dense element $a_1 \vee a_2$. Therefore $a_1 \wedge a_2 = 0$ and $a_1 \vee a_2$ is dense. So that $a_1 \in B_D$ and $a_1 \wedge x = x$. So that $a_1 \wedge {}^dx = {}^dx$. Now, $(a_1 \wedge {}^dx] = (a_1] \wedge (^dx] = (^dx]$. Hence $(^dx] \subseteq (a_1]$ has $PI(A)$ is a dual D-pseudo supplemented lattice.
Conversely suppose that $PI(A)$ is dual D-pseudo supplemented lattice. For any $x \in A$, $^d(x] \in B_D PI(A)$. Take $^d(x] = (a_1]$, for some $a_1 \in A$ and $a_1 \in B_D(A)$. Then $(x] \subseteq^d (x] = (a_1]$. So that $x \in (a_1]$ and hence $a_1 \wedge x = x$. Let $a_2 \in B_D(A)$ such that $a_2 \wedge x = x$. Then $(a_2] \in B_D PI(A)$ and $(a_2] \wedge (x] = (x]$. Therefore $(x] \subseteq (a_2]$. So that $(^dx] =^d (x] = (a_1] \subseteq (a_2]$. We get $a_2 \wedge a_1 = a_1$. Hence A is dual D-pseudo supplemented ADL. □

4. Conclusions and Future Work

We define D-pseudo supplementation (D-pseudo supplemented ADLs) which is a generalization of pseudo supplementation (pseudo supplemented ADLs) and characterized in terms of weakly relatively complemented ADLs and dense center.

The study of class of D-pseudo complementation as a key role in the class of ADLs in which there is no maximal element. We wish to work on this D-pseudo complementation on stone ADLs.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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