Asia Pac. J. Math. 2024 **11**:101

SOME NEW RESULTS OF FIXED POINTS ON FUZZY RECTANGULAR b-METRIC LIKE SPACES

AMENAH KAREEM YOUSIF[1,2,*], MOHAMMED JASSIM MOHAMMED[2]

[1]Department of Mathematics, College of Sciences, University of Basrah, Basrah, Iraq

[2]Department of Mathematics, College of Education for Pure Sciences, University of Thi-Qar, Thi-Qar, Iraq

[*]Corresponding author: amenahkareem9@gmail.com

Received Sep. 03, 2024

Abstract. In the present study, we prove a few fixed point results for contractive maps in the fuzzy rectangular b-metric like spaces which introduced recently. Additionally, we provide an example to show the theory, some findings in the literature are expanded upon and generalized by our studies.

2020 Mathematics Subject Classification. 47H10.

Keywords and phrases: fixed point results; rectangular b-metric like mappings; fuzzy rectangular b-metric like mappings; b-metric like mappings.

1. Introduction

Due to its numerous scientific applications, fixed point theory has been utilized extensively. The iterative technique for finding the fixed point is provided via the Fixed Point Theorem of Banach [3] that deals with self-mappings across all of metric space. The Banach Fixed Point Theorem has been demonstrated by a lot of writers who have expanded the Banach contraction in various ways (see; [4,6,7,11,23,24] and [12]). As an alternative, Zadeh [25] introduced the fuzzy set theory. By giving each set element a membership grade, it expands on standard set theory. K. and M. in [16] presented the idea of fuzzy metric space in 1975. In 1994 the concept of fuzzy metric space was refined by G. and V. [9] through the use of continuous triangular norms.

In order to generalize a metric space, Branciari [5], in 2000, devised a rectanglular metric space. Furthermore, the authors in [10] presented the idea of b-rectangular metric space. In this type of metric, they illustrated the contraction theorems of the Banach and Kannan types. Ding et al. [8] investigated, improved, and generalized several fixed point approaches for mappings in b-rectangular metric spaces, also they discussed these results in the space of rectangular metric. Using the Pata-type

DOI: 10.28924/APJM/11-101

contraction, Kadelburg et al. [15] were able to achieve fixed point conclusions in metric spaces including b-metric and b-rectangular. The basic idea of b-metric was developed by Nadaban [19] using a fuzzy set approach in the fuzzy mathematical sense. The researchers in [18] enhanced the notion of metric-like space by introducing the idea of rectangular metric-like space. M. et al. established the idea of fuzzy rectangular b-metric spaces in [17] and demonstrated the Banach contraction principle in this setting.

To expand on the notion of a metric-like and b-metric space, Alghamdi [1] proposed the concept of b-metric-like space. Additional fixed point conclusions for different kinds of contractions on a self mapping in b-metric-like space have been developed by Prakasam et al. [20]. Shukla and G. [22] invented the fuzzy metric like spaces technique. Fuzzy b-metric like spaces were introduced by Javed et al. [14] and many fixed point results were shown. Several fixed point findings in recently discovered spaces called fuzzy rectangle metric-like spaces was established in 2022 by U. Ishtiaq et al. in [13]. They additionally discovered the space of fuzzy b-rectangle metric-like spaces.

We demonstrate certain fixed point results on fuzzy rectangular b-metric-like spaces in this study, which were first introduced by U. Ishtiaq [13], we also provide an example to show the main theory. some findings in the literature are expanded upon and generalized by our studies.

2. Prelemnaries

The definitions and outcomes linked to our major findings are given in the section that follows.

Definition 2.1. [5] *Given $\psi \neq \emptyset$, the mapping $\kappa : \psi \times \psi \to [0, \infty)$ is referred to be a rectangular metric if it meets the subsequent criteria:*

(1) $\kappa(f, \omega) = 0$ *if and only if* $f = \omega$;

(2) $\kappa(f, \omega) = \kappa(\omega, f)$ *for all* $f, \omega \in \psi$;

(3) $\kappa(f, \omega) \leq \kappa(f, e) + \kappa(e, v) + \kappa(v, \omega)$ *for all* $f, e, v, \omega \in \psi$.

The pair (ψ, κ) is called rectangular metric space.

Definition 2.2. [10] *Let $\psi \neq \emptyset$, The rectangular b-metric (for simply, R.b.m) is the mapping $\mathfrak{A} : \psi \times \psi \to [0, \infty)$, fulfills the requirements for any $s \geq 1$:*

(1) $\mathfrak{A}(f, \omega) = 0$ *if and only if* $f = \omega, \forall f, \omega \in \psi$;

(2) $\mathfrak{A}(f, \omega) = \mathfrak{A}(\omega, f)$ *for all* $f, \omega \in \psi$;

(3) $\mathfrak{A}(f, \omega) \leq s[\mathfrak{A}(f, \mathfrak{r}) + \mathfrak{A}(\mathfrak{r}, v) + \mathfrak{A}(v, \omega)], \forall f, \mathfrak{r}, v, \omega \in \psi$.

Then the pairs $(\psi, \mathfrak{A})$ is represent R.b.m space.

Definition 2.3. [2] *Whenever $\psi \neq \emptyset$ and $\mathfrak{Q} : \psi \times \psi \to [0, \infty)$ is a function that ensures the following criteria apply for every $f, \omega, v \in \psi$, the pair $(\psi, \mathfrak{Q})$ represents a metric like-space,*

(1) $\mathfrak{Q}(f, \omega) = \mathfrak{Q}(\omega, f)$;

(2) $\mathfrak{Q}(f,\omega) \geq 0$, *and* $\mathfrak{Q}(f,\omega) = 0$ *then* $f = \omega$;

(3) $\mathfrak{Q}(f,\nu) \leq \mathfrak{Q}(f,\omega) + \mathfrak{Q}(\omega,\nu)$.

Definition 2.4. [1] *The pair* $(\psi,\aleph)$ *represents a b-metric like-space (or simply, b-m.l-s) on any set* $\psi \neq \emptyset$ *and a function* $\aleph : \psi \times \psi \to [0,\infty)$ *such that it satisfy the following assumptions for any* $f,\omega,v \in \psi$ *and* $\rho \geq 1$:

(1) *If* $\aleph(f,\omega) = 0$, *then* $f = \omega$;

(2) $\aleph(f,\omega) = \aleph(\omega,f)$;

(3) $\aleph(f,\omega) \leq \rho[\aleph(f,\nu) + \psi(\nu,\omega)]$.

Example 2.5. [1] *Let* $\psi = [0,\infty)$. *Define* $\aleph : \psi \times \psi \to [0,+\infty)$ *by* $\aleph(\epsilon,f) = (\epsilon + f)^2$. *Then* $(\psi,\aleph)$ *is a b-m.l-s with* $\rho = 2$.

Definition 2.6. [18] *Suppose* $\varphi : \psi \times \psi \to [0,\infty)$ *is a function and* ψ *is a nonempty set. When* φ *meets the specified criteria, it's referred to be a rectangular metric like-space, given each* $f,\omega,\kappa,v \in \psi$,

(1) *If* $\varphi(f,\omega) = 0$, *then* $f = \omega$;

(2) $\varphi(f,\omega) = \varphi(\omega,f)$;

(3) $\varphi(f,\omega) \leq \varphi(f,\kappa) + \varphi(\kappa, v) + \varphi(v,\omega)$.

The pair (ψ,φ) *is called a R.m.l-space.*

Definition 2.7. [25] *Consider* I *to be the real line's closed interval* $[0,1]$ *and let* ψ *be a non-empty set. The membership function* $\mu : \psi \longrightarrow I$, *which associates each point* $f \in \psi$ *with its grade or degree of membership* $\mu(f) \in I$, *characterizes a fuzzy set (F. set, for simply)* μ *in* ψ.

Definition 2.8. [21] *If a binary operation* $\circledast : I^2 \to I$ *satisfies the following axioms, then* $\circledast$ *is considered a continuous triangular norm (C.T-n. for simply),*

(1) $\zeta \circledast o = o \circledast \zeta$, *for all* $\zeta, o \in I$,

(2) $(\zeta \circledast o) \circledast \iota = \zeta \circledast (o \circledast \iota)$, *for all* $\zeta, o, \iota \in I$,

(3) $\zeta \circledast 1 = \zeta, \forall \zeta \in I$,

(4) $\zeta \circledast o \leq \iota \circledast \kappa$ *whenever* $\zeta \leq \iota$ *and* $o \leq \kappa$, *for all* $\zeta, o, \iota, \kappa \in I$,

(5) $\circledast$ *is continuous.*

Example 2.9. [21] $\zeta \circledast o = \zeta \cdot o$ *and* $\zeta \circledast o = \min\{\zeta, o\}$ *are examples of C.T-ns.*

Definition 2.10. [9] *If* $\psi \neq \emptyset, \circledast$ *be a C.T-n. and the mapping* $\mathfrak{P} : \psi^2 \times [0,\infty) \to I$ *is a F. set meeting the conditions required listed below,* $\forall f,\omega,v \in \psi, s, \Gamma > 0$, *then a fuzzy metric space is a triple* $(\psi,\mathfrak{P},\circledast)$,

(1) $\mathfrak{P}(f,\omega,\Gamma) > 0$,

(2) $\mathfrak{P}(f,\omega,\Gamma) = 1$ *if and only if* $f = \omega$,

(3) $\mathfrak{P}(f,\omega,\Gamma) = \mathfrak{P}(\omega,f,\Gamma)$,

(4) $\mathfrak{P}(f,\nu,\Gamma+s) \geq \mathfrak{P}(f,\omega,\Gamma) \circledast \mathfrak{P}(\omega,\nu,\ s)$,

(5) $\mathfrak{P}(f,\omega,\ .) : (0,\infty) \to I$ *is continuous mapping.*

Definition 2.11. [17] *If $\mathfrak{D}$ be a F. set on $\psi \times \psi \times [0,+\infty)$ meeting the following requirements, $\forall \varrho, e, g, \nu \in \psi, \Gamma, s, w > 0$ and $\circledast$ is a C.T-n., then a triplet $(\psi\ ,\mathfrak{D},\circledast)$ is referred to as a fuzzy rectangle metric space:*

(1) $\mathfrak{D}\ (\varrho, e, 0) = 0$;

(2) $\mathfrak{D}\ (\varrho, e, \Gamma) = 1 \iff \varrho = e$;

(3) $\mathfrak{D}\ (\varrho, e, \Gamma) = \mathfrak{D}\ (e, \varrho, \Gamma)$;

(4) $\mathfrak{D}(\varrho,\ g, \Gamma + s + w) \geq \mathfrak{D}\ (\varrho, e, \Gamma) \circledast \mathfrak{D}\ (e, \nu, s) \circledast \mathfrak{D}\ (\nu, g, w)\ \forall\ e, \nu \in \psi \backslash \{\varrho, g\}$;

(5) $\mathfrak{D}\ (\varrho, e, .) : (0, +\infty) \to I$ *is left continuous,*

(6) $\lim_{\Gamma \to +\infty} \mathfrak{D}\ \ (\varrho, e, \Gamma) = 1$.

Definition 2.12. [17] *A triplet $(\psi, \mathfrak{V}, \circledast)$ is called fuzzy R.b-m space if $\psi \neq \emptyset, \mathfrak{r} \geq 1$, $\circledast$ is a C.T-n., $\mathfrak{V}$ is a F. set on $\psi \times \psi \times [0,+\infty)$ meeting the following requirements, $\forall \sigma, e, g, \nu \in \psi$ and $\Gamma, s, w > 0$,*

(1) $\mathfrak{V}\ (\sigma, e, 0) = 0$;

(2) $\mathfrak{V}\ (\sigma, e, \Gamma) = 1$ *if and only if* $\sigma = e$;

(3) $\mathfrak{V}\ (\sigma, g,\ \mathfrak{r}(\Gamma + s + w)) \geq \mathfrak{V}\ (\sigma, e, \Gamma) \circledast \mathfrak{V}\ (e, \nu, s) \circledast \mathfrak{V}(\nu, g, w)$ *for all* $e, \nu \in \psi \backslash \{\sigma, g\}$;

(4) $\mathfrak{V}\ (\sigma, e, .) : (0, +\infty) \to I$ *is left continuous;*

(5) $\lim_{\Gamma \to +\infty} \mathfrak{V}\ (\sigma, e, \Gamma) = 1$

Definition 2.13. [13] *If $\psi\ \neq \emptyset, \circledast$ be a C.T-n. and $\mathfrak{R}$ is a F. set on $\psi \times \psi \times [0,+\infty)$ meeting the conditions required listed below, then a triple $(\psi\ ,\mathfrak{R},\circledast)$ is regarded as a fuzzy R.m.l-space, $\forall \zeta, \varpi, g \in \psi$ and $\Gamma, s, w > 0$,*

(1) $\mathfrak{R}(\zeta, \varpi,\ 0) = 0$;

(2) $\mathfrak{R}(\zeta, \varpi, \Gamma) = 1$ *implies* $\zeta = \varpi$;

(3) $\mathfrak{R}(\zeta, \varpi, \Gamma) = \mathfrak{R}(\varpi, \zeta, \Gamma)$;

(4) $\mathfrak{R}(\zeta, g, \Gamma + s + w) \geq \mathfrak{R}(\zeta, \varpi, \Gamma) \circledast \mathfrak{R}(\varpi, \nu, s) \circledast \mathfrak{R}(\nu, g, w)$ *for all* $\varpi, \nu \in \psi \backslash \{\zeta, g\}$;

(5) $\mathfrak{R}\ (\zeta, \varpi, .) : (0, +\infty) \to I$ *is left continuous;*

(6) $\lim_{\Gamma \to +\infty} \mathfrak{R}(\zeta, \varpi, \Gamma) = 1$.

Definition 2.14. [13] *If $\psi\ \neq \emptyset,\ \lambda \geq 1, \circledast$ be a C.T-n., and Ç is a F. set on $\psi\ \times \psi\ \times [0,+\infty)$ meeting the conditions required listed below, then a triplet $(\psi, Ç, \circledast)$ is considered as fuzzy rectangular b-metric like-space (or, to put it simply, F.R.b-m.l-s), $\forall \zeta, \varpi, g, \nu \in \psi$ and $\Gamma, s, w > 0$,*

(1) $Ç\ (\zeta, \varpi, 0) = 0$;

(2) $Ç\ (\zeta, \varpi, \Gamma) = 1$ *implies* $\zeta = \varpi$;

(3) $Ç\ (\zeta, \varpi, \Gamma) =\ Ç\ (\varpi, \zeta, \Gamma)$;

(4) $Ç(\zeta, g, \lambda(\Gamma + s + w))\ \geq\ Ç\ (\zeta, \varpi, \Gamma) \circledast Ç\ (\varpi, \nu, s) \circledast Ç\ (\nu, g, w)$ *for all* $\varpi, \nu \in \psi \backslash \{\zeta, g\}$;

(5) $Ç\ (\zeta, \varpi, .) : (0, +\infty) \to I$ *is left continuous;*

(6) $\lim_{\Gamma\to+\infty} Ç(\zeta,\varpi,\Gamma)=1$.

Example 2.15. [13] *Suppose* (ψ,κ) *be a R.b-m.l-s, define* $Ç:\psi\times\psi\times[0,+\infty)\to I$ *by*

$$Ç(\zeta,\varpi,\Gamma)=\frac{\Gamma}{\Gamma+\kappa(\zeta,\varpi)},$$

for all $\zeta,\varpi\in\psi$ *and* $\Gamma>0$, *with C.T-n.* $\circledast$. *Therefore,* $(\psi,Ç,\circledast)$ *is a F.R.b-m.l-s.*

Definition 2.16. [13] *Let* $(\psi,Ç,\circledast)$ *be a F.R.b-m.l-s and assume* $\{a_e\}$ *is a sequence in* ψ *such that* $e=1,2,\ldots$, *then*

(1) *The sequence* $\{a_e\}$ *can be considered convergent if* $\zeta\in\psi$, *such that*

$$\lim_{e\to\infty} Ç(\zeta_e,\zeta,\Gamma)=Ç(\zeta,\zeta,\Gamma),\ for\ all\ \Gamma>0$$

(2) *If for every* $\Gamma>0, q=1,2,\ldots,\lim_{e\to\infty}Ç(\zeta_e,\zeta_{e+q},\Gamma)$ *exists and is finite, then* $\{a_e\}$ *is considered a Cauchy sequence.*

(3) $(\psi,Ç,\circledast)$ *is complete F.R.b-m.l-s if all of the Cauchy sequence converges in* ψ, *such that*

$$\lim_{e\to\infty} Ç(\zeta_e,\zeta,\Gamma)=Ç(\zeta,\zeta,\Gamma)=\lim_{e\to\infty}Ç(\zeta_e,\zeta_{e+q},\Gamma)$$

for all $\Gamma>0$.

Lemma 2.17. [13] *Suppose that* $(\psi,Ç,\circledast)$ *be a complete F.R.b-m.l.s and*

$$Ç(f,\omega,\hslash\Gamma)\geq Ç(f,\omega,\Gamma),$$

for all $f,\omega\in\psi,\Gamma>0$ *and* $\hslash\in(0,1)$, *then* $f=\omega$.

Proof. $Ç(f,\omega,\Gamma)\geq Ç\left(f,\omega,\frac{\Gamma}{\hslash}\right)\geq Ç\left(f,\omega,\frac{\Gamma}{\hslash^2}\right)\geq\ldots\geq Ç\left(f,\omega,\frac{\Gamma}{\hslash^e}\right)$ If we take the limit as $e\to\infty$ and since $\hslash\in(0,1)$,we get $Ç(f,\omega,\Gamma)\geq 1\Rightarrow Ç(f,\omega,\Gamma)=1$.

Therefore, $f=\omega$. □

3. Main Result

Several fixed point results on F.R.b-m.l.s were presented in this part. To further illustrate the theory, we offer an example and our conclusions generalize and extend some discoveries in previous research.

Theorem 3.1. *Let* $(\psi,Ç,\circledast)$ *be a complete F.R.b-m.l.s with* $o\geq 1$ *and* $\ell:\psi\to\psi$ *be a map satisfying the conditions:*

$$Ç(f,\omega,\Gamma)=1$$

and

$$Ç(\ell f,\ell\omega,\hslash\Gamma)\geq\min\left\{\frac{Ç(\omega,\ell\omega,\Gamma)(1+Ç(f,\ell f,\Gamma))}{1+Ç(f,\omega,\Gamma)},Ç(f,\omega,\Gamma)\right\} \quad (1)$$

for all $f,\omega\in\psi,\Gamma>0$ *and* $\hslash\in(0,\frac{1}{o})$, *then* ℓ *has a unique fixed point.*

Proof. Let $f \in \psi$ and f_e is a sequence in ψ such that $\ell f_e = f_{e+1},\ \forall f \in \psi$. Now, we will show that f_e is Cauchy sequence. By using 1, we get

$$\begin{aligned}
\text{Ç}\left(\ell f_{e-1},\ \ell f_e,\ \text{ƙ}\Gamma\right) &= \text{Ç}\left(f_e, f_{e+1},\ \Gamma\right) \\
&\geq \min\left\{\frac{\text{Ç}\left(f_e,\ \ell f_e,\ \Gamma\right)\left(1+\text{Ç}\left(f_{e-1},\ \ell f_{e-1},\ \Gamma\right)\right)}{1+\text{Ç}\left(f_{e-1}, f_e, \Gamma\right)},\ \text{Ç}\left(f_{e-1}, f_e, \Gamma\right)\right\} \\
&\geq \min\left\{\frac{\text{Ç}\left(f_e, f_{e-1},\ \Gamma\right)\left(1+\text{Ç}\left(f_{e-1}, f_e,\ \Gamma\right)\right)}{1+\text{Ç}\left(f_{e-1}, f_e, \Gamma\right)},\ \text{Ç}(f_{e-1}, f_e, \Gamma)\right\} \\
&\geq \min\left\{\text{Ç}\left(f_e, f_{e+1},\ \Gamma\right),\ \text{Ç}(f_{e-1}, f_e, \Gamma)\right\}
\end{aligned}$$

If $\text{Ç}\left(f_e, f_{e+1},\ \Gamma\right) \leq\ \text{Ç}(f_{e-1}, f_e, \Gamma)$, then

$$\text{Ç}\left(f_e, f_{e+1},\ \text{ƙ}\Gamma\right) \geq \text{Ç}\left(f_e, f_{e+1},\ \Gamma\right)$$

By lemma 2.17, we get $f_e = f_{e+1} = \ell f_e$ and so f_e is fixed point of ℓ. If $\text{Ç}\left(f_{e-1}, f_e,\ \Gamma\right) \leq\ \text{Ç}(f_e, f_{e+1}, \Gamma)$, then

$$\begin{aligned}
\text{Ç}\left(f_e, f_{e+1},\ \text{ƙ}\Gamma\right) &\geq \text{Ç}\left(f_{e-1}, f_e,\ \Gamma\right) \\
&\geq \text{Ç}\left(f_{e-2}, f_{e-1},\ \frac{\Gamma}{\text{ƙ}}\right) \\
&\geq \text{Ç}\left(f_{e-3}, f_{e-2},\ \frac{\Gamma}{\text{ƙ}^2}\right) \\
&\geq \ldots \geq \text{Ç}\left(f_0, f_1,\ \frac{\Gamma}{\text{ƙ}^e}\right) \qquad (2)
\end{aligned}$$

Now, for all $s \in Z^+$, we have

If s is odd, then $s = 2p+1$ where $p = 1, 2, \ldots,$

$$\begin{aligned}
&\text{Ç}\left(f_e, f_{(e+2p+1)}, \Gamma\right) \geq \text{Ç}\left(f_e, f_{e+1}, \frac{\Gamma}{3o}\right) \circledast \text{Ç}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{3o}\right) \circledast \\
&\text{Ç}\left(f_{e+2}, f_{e+2p+1}, \frac{\Gamma}{3o}\right) \geq \text{Ç}\left(f_e, f_{e+1}, \frac{\Gamma}{3o}\right) \circledast \text{Ç}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{3o}\right) \circledast \\
&\text{Ç}\left(f_{e+2}, f_{e+3}, \frac{\Gamma}{(3o)^2}\right) \circledast \text{Ç}\left(f_{e+3}, f_{e+4}, \frac{\Gamma}{(3o)^2}\right) \circledast \text{Ç}\left(f_{e+4}, f_{e+2p+1}, \frac{\Gamma}{(3o)^2}\right) \geq \\
&\text{Ç}\left(f_e, f_{e+1}, \frac{\Gamma}{(3o)}\right) \circledast \text{Ç}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{(3o)}\right) \circledast \text{Ç}\left(f_{e+2}, f_{e+3}, \frac{\Gamma}{(3o)}\right) \circledast \\
&\text{Ç}\left(f_{e+3}, f_{e+4}, \frac{\Gamma}{(3o)^2}\right) \circledast \text{Ç}\left(f_{e+4}, f_{e+5}, \frac{\Gamma}{(3o)^2}\right) \circledast \ldots \circledast \text{Ç}\left(f_{e+2p}, f_{e+2p+1}, \frac{\Gamma}{(3o)^p}\right) \\
&\geq \text{Ç}\left(f_0, f_1, \frac{\Gamma}{3o\text{ƙ}^e}\right) \circledast \text{Ç}\left(f_0, f_1, \frac{\Gamma}{3o\text{ƙ}^{e+1}}\right) \circledast \text{Ç}\left(f_0, f_1, \frac{\Gamma}{(3o)^2\,\text{ƙ}^{e+2}}\right) \\
&\circledast \text{Ç}\left(f_0, f_1, \frac{\Gamma}{(3o)^2\,\text{ƙ}^{e+3}}\right) \circledast \text{Ç}\left(f_0, f_1, \frac{\Gamma}{(3o)^3\,\text{ƙ}^{e+4}}\right) \circledast \ldots \circledast \text{Ç}\left(f_0, f_1, \frac{\Gamma}{(3o)^p\,\text{ƙ}^{e+2p}}\right)
\end{aligned}$$

$$= \c{C}\left(f_0, f_1, \frac{\Gamma}{3o\hbar^e}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)\,\hbar^e}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^2\,\hbar^e}\right)$$

$$\circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^2\,\hbar^{e+1}}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^3\,\hbar^{e+1}}\right) \circledast \ldots \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^p\,\hbar^{e+p}}\right)$$

From $\lim_{\Gamma\to\infty} \c{C}(f, \omega, \Gamma) = 1$, we have

$$\lim_{e\to\infty} \c{C}\left(f_e, f_{e+2p+1},\ \Gamma\right) = 1$$

If s is even, $s = 2p$,

$$\c{C}\left(f_e, f_{e+2p}, \Gamma\right) \geq \c{C}\left(f_e, f_{e+1}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+2}, f_{e+2p}, \frac{\Gamma}{3o}\right)$$

$$\geq \c{C}\left(f_e, f_{e+1}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+2}, f_{e+3}, \frac{\Gamma}{(3o)^2}\right)$$

$$\circledast \c{C}\left(f_{e+3}, f_{e+4}, \frac{\Gamma}{(3o)^2}\right) \circledast \c{C}\left(f_{e+4}, f_{e+2p}, \frac{\Gamma}{(3o)^2}\right)$$

$$\geq \c{C}\left(f_e, f_{e+1}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+1}, f_{e+2}, \frac{\Gamma}{3o}\right) \circledast \c{C}\left(f_{e+2}, f_{e+3}, \frac{\Gamma}{(3o)^2}\right) \circledast$$

$$\c{C}\left(f_{e+3}, f_{e+4}, \frac{\Gamma}{(3o)^2}\right) \circledast \c{C}\left(f_{e+4}, f_{e+5}, \frac{\Gamma}{(3o)^3}\right) \circledast \ldots \circledast \c{C}\left(f_{e+2p-4}, f_{e+2p-3}, \frac{\Gamma}{(3o)^{p-1}}\right)$$

$$\circledast \c{C}\left(f_{e+2p-3}, f_{e+2p-2}, \frac{\Gamma}{(3o)^{p-1}}\right) \circledast \c{C}\left(f_{e+2p-2}, f_{e+2p}, \frac{\Gamma}{(3o)^{p-1}}\right)$$

$$\geq \c{C}\left(f_0, f_1, \frac{\Gamma}{3o\hbar^e}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{3o\hbar^{e+1}}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o)^2\,\hbar^{e+2}}\right)$$

$$\circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o)^2\,\hbar^{e+3}}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o)^3\,\hbar^{e+4}}\right) \circledast \ldots$$

$$\circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o)^{p-1}\,\hbar^{e+2p-2}}\right)$$

$$= \c{C}\left(f_0, f_1, \frac{\Gamma}{3o\hbar^e}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)\,\hbar^e}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^2\,\hbar^e}\right)$$

$$\circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^2\,\hbar^{e+1}}\right) \circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^3\,\hbar^{e+1}}\right) \circledast \ldots$$

$$\circledast \c{C}\left(f_0, f_1, \frac{\Gamma}{(3o\hbar)^{p-1}\,\hbar^{e+p-1}}\right)$$

From $\lim_{\Gamma\to\infty} \c{C}(f, \omega, \Gamma) = 1$, we have

$$\lim_{e\to\infty} \c{C}\left(f_e,\ f_{e+2p},\ \Gamma\right) = 1$$

Therefore, from the two cases, we have for all $s \in N$,

$$\lim_{e\to\infty} \c{C}\left(f_e,\ f_{e+s}, \Gamma\right) = 1$$

Then, the sequence $\{f_e\}$ is a Cauchy sequence. Since Ç is complete F.R.b-m.l.s, so we have an element $f \in Ç$ such that

$$\lim_{e\to\infty} Ç\left(f_e, f, \Gamma\right) = Ç\left(f, f, \Gamma\right) = \lim_{e\to\infty} Ç\left(f_e, f_{e+p}, \Gamma\right) = 1 \forall \Gamma > 0, p = 1, 2, \ldots.$$

Now, we will show that f is fixed point for ℓ,

$$\begin{aligned} Ç\left(\ell f, f, \Gamma\right) &\geq Ç\left(\ell f,\ f_{e+1}, \Gamma\right) \circledast Ç\left(f_{e+1},\ f_{e+2}, \Gamma\right) \circledast Ç\left(f_{e+2},\ f, \Gamma\right) \\ &\geq Ç\left(\ell f,\ \ell f_e, \Gamma\right) \circledast Ç\left(f_{e+1},\ f_{e+2}, \Gamma\right) \circledast Ç\left(f_{e+2},\ f, \Gamma\right) \\ &= Ç\left(\ell f_e\ , \ell f, \Gamma\right) \circledast Ç\left(f_{e+1},\ f_{e+2}, \Gamma\right) \circledast Ç\left(f_{e+2},\ f, \Gamma\right) \end{aligned} \tag{3}$$

$$\begin{aligned} Ç\left(\ell f_e, \ell f, \Gamma\right) &\geq \min\left\{\frac{Ç\left(f, \ell f, \frac{\Gamma}{k}\right)\left(1 + Ç\left(f_e, \ell f_e,\ \frac{\Gamma}{k}\right)\right)}{1 + Ç\left(f_e, f, \frac{\Gamma}{k}\right)}, Ç\left(f_e, f, \frac{\Gamma}{k}\right)\right\} \\ &= \min\left\{\frac{Ç\left(f, \ell f, \frac{\Gamma}{k}\right)\left(1 + Ç\left(f_e, f_{e+1},\ \frac{\Gamma}{k}\right)\right)}{1 + Ç\left(f_e, f, \frac{\Gamma}{k}\right)}, Ç\left(f_e, f, \frac{\Gamma}{k}\right)\right\} \end{aligned}$$

From (3) we get,

$$\begin{aligned} Ç\left(\ell f, f, \Gamma\right) \geq \min&\left\{\frac{Ç\left(f, \ell f,\ \frac{\Gamma}{k}\right)\left(1 + Ç\left(f_e, f_{e+1}, \frac{\Gamma}{k}\right)\right)}{1 + Ç\left(f_e, f, \frac{\Gamma}{k}\right)}, Ç\left(f_e, f, \frac{\Gamma}{k}\right)\right\} \\ &\circledast Ç\left(f_{e+1}, f_{e+2}, \Gamma\right) \circledast Ç\left(f_{e+2}, f, \Gamma\right) \end{aligned}$$

By taking the limit $e \to \infty$, we have

$$\begin{aligned} Ç\left(\ell f, f, \Gamma\right) &\geq \min\left\{\frac{Ç\left(f, \ell f, \frac{\Gamma}{k}\right)(1+1)}{1+1}, 1\right\} \circledast 1 \circledast 1 \\ &= \min\left\{Ç\left(f, \ell f, \frac{\Gamma}{k}\right), 1\right\} \\ Ç\left(\ell f, f, \Gamma\right) &\geq Ç\left(f, \ell f, \frac{\Gamma}{k}\right) \end{aligned}$$

By lemma 2.17, we have $\ell f = f$.

Assume that w is an additional fixed point of ℓ in order to demonstrate the uniqueness of f, that is, $\ell w = w$ and $f \neq w$.

$$\begin{aligned} Ç\left(w, f, \Gamma\right) = Ç\left(\ell w, \ell f, \Gamma\right) &\geq \min\left\{\frac{Ç\left(f,\ \ell f,\ \frac{\Gamma}{k}\right)\left(1 + Ç\left(w, \ell w, \frac{\Gamma}{k}\right)\right)}{1 + Ç\left(w, f, \frac{\Gamma}{k}\right)},\ Ç\left(w, f, \frac{\Gamma}{k}\right)\right\} \\ &= \min\left\{\frac{Ç\left(f, f, \frac{\Gamma}{k}\right)\left(1 + Ç\left(w, w, \frac{\Gamma}{k}\right)\right)}{1 + Ç\left(w, f, \frac{\Gamma}{k}\right)},\ Ç\left(w, f, \frac{\Gamma}{k}\right)\right\} \\ &= \min\left\{\frac{2}{1 + Ç\left(w, f, \frac{\Gamma}{k}\right)},\ Ç\left(w, f, \frac{\Gamma}{k}\right)\right\} = Ç\left(w, f, \frac{\Gamma}{k}\right) \end{aligned}$$

This is contraction, so $f = w$. Hence, f is a unique fixed point of ℓ. □

Corollary 3.2. *Let* $(\psi, \c{C}, \circledast)$ *be a complete F.R.b-m.l.s and* $g : \psi \to \psi$ *be a map satisfying the conditions*

$$\c{C}(f, \omega,\ \Gamma) = 1$$

and

$$\c{C}(gf,\ g\omega,\ \hbar\Gamma) \geq \min\left\{\frac{\c{C}(\omega,\ g\omega,\ \Gamma)(1+\c{C}(f,\ gf,\ \Gamma))}{1+\c{C}(f,\omega,\Gamma)},\ \c{C}(f,\omega,\Gamma)\right\}$$

for all $f,\ \omega \in \psi, 0 < \hbar < 1$*, then* g *has fixed point and it is unique.*

Proof. By Theorem 3.1, it is immediately apparent if we assume $o = 1$.

Example 3.3. *Suppose that* $\psi = \{0,\ 1,\ 2\}$*. A mapping* $\c{C} : \psi \times \psi \times [0, 1]$ *defined by*

$$\c{C}(f,\ \omega,\ \Gamma) = \frac{\Gamma}{\Gamma + \max\{f, \omega\}^2}, \forall f,\ \omega \in \psi$$

Then $\c{C}(f,\ \omega,\ \Gamma)$ *is a complete F.R.b-m.l.s. Define a map* $g : \psi \to \psi$ *by*

$$gf = \sqrt{\hbar}\, f, 0 < \hbar < 1.$$

Now, for all $\Gamma > 0$*, we get*

$$\c{C}(gf,\ g\omega, \hbar\, \Gamma) \geq \min\left\{\frac{\c{C}(\omega,\ g\omega,\ \Gamma)(1+\c{C}(f,\ gf,\ \Gamma))}{1+\c{C}(f,\omega,\Gamma)},\ \c{C}(f,\omega,\Gamma)\right\}$$

$$\c{C}\left(\sqrt{\hbar}\, f,\ \sqrt{\hbar}\,\omega, \hbar\,\Gamma\right) \geq \min\left\{\frac{\c{C}\left(\omega,\ \sqrt{\hbar}\,\omega,\ \Gamma\right)\left(1+\c{C}\left(f,\ \sqrt{\hbar}\, f,\ \Gamma\right)\right)}{1+\c{C}(f,\omega,\Gamma)},\ \c{C}(f,\omega,\Gamma)\right\}$$

$$\frac{\hbar\Gamma}{\hbar\Gamma + \max\sqrt{\hbar}f, \sqrt{\hbar}\omega^2} \geq \min\left\{\frac{\frac{\Gamma}{\Gamma+\max\{\omega,\sqrt{\hbar}\omega\}^2}\left(1+\frac{\Gamma}{\Gamma+\max\{f,\sqrt{\hbar}f\}^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}, \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}\right\}$$

$$\Rightarrow \frac{\hbar\Gamma}{\hbar\Gamma + \hbar.\max f, \omega^2} \geq \min\left\{\frac{\frac{\Gamma}{\Gamma+\omega^2\max\{1,\sqrt{\hbar}\}^2}\left(1+\frac{\Gamma}{\Gamma+f^2\max\{1,\sqrt{\hbar}\}^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}, \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}\right\}$$

$$\Rightarrow \frac{\Gamma}{\Gamma + \max f, \omega^2} \geq \min\left\{\frac{\frac{\Gamma}{\Gamma+\omega^2}\left(1+\frac{\Gamma}{\Gamma+f^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}, \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}\right\}$$

If

$$\min\left\{\frac{\frac{\Gamma}{\Gamma+\omega^2}\left(1+\frac{\Gamma}{\Gamma+f^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}, \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}\right\} = \frac{\frac{\Gamma}{\Gamma+\omega^2}\left(1+\frac{\Gamma}{\Gamma+f^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}$$

$$\Longrightarrow \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2} \geq \frac{\frac{\Gamma}{\Gamma+\omega^2}\left(1+\frac{\Gamma}{\Gamma+f^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}$$

$$\Longrightarrow \c{C}(gf,\ g\omega, \hbar\,\Gamma) \geq \min\left\{\frac{\c{C}(\omega,\ g\omega,\ \Gamma)(1+\c{C}(f,\ gf,\ \Gamma))}{1+\c{C}(f,\omega,\Gamma)},\ \c{C}(f,\omega,\Gamma)\right\}$$

If

$$\min\left\{\frac{\frac{\Gamma}{\Gamma+\omega^2}\left(1+\frac{\Gamma}{\Gamma+f^2}\right)}{1+\frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}}, \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}\right\} = \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}$$
$$\implies \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2} \geq \frac{\Gamma}{\Gamma+\max\{f,\omega\}^2}$$
$$\implies Ç(gf,\ g\omega,\hbar\Gamma) \geq \min\left\{\frac{Ç(\omega,\ g\omega,\ \Gamma)(1+Ç(f,\ gf,\ \Gamma))}{1+Ç(f,\omega,\Gamma)},\ Ç(f,\omega,\Gamma)\right\}$$

Thus all the conditions of Theorem 3.1 are satisfied,

$$gf = \sqrt{\hbar} f \implies g0 = 0.$$

So, 0 is the fixed point of g.

□

4. Conclusion

It is exceedingly challenging to turn crisp metric contraction conditions into fuzzy contractions, yet fuzzy mathematics principles allow one to adapt current results from the fixed point theory publications into fuzzy settings. However, compared to crisp metric fixed point theory, fuzzy fixed point theory becomes more generalized. Additionally, we can guarantee the existence of solutions to the different mathematical issues that researchers encounter due to symmetry in fuzzy metric spaces and their generalized spaces. We developed various fixed point results in U. Ishtiaq's newly introduced F.R.b-m.l.s in this work. A non-trivial case confirms the solution's uniqueness as well.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.A. Alghamdi, N. Hussain, P. Salimi, Fixed point and coupled fixed point theorems on b-metric-like spaces, J. Ineq. Appl. 2013 (2013), 402. https://doi.org/10.1186/1029-242X-2013-402.

[2] A. Amini-Harandi, Metric-like spaces, partial metric spaces and fixed points, Fixed Point Theory Appl. 2012 (2012), 204. https://doi.org/10.1186/1687-1812-2012-204.

[3] S. Banach, Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales, Fund. Math. 3 (1922), 133–181. https://doi.org/10.4064/fm-3-1-133-181.

[4] V. Berinde, M. Pacurar, Fixed points and continuity of almost contractions, Fixed Point Theory, 9 (2008), 23–34.

[5] A. Branciari, A fixed point theorem of Banach-Caccioppoli type on a class of generalized metric spaces, Publ. Math. Debrecen, 57 (2000), 31–37.

[6] L.B. Ćirić, A generalization of Banach's contraction principle, Proc. Amer. Math. Soc. 45 (1974), 267–273.

[7] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Univ. Ostrav. 1 (1993), 5–11. http://dml.cz/dmlcz/120469.

[8] H. S. Ding, M. Imdad, S. Radenović, J. Vujaković, On some fixed point results in b-metric, rectangular and b-rectangular metric spaces, Arab J. Math. Sci. 22 (2016), 151–164. https://doi.org/10.1016/j.ajmsc.2015.05.003.

[9] A. George, P. Veeramani, On some results in fuzzy metric spaces, Fuzzy Sets Syst. 64 (1994), 395–399. https://doi.org/10.1016/0165-0114(94)90162-7.

[10] R. George, S. Radenović, K.P. Reshma, S. Shukla, Rectangular b-metric space and contraction principles, J. Nonlinear Sci. Appl. 08 (2015), 1005–1013. https://doi.org/10.22436/jnsa.008.06.11.

[11] A.M. Hashim, S.L. Singh, New fixed point theorem for weak* compatible maps in rectangular metric space, Jnanabha 47 (2017), 51–62.

[12] A. M. Hashim, D. L. Baqair, On a fixed point theorems for multivalued maps in b-metric space, Basrah J. Sci. 33 (2015), 126–136.

[13] U. Ishtiaq, A. Hussain, H. Al Sulami, Certain new aspects in fuzzy fixed point theory, AIMS Math. 7 (2022), 8558–8573. https://doi.org/10.3934/math.2022477.

[14] K. Javed, F. Uddin, H. Aydi, M. Arshad, U. Ishtiaq, H. Alsamir, On fuzzy b-metric-like spaces, J. Function Spaces 2021 (2021), 6615976. https://doi.org/10.1155/2021/6615976.

[15] Z. Kadelburg, S. Radenović, Pata-type common fixed point results in b-metric and b-rectangular metric spaces, J. Nonlinear Sci. Appl. 08 (2015), 944–954. https://doi.org/10.22436/jnsa.008.06.05.

[16] I. Kramosil, J. Michálek, Fuzzy metrics and statistical metric spaces, Kybernetika 11 (1975), 336–344. http://dml.cz/dmlcz/125556.

[17] F. Mehmood, R. Ali, N. Hussain, Contractions in fuzzy rectangular b-metric spaces with application, J. Intell. Fuzzy Syst. 37 (2019), 1275–1285. https://doi.org/10.3233/JIFS-182719.

[18] N. Mlaiki, K. Abodayeh, H. Aydi, T. Abdeljawad, M. Abuloha, Rectangular metric-like type spaces and related fixed points, J. Math. 2018 (2018), 3581768. https://doi.org/10.1155/2018/3581768.

[19] S. Nădăban, Fuzzy b-metric spaces, Int. J. Comp. Commun. Control 11 (2016), 273–281.

[20] S.K. Prakasam, A.J. Gnanaprakasam, O. Ege, G. Mani, S. Haque, N. Mlaiki, Fixed point for an Ogf-c in O-complete b-metric-like spaces, AIMS Math. 8 (2023), 1022–1039. https://doi.org/10.3934/math.2023050.

[21] B. Schweizer, A. Sklar, Statistical metric spaces, Pac. J. Math. 10 (1960), 313–334.

[22] S. Shukla, M. Abbas, Fixed point results in fuzzy metric-like spaces, Iran. J. Fuzzy Syst. 11 (2014), 81–92. https://doi.org/10.22111/ijfs.2014.1724.

[23] T. Suzuki, A generalized banach contraction principle that characterizes metric completeness, Proc. Amer. Math. Soc. 136 (2007), 1861–1870. https://doi.org/10.1090/S0002-9939-07-09055-7.

[24] D. Wardowski, Fixed points of a new type of contractive mappings in complete metric spaces, Fixed Point Theory Appl. 2012 (2012), 94. https://doi.org/10.1186/1687-1812-2012-94.

[25] L.A. Zadeh, Fuzzy sets, Inf. Control 8 (1965), 338–353. https://doi.org/10.1016/S0019-9958(65)90241-X.