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INTERCONNECTION BETWEEN H-STABLE, $D(\alpha)-$STABLE, D-SEMISTABLE MATRICES, AND $\mu-$VALUES

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Abstract. In this article, we present some new results on the interconnections between the notations of H-stability, $D(\alpha)$-stability, a rank-1 perturbation to D-semistable matrices, and the $\mu-$values. The $\mu-$value known as structured singular value is a well-known tool in system theory, and analyze the robustness and performance of linear time invariant systems. The exact computation of $\mu-$value is an NP-hard problem.

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1. Introduction

The D-stability for a class of square matrices was introduced by Arrow and McManus [1], and Enthovan and Arrow [4] for analysis of dynamic models of competitive markets. The D-stable, and additive D-stable matrices are of great interest of research in various areas of science and engineering, for instance, mathematical economics, dynamics of population, neural networks, electrical and control engineering [2, 6–9, 12, 14, 16, 17].

The H-stability and H-semistability as a two strong notions of matrix stability were studied in [3], and necessary and sufficient conditions for a matrix to be H-stable or H-semistable were also presented.

The notion of $D(\alpha)-$stability for $\alpha = (\alpha_1, \alpha_1, \cdots, \alpha_p)$ was studied by Khalil and Kokotovic [10, 11] and is useful to study the problems such as time-invariant multi-parameter singular perturbations. In particular, the study of boundary layer systems of the form $E(\epsilon)\dot{z} = Dz$.

The structured singular value, also knows as μ-value [13], is a well-known mathematical tool in control to study the robustness, performance, and stability of linear time invariant systems. The necessary and sufficient conditions on structured singular values allow us to develop its connection with D-stability theory.

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In this article, we have developed some new but interesting results on the interconnections between structured singular values, that is, μ-values and H-stable matrices, $D(\alpha)$-stable matrices. Furthermore, we present some new results on the rank-1 perturbations to D-semistable matrices and μ-values.

2. Preliminaries

Throughout this paper, $\mathbb{R}$, and $\mathbb{C}$ denotes the field of real and complex numbers. The notations $\mathbb{R}^{n\times n}$, and $\mathbb{C}^{n\times n}$ represents n-dimensional real and complex valued matrices. The symbols $\lambda(\cdot)$, and $\mu(\cdot)$ denotes eigenvalues and structured singular values corresponding to a given matrix. The real part of the eigenvalues is denoted by $Re(\lambda(\cdot))$. The family of positive definite matrices is denoted by H^+, and $H > 0$ means that H is a positive definite matrix.

The following definitions $1-3$, Theorem-1, and Theorem-2 are taken from [15].

Definition 1. *Let $\alpha = \{p_1, p_2, ..., p_r\}$ be the partition of $\{1, 2, ..., n\}$. A diagonal matrix having diagonal blocks indexed by $p_1, p_2, ..., p_r$ is called $\alpha-$diagonal.*

Definition 2. *Let $\alpha = \{p_1, p_2, ..., p_r\}$ be the partition of $\{1, 2, ..., n\}$. An $n \times n$ complex valued matrix is called $H(\alpha)-$stable matrix if the product AH is stable for each $\alpha-$diagonal positive definite matrix H.*

Definition 3. *Let $\alpha = \{p_1, p_2, ..., p_r\}$ be the partition of $\{1, 2, ..., n\}$. An $n \times n$ dimensional matrix A is said to be real Lyapunov $\alpha-$ scalar stable matrix if there exists a positive definite matrix D such that $(AD + DA^*) > 0$, where $*$ is the complex conjugate transpose of A.*

Theorem 1. *Let α be the partition of $\{1, 2, ..., n\}$. Then real Lyapunov $\alpha-$ scalar stable matrix is $H(\alpha)$ stable matrix.*

Theorem 2. *$A \in \mathbb{C}^{n\times n}$ is $H-$stable matrix if and olny if following conditions hold:*
(i) $(A + A^*)$ is positive semidefinite matrix.
(ii) $x^*(A + A^*)x = 0$. In turn, $x^*(A - A^*)x = 0$ for every vector x.
(iii) $\lambda_i(A) \neq 0, \ \forall i = 1 : n$, where $\lambda(\cdot)$ is an eigenvalue of A.

3. New results on interconnection between H-stable matrices and structured singular values

In this section, we present some new results on the interconnection between H-stability and structured singular values for a family of squared real or complex valued matrices.

The following Definition 4 is taken from [1].

Definition 4. *$A \in \mathbb{R}^{n\times n}$ is called $(multiplicative)$ H-stable if HA is stable for every symmetric positive-definite matrix H.*

Theorem 3 shows that $A \in \mathbb{C}^{n\times n}$ is H-stable matrix whenever $H > 0$, a positive definite matrix.

Theorem 3. *Let $A, H \in \mathbb{C}^{n\times n}$, $H > 0$, a positive definite matrix. If A is H-stable for every $H > 0$, then $H > 0$ whenever A is $H-$stable.*

Proof. Suppose that given $A \in \mathbb{C}^{n\times n}$ is H-stable for every H, a positive definite matrix. This means that $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$. Infact, $H > 0$ causes $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ for given $A \in \mathbb{C}^{n\times n}$. We conclude that for given $A \in \mathbb{C}^{n\times n}$, $\mathrm{Re}(\lambda_i(AH)) > 0, \forall i$ and this is very much possible only if $H > 0$, a positive definite matrix. This complete the proof. □

The following Theorem 4 gives an interaction between H-stability and μ-values.

Theorem 4. *Let $A \in \mathbb{R}^{n\times n}$ be H-stable. Then for every $H \in H^+, 0 \leq \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$, with*

$$H^+ = \{H : \lambda_i(H) > 0, \forall i = 1 : n\},$$

and $\mathbb{B}$ denotes the set of block-diagonal matrices.

Proof. We aim to show that $0 \leq \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1$ for every $H \in H^+$. As D-stability of a matrix does imply its stability. We make use of this fact to prove our result. $A \in \mathbb{R}^{n\times n}$ is $D-$stable iff A is stable and

$$\lambda_i \begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0, \quad \forall i.$$

Since, $\lambda_i \begin{pmatrix} A & -H \\ H & A \end{pmatrix} \neq 0,\ \forall i$. In turn this implies that

$$\lambda_i\left(A^2 - HA^{-1}HA\right) \neq 0, \forall i.$$

Furthermore,

$$\lambda_i\left(A^2 - HA^{-1}HA\right) \neq 0 \Rightarrow \lambda_i\left(I_n - \frac{1}{A^2}H\right) \neq 0, \forall H \in H^+.$$

Finally, we conclude that

$$\lambda_i\left(I_n - \frac{1}{A^2}H\right) \neq 0, \Rightarrow 0 \leq \mu_{\mathbb{B}}\left(\frac{1}{A^2}\right) < 1.$$

This complete the proof. □

Theorem 5. *Let $A, H \in \mathbb{C}^{n\times n}$, $H \geq 0$, a Hermitian positive semi-definite matrix. If $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H))$ for every $H \geq 0$, then $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ and*

$$0 \leq \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + A)^{-1}(\boldsymbol{i}I_n - A)\right) < 1,\ \boldsymbol{i} = \sqrt{-1}.$$

Proof. Firstly, we aim to show that $\mathrm{Re}(\lambda_i(A)) > 0, \forall i$ if $\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i, \forall H \geq 0$. Furthermore, we have that, if $\mathrm{Re}(\lambda_i(A)) \geq 0, \forall i$ and $A \in \mathbb{C}^{n\times n}$ is $n \times n$-singular matrix, then $\exists U$, a unitary matrix such that

$$U^*AU = \begin{pmatrix} M_{11} + \mathbf{i}N_{11} & \mathbf{i}N_{12} \\ \mathbf{i}N_{21} & \cdot \end{pmatrix},$$

with $M_{11} > 0$; for $U^*AU = \begin{pmatrix} \cdot & \cdot \\ \cdot & I_n \end{pmatrix} \geq 0$. In turn, this yields

$$\mathrm{Re}(\lambda_i(AH)) = \mathrm{Re}(\lambda_i(H)), \forall i.$$

Secondly, the aim is to show that

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

Consider $A = e^H$, a stable matrix where $H \geq 0$, a positive semi-definite matrix. Let $P > 0$, a positive definite such that $\lambda_i(\mathbf{i}I_n + e^H P) \neq 0, \forall i$ where $P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)$ for all $\Delta \in \mathbb{B}$. This allows as to have that

$$\lambda_i\left(\mathbf{i}I_n + e^H(iI_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \neq 0 \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Furthermore,

$$\lambda_i\left((\mathbf{i}I_n + e^H)^{-1}(\mathbf{i}I_n - e^H)\Delta\right) \neq 0 \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Finally, this implies that

$$\lambda_i\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\Delta\right) \neq 0 \quad \forall i, \forall \Delta \in \mathbb{B}.$$

Thus,

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + A)^{-1}(\mathbf{i}I_n - A)\right) < 1.$$

This complete the proof. □

The following Definitions are taken from [10], and [11], respectively.

Definition 5. *$A \in \mathbb{R}^{n\times n}$ is called $D(\alpha)$-stable if DA is stable for every positive α-scalar matrix D.*

Definition 6. *A diagonal matrix D is called an α-scalar matrix if $D[\alpha_k]$ is a scalar matrix for every $k = 1 : p$, that is,*

$$D = \mathrm{diag}(d_{11}I[\alpha_1], ..., d_{11}I[\alpha_p]).$$

Note: In Definition 6, $D[\alpha_k]$ denotes the principal submatrix spanned by rows and columns having indices from α_k where $\alpha = (\alpha_1, ..., \alpha_p)$ be a partition of set of indices $\{1, ..., n\}$ and $1 \leq p \leq n$ being as a positive integer.

4. New results on interconnection between $D(\alpha)$-stable matrices and structured singular matrices

The following theorem links the bridge between $D(\alpha)$-stable matrices and structured singular values.

Theorem 6. *Let $A \in \mathbb{R}^{n\times n}$. Then A is $D(\alpha)$-stable if and only if*

$$\mathrm{Re}\left(\lambda_i(\mathrm{diag}(d_{kk}I[\alpha_k]))A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right) > 0,\ \forall i = 1:n, \forall k = 1:p, \quad 1 \leq p \leq n,$$

and $0 \leq \mu_{\mathbb{B}}(M) < 1$, *where M is obtained from A as*

$$M = \left(\boldsymbol{i}I_n + \mathrm{diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right)^{-1}\left(\boldsymbol{i}I_n - \mathrm{diag}(d_{kk}I[\alpha_k])A - A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right).$$

Proof. We aim to prove that $A \in \mathbb{R}^{n\times n}$ is $D(\alpha)$-stable iff $0 \leq \mu_{\mathbb{B}}(M) < 1$. Let $\Delta \in \mathbb{B}$ a block-diagonal structure, that is,

$$\Delta = (\mathbf{i}I_n - \mathrm{diag}(d_{kk}I[\alpha_k]))\,(\mathbf{i}I_n + \mathrm{diag}(d_{kk}I[\alpha_k]))^{-1},\ \forall k = 1:p, 1 \leq p \leq n.$$

As,

$$\lambda_i\left(\mathrm{diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k]))\right) \neq 0\ \forall i, \forall k = 1:p, 1 \leq p \leq n.$$

This can be written as

$$\lambda_i\big(\mathrm{diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k])) + (\mathrm{diag}(d_{kk}I[\alpha_k]))\big) \neq 0\ \forall i, \forall k = 1:p, 1 \leq p \leq n,$$

and this is true if and only if

$$\lambda_i\big(\mathrm{diag}(d_{kk}I[\alpha_k])A + A^T(\mathrm{diag}(d_{kk}I[\alpha_k])) + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\big) \neq 0\ \forall i, \forall k = 1:p, 1 \leq p \leq n.$$

This further implies that

$$\lambda_i\big((\mathbf{i}I_n + \mathrm{diag}(d_{kk}I[\alpha_k])A + A^T\mathrm{diag}(d_{kk}I[\alpha_k])A) - (\mathbf{i}I_n - \mathrm{diag}(d_{kk}I[\alpha_k])A - A^T\mathrm{diag}(d_{kk}I[\alpha_k]))\Delta\big)$$

$\neq 0,\ \ \forall i, \forall k = 1:p, 1 \leq p \leq n; \forall \Delta \in \mathbb{B}$.

Thus,

$$\lambda_i\big(I_n - \left(\mathbf{i}I_n + \mathrm{diag}(d_{kk}I[\alpha_k])A + A^T\mathrm{diag}(d_{kk}I[\alpha_k])\right)^{-1}(\mathbf{i}I_n - \mathrm{diag}(d_{kk}I[\alpha_k])A - A^T\mathrm{diag}(d_{kk}I[\alpha_k]))\Delta\big) \neq 0.$$

$\forall i, \forall k = 1:p, 1 \leq p \leq n; \forall \Delta \in \mathbb{B}$. The final expression for λ_i, $\forall i$ is equivalent to the fact that $0 \leq \mu_{\mathbb{B}}(M) < 1$, see [5]. This complete the proof. □

Theorem 7 gives the necessary condition for $A \in \mathbb{C}^{n,n}$ to be $D(\alpha)$-stable.

Theorem 7. *Let $A \in \mathbb{C}^{n,n}$. A necessary condition for A to be $D(\alpha)$-stable matrix is that $A = e^B$ (Hermitian matrix $B \in \mathbb{C}^{n,n}$) is stable and* $0 \leq \mu_{\mathbb{B}}\left((\boldsymbol{i}I_n + e^B)^{-1}(\boldsymbol{i}I_n - e^B)\right) < 1$.

Proof. We aim to show that A is $D(\alpha)$-stable if $\lambda_k(\mathbf{i}I_n + e^B\, diag(d_{kk}I[\alpha_k])) \neq 0,\ \forall\, k = 1:p,\, 1 \leq p \leq n$. Let $\Delta \in \mathbb{B}$ with a block-diagonal structure, and let

$$diag(d_{kk}I[\alpha_k]) = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

Then, we have that

$$\lambda_k(\mathbf{i}I_n + e^B(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)) \neq 0,\ \forall\, k = 1:p,\, 1 \leq p \leq n.$$

In turn, this further reduces to

$$\lambda_k((\mathbf{i}I_n + e^B)^{-1}(\mathbf{i}I_n - e^B)\Delta) \neq 0,\ \forall\, k = 1:p,\, 1 \leq p \leq n,\ \forall\, \Delta \in \mathbb{B}.$$

Finally, the definition of structured singular values allows us to conclude that

$$0 \leq \mu_{\mathbb{B}}\left((\mathbf{i}I_n + e^B)^{-1}(\mathbf{i}I_n - e^B)\right) < 1.$$

This complete the proof. □

5. The rank-1 perturbation to D-semistable matrices

In this section, we present some new results to establish the connection between D-semistable matrices and structured singular values. Theorem 8 shows that a square complex valued matrix A is D-semistable if and only if it is semi-stable and all the eigenvalues of rank-1 perturbation to A, that is, $A + v\omega^*$ are non-zero.

Theorem 8. *Let $A \in \mathbb{C}^{n,n}$. Then A is D-semistable if and only if A is semi-stable and $\lambda_k(A + v\omega^*) \neq 0, \forall k$, with $v\omega^*$, a rank-1 matrix.*

Proof. Suppose that A is D-semistable, that is, $\mathrm{Re}(\lambda_k(AD)) \geq 0, \forall k, \forall D \in \Omega$. We aim to show that A is semi-stable, that is, $\mathrm{Re}(\lambda_k(A)) \geq 0, \forall k$, and $\lambda_k(A + v\omega^*) \neq 0, \forall k,\, v\omega \in \mathbb{C}^{n,1}$. Since, $\mathrm{Re}(\lambda_k(AD)) \geq 0, \forall k, \forall D \in \Omega$. This implies that $\mathrm{Re}(\lambda_k(AI_n)) \geq 0, \forall k$ or $\mathrm{Re}(\lambda_k(A)) \geq 0$. This implies that A is semi-stable matrix. Furthermore, for a rank-1 perturbation to A, that is, $(A + v\omega^*)$, we aim that

$$\mathrm{Re}(\lambda_k(A + v\omega^*)) \neq 0,\ \ \forall k.$$

We assume that for $v, \omega \in \mathbb{C}^{n,1}$, $z_l^* v \neq 0$ and $\omega^* z_r \neq 0$. Here, z_l and z_r are the left hand side and sight hand side eigenvectors corresponding to a simple eigenvalue of A. Additionally, suppose that $y \in \mathrm{Ker}(A + v\omega^*)$. This allows to have homogeneous system of linear equations of the form

$$z_l^* v\omega^* y = z_l^*(A + v\omega^*)y = 0.$$

This implies that $\omega^* y = 0$ because $z_l^* v \neq 0$. On the other hand, $(A + v\omega^*)y = 0$, so that $y = \alpha z_r$ for some $\alpha \in \mathbb{C}$. This, in turn implies that $\omega^* y = \alpha\omega^* z_r = 0 \Rightarrow \alpha = 0, y = 0$. Conversely, suppose that if $\omega^* z_r = 0$, then

$$(A + v\omega^*)z_r = 0.$$

Also, $z_l^*(A + v\omega^*) = 0$ if $z_e^* v = 0$. This complete the proof. □

The following theorem 9 show that a square complex valued matrix is D-semistable iff it is semi-stable and the structured singular value is bounded above by 1.

Theorem 9. *Let $A \in \mathbb{C}^{n,n}$. Then A is D-semistable if and only if A is semi-stable and $0 \leq \mu_{\mathbb{B}}(M) < 1$, where*

$$M = (\boldsymbol{i}I_n + \widehat{A})^{-1}(\boldsymbol{i}I_n - \widehat{A}),$$

with $\widehat{A} = A + v\omega^*$ for $v, \omega \in \mathbb{C}^{n,1}$.

Proof. The matrix A is D-semistable if and only if A is semi-stable and $\lambda_k(A + v\omega^* + \mathbf{i}P) \neq 0, \forall k, \forall P \in \Omega$. To prove that $\lambda_k(A + v\omega^* + \mathbf{i}P) \neq 0, \forall k, \forall P \in \Omega$. Let $\Delta = (\mathbf{i}I_n - P)(\mathbf{i}I_n + P)^{-1}$ be a diagonal matrix and $\Delta \in \mathbb{B}$. The positive diagonal matrix P takes the following form

$$P = (\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta).$$

Since, $\lambda_k(A + v\omega^* + \mathbf{i}P) \neq 0$, this implies that

$$\lambda_k\left(A + v\omega^* + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \neq 0, \quad \forall\Delta \in \mathbb{B}.$$

Furthermore,

$$\operatorname{rank}\left(A + v\omega^* + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) = \operatorname{rank}\left((\mathbf{i}I_n + A + v\omega^*) - (\mathbf{i}I_n - A + v\omega^*)\Delta\right), \quad \forall\Delta \in \mathbb{B}.$$

In turn, this yields

$$\left(A + v\omega^* + \mathbf{i}(\mathbf{i}I_n + \Delta)^{-1}(\mathbf{i}I_n - \Delta)\right) \sim \left((\mathbf{i}I_n + A + v\omega^*) - (\mathbf{i}I_n - A + v\omega^*)\Delta\right), \quad \forall\Delta \in \mathbb{B}.$$

Also,

$$\left((\mathbf{i}I_n + A + v\omega^*) - (\mathbf{i}I_n - A + v\omega^*) - (\mathbf{i}I_n - A + v\omega^*)\Delta\right), \quad \forall\Delta \in \mathbb{B}.$$

Finally, We conclude that

$$\lambda_k\left(I_n - (\mathbf{i}I_n + A + v\omega^*)^{-1}(\mathbf{i}I_n - A + v\omega^*)\Delta\right) \neq 0, \quad \forall\Delta \in \mathbb{B}.$$

This is necessary condition that $0 \leq \mu_{\mathbb{B}}(A) < 1$. This complete the proof. □

6. Conclusion

In this article, we have considered the problem on the interconnection between D-stability theory and μ-theory. We have developed some new mathematical results linking the bridge between the notation of H-stability, $D(\alpha)$-stability, a rank-1 perturbation to D-semistable matrices, and structured singular values, that is, μ-values.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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