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INTRODUCING AN EFFECTIVE EIGHTH-ORDER ITERATIVE METHOD UTILIZING NEWTON'S INTERPOLATION TO ADDRESS CHALLENGES IN MATHEMATICAL PHYSICS

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Abstract. A variety of advanced iterative techniques involving derivative evaluations have been explored in literature to compute simple zeros. However, methodologies that achieve higher order accuracy without relying on derivatives are scarce in the existing literature. Recognizing this gap, we develop a novel family of three step Ostrwoski's type derivative-free method. The order of convergence of the proposed family is eight, according to the convergence analysis performed using computational algebra system CAS Maple-18. Notably, this method demand only four function evaluations per iteration, thereby demonstrating optimality as per to the Kung-Traub conjecture. To compare and examine the performance of the proposed and already well-known iterative schemes, some real world problems are considered, such as parachute problem, continuous stirred tank reactor (CSTR), van der waals equation, and probability of a shutout in a racquetball. Additionally, we employ a dynamical tool, i.e., stereographic projections, to investigate the regions and stability of the proposed schemes. Both practical and theoretical analysis shows that newly developed scheme is an alternate to the existing schemes in their respective domain.

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1. Introduction

Solving nonlinear equations is a significant challenge across various scientific and technological domains [5,20,22]. Exact solutions for these equations are often hard to find, so we turn to iterative schemes to find close approximations. In the realm of research, numerous iterative techniques have

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been devised to approximate roots of nonlinear equations of the form:

$$G(t) = 0. \tag{1}$$

Therefore, in order to achieve the convergence and enhance the efficiency index of their proposed new or modified schemes, researchers have focused on solving these nonlinear equations employing modern, diverse strategies or modifying existing ones. A lot of research effort has been focused on improving the speed of the convergence of existing algorithms to the desired solutions. Among the early numerical methods proposed and studied in the literature are Newton method, Secant method, and their modifications which are very attractive because of their quick convergence. The most well-known method is unquestionably Newton's method with quadratic convergence [28].

$$t_{k+1} = t_k - \frac{G(t_k)}{G'(t_k)}, k = 0, 1, 2, \cdots,$$

where t_k is the current approximation and $G'(t_k)$ represents the derivative of the function at t_k. Here, t_0 serves as the initial guess for the root α. A root α is deemed simple if $G(\alpha) = 0$ but $G'(\alpha) \neq 0$. It requires two evaluations per iteration, namely G and G', to achieve second order of convergence. However, there are some challenges associated with applying Newton's method. One major difficulty is the need to compute the first-order derivative at each step. In many practical situations, calculating these derivatives is either costly or very time-consuming. To address this issue, Traub–Steffensen method (see [28], which is expressed as:

$$t_{k+1} = t_k - \frac{G(t_k)}{G[t_k, w_k]}, \tag{2}$$

where $w_k = t_k + \beta G(t_k)$, $\beta \neq 0$ is any real constant and $G[t_k, w_k] = \frac{G(w_k) - G(t_k)}{w_k - t_k}$ is the first order divided difference, is a noticeable improvement of Newton's method, because it maintains quadratic convergence without using any derivative. For $\beta = 1$, this method reduces to the well-known Steffensen's method [26].

Indeed, many researchers have presented fourth-order methods for solving nonlinear equations, see [6,14,19,24,29]. Similarly, optimal higher order convergence has been introduced by many authors also, for instance [1,2,7,30].

Derivative-free methods have gained a great deal of attention in past years. These methods are useful when the derivative of function F is difficult to evaluate, expensive to compute, or does not exist. Some researchers have proposed derivative free techniques for simple roots of the nonlinear equations [4,8,9,16,31]. Our prime motive is to develop an efficient and optimal three-step derivative-free eighth-order convergent method for solving nonlinear equations by using newtonian interpolation [21].

Over time, various adaptations of numerical techniques have emerged, contributing to the development of innovative variety in the literature. Building upon previous findings, particularly inspired by the works mentioned above, our interest was sparked in developing a derivative-free Ostrowski-type iterative scheme with optimal eighth-order convergence. The eighth-order method $(AMK-1)$ introduced in this paper is derivative-free and requires only four evaluations of the function per iteration. We have achieved the optimal order of convergence, supporting the Kung-Traub conjecture. Traub [27] stated that multipoint iteration methods, with $\wp$ evaluations of the function, could achieve optimal convergence order $2^{\wp-1}$.

We chose wide variety of real-life applications to examine precision and reliability of the suggested method.e-g Parachute problem [15,17,18], continuous stirred tank reactor [3], van der waals equation, Probability of a Shutout in Racquetball [13]. Numerical experiments show that our method outperforms the existing ones of the same class. As a consequence, we have found that the new eighth order derivative-free method is consistent and well constructed, and its dynamical study extends significantly into theoretical aspects.

The paper is organized as follows. Section 2 deals with the development of the derivative-free approximation technique and convergence analysis. Numerical illustrations of derivative-free methods are given in Section 3. Section 4 is devoted to the stereographic projections. Finally, concluding remarks are given in Section 5.

2. Construction of Suggested Higher Order Scheme

In this section, we develop our optimal eighth-order derivative-free scheme for solving nonlinear equation (1). Sharma et al. [23] derived a three-point Ostrowski's type iterative method and has a convergence order eight which is expressed as:

$$\begin{aligned}
y_k &= t_k - \frac{G(t_k)}{G'(t_k)},\\
s_k &= t_k - \frac{G(y_k)-G(t_k)}{G'(t_k)G(y_k)-G[y_k,t_k]G(t_k)}G(t_k),\\
t_{k+1} &= t_\wp - \frac{(G(s_k)-G(y_k))(G(y_k)-G(t_k))(G(t_k)-G(s_k))}{p+q+r}G(t_k). \qquad (3)
\end{aligned}$$

Here,

$$\begin{aligned}
p &= G(s_k)G(t_k)(G(s_k)-G(t_k))G[y_k,t_k],\\
q &= G(t_k)G(y_k)(G(t_k)-G(y_k))G[s_k,t_k],\\
r &= G(y_k)G(s_k)(G(y_k)-G(s_k))G'(t_k).
\end{aligned}$$

Our main goal is to develop a derivative-free eighth order iterative scheme. This modification offers the advantage of reducing the computational cost associated with derivatives at each step of the method. For developing a newly derivative-free scheme, we used interpolating polynomials of the 2^{nd} and 3^{rd}

degree. Thus, the derivatives of the second and third steps of (3) are replaced by:

$$\begin{aligned} G'(t_k) &\approx G[t_k, w_k] + G[t_k, w_k, y_k](t_k - w_k), \\ G'(t_k) &\approx G[t_k, w_k] + G[t_k, w_k, y_k](t_k - w_k) \\ &\quad + G[t_k, w_k, y_k, s_k](t_k - w_k)(t_k - y_k). \end{aligned} \tag{4}$$

By employing the above approximation, we get the eighth-order derivative-free iterative scheme with four function evaluations per iterative cycle and is given by:

$$\begin{aligned} y_k &= t_k - \frac{G(t_k)}{G[t_k, w_k]}, w_k = t_k + \gamma G(t_k)^3, \\ s_k &= t_k - \frac{G(y_k) - G(t_k)}{(G[t_k, w_k] + G[t_k, w_k, y_k](t_k - w_k))G(y_k) - G[y_k, t_k]G(t_k)} G(t_k), \\ t_{k+1} &= t_k - \frac{(G(s_k) - G(y_k))(G(y_k) - G(t_k))(G(t_k) - G(s_k))}{p + q + r} G(t_k), \end{aligned} \tag{5}$$

where

$$\begin{aligned} p &= G(s_k)G(t_k)(G(s_k) - G(t_k))G[y_k, t_k], \\ q &= G(t_k)G(y_k)(G(t_k) - G(y_k))G[s_k, t_k], \\ r &= G(y_k)G(s_k)(G(y_k) - G(s_k))(G[t_k, w_k] + G[t_k, w_k, y_k](t_k - w_k) \\ &\quad + G[t_k, w_k, y_k, s_k](t_k - w_k)(t_k - y_k)). \end{aligned}$$

The method (5) is our new derivative-free method, denoted by AMK-1. The following theorem demonstrates the eighth-order of convergence of the developed scheme (5).

Theorem 1 Suppose ϖ be a root of a real-valued differentiable function $G : I \subseteq \mathbb{C} \longrightarrow \mathbb{C}$ within an open interval I, where ϖ is a simple root in I and k_0 is an initial approximation close to ϖ for guaranteed convergence, In this scenario, the scheme defined in (3) demonstrates the convergence order eight, and it possesses by the following error equation:

$$e_{k+1} = C_2^2(-C_3C_4 - 5C_2^3C_3 + 3C_2C_3^2 + C_4C_2^2 + 2C_2^5)e_k^8 + O(e_k^9).$$

Here, $t = \varpi$ with $C_k = \frac{G^{(\wp)}(\varpi)}{\wp!}, C_1 = 1$, and $k = 1, 2, 3, \cdots$.

Proof. Assume that e_k represent the error of the k^{tG} step, defined as:

$$e_k = t_k - \varpi. \tag{6}$$

Expanding the function $G(t_k)$ about ϖ using Taylor series and considering $G(\varpi) = 0$, we obtain:

$$G(t_k) = C_1e_k + C_2^2e_k + C_3^3e_k + C_4^4e_k + C_5^5e_k + C_6^6e_k + C_7^7e_k + C_8^8e_k + \cdots. \tag{7}$$

Additionally, we introduced a new variable w_k which combines $t_\wp$ and $G(t_\wp)$ as follows:

$$w_k = t_k + \gamma G(t_k)^3 = \varpi + e_k + \gamma e_k^3 + 3\gamma C_2 e_k^4 + 3\gamma(C_2^2 + C_3)e_k^5 + \cdots. \tag{8}$$

After expanding $G(w_k)$ about ϖ we get:

$$G(w_k) = e_{_k} + C_2 e_{_k}^2 + (\gamma + C_3) e_{_k}^3 + (5\gamma C_2 + C_4) e_{_k}^4 + \cdots . \tag{9}$$

Thus, using (7)-(9) to calculate the t_k and w_k. Subsequently, we employed the divided difference (4) instead of derivative in the first step of iterative scheme (5), to develop a derivative-free approach.

$$G[t_k, w_k] = (G(t_k) - G(w_k))/(t_k - w_k), = 1 + 2C_2 e_{_k} + 3C_3 e_{_k}^2 + \cdots . \tag{10}$$

Now, replacing equations (7) and (10) in the first step of three-step scheme (5), the following term is obtained:

$$y_k - \varpi = C_2 e_k^2 + (2C_3 - 2C_2^2) e_k^3 + (3C_4 - 7C_2C_3 + 4C_2^3 + \gamma C_2) e_k^4 + \cdots . \tag{11}$$

By expanding Taylor series, we find $G(y_k)$ about ϖ, which can be expressed as:

$$G(y_k) = C_2 e_{_k}^2 + (-2C_2^2 + 2C_3) e_{_k}^3 + (-7C_3C_2 + 5C_2^3 + 3C_4 + \gamma C_2) e_{_k}^4 + \cdots . \tag{12}$$

Also, by utilizing equations. (7), (11) and (12) we get the divided difference of $G[t_\wp, y_\wp]$, as follows:

$$G[t_k, y_k] = 1 + C_2 e_k + (C_2^2 + C_3) e_k^2 + (-2C_2^3 + C_4 + 3C_3C_2) e_k^3 + \cdots , \tag{13}$$

Combining equations. (7)-(12), we get $G'(t_k)$ which is the derivative approximation of the second step iterative method (3), as shown below:

$$G'(t_k) = G[t_k, w_k] + G[t_k, w_k, y_k](t_k - w_k), = 1 + 2C_2 e_{_k} + 3C_3 e_k^2 + 4C_4 e_{_k}^3 + \cdots . \tag{14}$$

So, convergence order four is obtained by using equations (7) and (11)-(14) in second step of iterative method (5).

$$s_\wp - \varpi = (C_2^3 - C_3^2 C_2) e_{_k}^4 + (-2C_3^2 - 2C_4C_2 + 8C_3C_2^2 - 4C_2^4) e_{_k}^5 + \cdots . \tag{15}$$

Expanding $G(s_k)$ about ϖ yields:

$$G(s_k) = C_2(C_2^2 - C_3) e_{_k}^4 + (-4C_2^4 - 2C_3^2 - 2C_4C_2 + 8C_3C_2^2) e_{_k}^5 + \cdots . \tag{16}$$

Utilizing equations. (7), (15) and (16), we calculated the divided difference of t_k and s_k given as:

$$G[t_k, s_k] = 1 + C_2 e_{_k} + C_3 e_{_k}^2 + C_4 e_{_k}^3 + (C_5 + C_2^4 - C_3 C_2^2) e_{_k}^4 + \cdots .$$

By using equations (7)- (16), we get the Newton's interpolation for the derivative-free third step iterative scheme (5), as shown below:

$$\begin{aligned} G'(t_k) &= G[t_k, w_{_k}] + G[t_k, w_k, y_k](t_k - w_{_k}) + G[t_k, w_{_k}, y_k, s_k](t_k - w_{_k})(t_k - y_{_k}), \\ &= 1 + 2C_2 e_k + \cdots . \end{aligned} \tag{17}$$

Through equations (7), (11) and (13)-(17), we obtain the values of p, q and r for substitution in the third step iterative scheme (5).

$$\begin{aligned} p &= G(s_k)F(t_k)(G(s_k)-G(t_k))G[y_k,t_k] = -C_2(C_2^2-C_3)e_k^6 + + \cdots, \\ q &= G(t_k)F(y_k)(G(t_k)-G(y_k))G[s_k,t_k] = C_2e_k^4 + 2C_3e_k^5 + \cdots, \\ r &= G(y_k)F(s_k)(G(y_k)-G(s_k))G[t_k,w_k] + G[t_k,w_k,y_k](t_k-w_k) \\ &\quad +G[t_k,w_k,y_k,s_k](t_k-w_k)(t_k-y_k)) \\ &= C_2^3(C_2^2-C_3)e_k^8 + \cdots. \end{aligned} \tag{18}$$

By substituting equation (18), into the third step of iterative scheme (5), we achieve eighth-order convergence given as:

$$e_{k+1} = C_2^2(-C_3C_4 - 5C_2^3C_3 + 3C_2C_3^2 + C_4C_2^2 + 2C_2^5)e_k^8 + O(e_k^9). \tag{19}$$

□

3. Remark

Let us remark that, the above expression of equation (19) assures two things about scheme. Firstly, (AMK-1) (5) has convergence order eight and secondly it is optimal by the conjecture of Kung-Traub, as it requires four evaluations of function.

4. Computational Results

Here, we assess the efficiency of our developed scheme in comparison to already known eighth-order derivative-free schemes. To achieve this goal, we apply our schemes on different nonlinear test functions. Furthermore, we examine a selection of some real-world applications. The computational order of convergence (COC) is expressed as:

$$COC = \frac{\log(t_k - t_{k-1})/(t_{k-1} - t_{k-2})}{\log(t_{k-1} - t_{k-2})/(t_{k-2} - t_{k-3})}.$$

We use a computer programming system i.e., Maple 18 for numerical computation. To check the accuracy, we calculate the absolute error between t_{k+1} and t_k for three consecutive iterations, each with 3000 digits of mantissa. Our aim is to examine the efficiency and effectiveness of newly developed scheme with the already known derivative-free schemes. The following three step eighth order existing schemes are chosen for comparison.

Modified King's method $\mathbf{MK}_{8a}$ by Obadah et al. [25]:

$$\begin{aligned} y_k &= t_k - \frac{G(t_k)}{G[t_k, w_k]}, w_k = t_k + G(t_k), \\ z_k &= y_k - \frac{G(y_k)}{G(t_k)} \frac{G(t_k) + \beta G(y_k)}{G(t_k) + (\beta - 2)G(y_k)}, \\ t_{k+1} &= t_k - \frac{G(t_k)(m_1 + m_2 + m_3)}{m_1 G[w_k, t_k] + m_2 G[y_k, t_k] + m_3 G[z_k, t_k]}, \\ m_1 &= G(y_k)G(z_k)(z_k - y_k), \\ m_2 &= G(w_k)G(z_k)(w_k - z_k), \\ m_3 &= G(w_k)G(y_k)(y_k - w_k). \end{aligned} \tag{20}$$

Derivative for 2^{nd} *and* 3^{rd} step of (20) is approximated as follows:

$$\begin{aligned} G'(t_k) &= G[t_k, w_k], \\ G'(t_k) &= G[t_k, w_k] + 2(w_k - t_k) G[t_k, w_k, y_k] - G[y_k, w_k] + G[t_k, y_k], \\ G[t_k, y_k] &= \frac{G(t_k) - G(y_k)}{t_k - y_k}, G[w_k, t_k, y_k] = \frac{G[w_k, t_k] - G[t_k, y_k]}{w_k - y_k}. \end{aligned}$$

Modified King's method $\mathbf{MK}_{8b}$ by Obadah et al. [25]:

$$\begin{aligned} y_k &= t_k - \frac{G(t_k)}{G[t_k, w_k]}, w_k = t_k + G(t_k), \\ z_k &= y_k - \frac{G(y_k)}{G(t_k)} \frac{G(t_k) + \beta G(y_k)}{F(t_k) + (\beta - 2)G(y_k)}, \\ t_{k+1} &= z_{\wp} - \frac{G(z_k)}{C_2 - C_1 C_4}, \end{aligned} \tag{21}$$

where,

$$\begin{aligned} c_1 &= G(z_k), \\ c_2 &= G[y_k, z_k] - C_3(y_k - z_k) + C_4 G(y_k), \\ c_3 &= G[y_k, z_k, w_k] + c_4 G[y_k, w_k], \\ c_4 &= \frac{G[y_k, z_k, t_k] - G[y_k, z_k, w_k]}{G[y_k, w_k] - G[y_k, t_k]}. \end{aligned}$$

Derivative approximation for 2^{nd} and 3^{rd} step of (21) is made as:

$$\begin{aligned} G'(t_k) &= f[t_k, w_k], \\ G'(t_k) &= G[t_k, w_k] + 2(w_k - t_k) G[t_k, w_k, y_k] \\ &\quad - G[y_k, w_k] + G[t_k, y_k], \\ G[t_k, y_k] &= \frac{G(t_k) - G(y_k)}{t_k - y_k}, G[w_k, t_k, y_k] = \frac{G[w_k, t_k] - G[t_k, y_k]}{w_k - y_k}. \end{aligned} \tag{22}$$

Derivative-free scheme of zafar et al. [12](**MFN**):

$$G^{'}(t_k) \approx G[t_k, z_k], z_k = t_k + \gamma G(t_k)^m, m \geq q, \gamma \in R - \{0\}, \tag{23}$$

$$\begin{aligned}
y_k &= t_k - \frac{G(t_k)}{G[z_k, t_k]}, z_k = t_k + G(t_k)^3, k \gneq 0, \\
w_k &= y_k + g_3 G(t_k)^2, \\
t_{k+1} &= y_k + b_3 G(t_k)^2 - d_3 G(t_k)^3,
\end{aligned}$$

where,

$$\begin{aligned}
g_3 &= \frac{1}{[G(y_k) - G(t_k)]G[y_k, t_k]} - \frac{1}{[G(y_k) - G(t_k)]G[z_k, t_k]}, \\
b_3 &= \frac{1}{[G(y_k) - G(t_k)][G(y_k) - G(w_k)]G[y_k, t_k]} \\
&\quad - \frac{1}{[G(w_k) - G(t_k)][G(y_k) - G(w_k)]G[w_k, t_k]} \\
&\quad - \frac{1}{[G(w_k) - G(t_k)][G(y_k) - G(w_k)]G[z_k, t_k]} \\
&\quad - \frac{1}{[G(y_k) - G(t_k)][G(y_k) - G(w_k)]G[z_k, t_k]}, \\
d_3 &= \frac{1}{[G(y_k) - G(t_k)]G[y_k, t_k]} - \frac{1}{G[z_k, t_k][G(y_k) - G(t_k)]} - g_3[G(y_k) - G(t_k)].
\end{aligned}$$

Three-step eighth order derivative-free scheme $SJ8$ proposed by Jamali et al. [11], which is given below:

$$\begin{aligned}
y_k &= t_k - \frac{G(t_k)}{G[z_k, t_k]}, z_k = t_k + G(t_k)^3, k \gneq 0, \\
w_k &= y_k - \frac{G(y_k)}{G[z_k, t_k]} \times \frac{G(t_k)^2}{G(t_k)^2 - 2 * G(t_k)G(y_k) + G(y_k)^2}, \\
t_{k+1} &= z_k - (A(t_1) + B(t_2) + C(t_3)) \times \frac{G(w_k)}{G[y_k, z_k]}, \\
A(t_1) &= 1 + t_1^2, B(t_2) = -1, C(t_3) = 1 + 2t_3, \\
t_1 &= \frac{G(y_k)}{G(t_k)}, t_2 = \frac{G(w_k)}{G(y_k)}, t_1 = \frac{G(w_k)}{G(t_k)}.
\end{aligned}$$

4.1. Some Real World Applications. Problem 1 Continuous Stirred Tank Reactor (CSTR) (see [10]):

Industries frequently utilize continuous stirred-tank reactors (CSTRs). These reactors are integral to engineering problems involving feedback control systems and are often utilized to model chemical processes. They serve as a tool for understanding principles of chemical reactor modeling.

Consider a basic, irreversible fluid stage compound response in which chemical species M responds with chemical species N. The reaction for an isothermal nonstop mixed tank reactor (CSTRs) issue can be composed as $M \to N$, which is the first order rate of reaction. Now, imagine we have two different substances, S_1 and S_2, being fed into two reactors, let's call them B_1 and B_2. In these reactors, various reactions occur. For instance:

$$\begin{aligned} S_1 + S_2 &\rightarrow B_1, \\ B_1 + S_2 &\rightarrow C_1, \\ Q_1 + S_2 &\rightarrow D_1, \\ Q_1 + S_2 &\rightarrow E_1. \end{aligned}$$

A straightforward model for criticism control frameworks was planned while concentrating on the

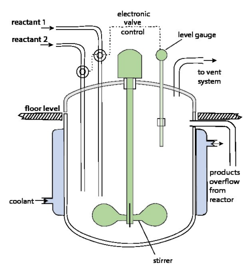


FIGURE 1. Continuous Stirred Tank Reactor (CSTR)

above model. The accompanying numerical expression is the consequence of changing over the above model:

$$RQ_1 = \frac{2.98(w + 2.25)}{(w + 1.45)(w + 2.85)^2(w + 4.35)} = -1.$$

Here, RQ_1 represents something called the gain of a proportional controller. It helps us control the system effectively. By setting RQ_1 to zero, we get another equation in term of t:

$$G_1(t) = t^4 + 11.50t^3 + 47.49t^2 + 83.06325t + 51.232668, \tag{24}$$

This equation helps us to understand how the system behaves over time. G_1 is a non-linear function with four roots $\varpi = 1.45, 2.85, -2.85, -4.35$. Let's consider $\varsigma_0 = -1.451$ as the initial approximation and $\varpi = -1.45$ as the exact root for the non-linear polynomial $G_1(t)$. The results we computed are given in Table II. We found that the developed scheme $(AMK - 1)$ works better than other methods of the existing domain when we look at the absolute difference between two iterations.

Problem 2 Van der Waals Equation (see [15]):

A key model in chemistry and engineering for figuring out a gas's specific volume is the Van der Waals equation. It explains the attraction interactions between gas molecules and the excluded volume of gas particles, two significant characteristics of real gases. The equation can be expressed as:

$$(W + \frac{iu^2}{G})(G - ub) = uQS.$$

incorporates constants i and u representing the strength of intermolecular attraction and the excluded volume, respectively. Here, G stands for the gas's volume in moles, S for temperature, W for measured pressure, and Q for the real gas constant, which has a value of $0.08206L$ atm $mol^{-1}K^{-1}$. According to the Van der Waals equation, each pressure value at a particular temperature corresponds to three specific volume values, or a cubic equation with regard to specific volume G. Upon solving equation or G, the volume of gases is obtained. The equation can be simplified to:

$$WG^3 - (ubW + uQS)G^2 + iu^2G - iu^2b = 0.$$

Through the particular value selection, the above equation can be further simplified into a form like:

$$G_2(t) = t^3 - 5.22t^2 + 9.0825t - 5.2675, \varsigma_0 = 1.719991. \tag{25}$$

$\varpi = 1.7199$ is the exact root of the function G_2. Let us assume that $\varsigma_0 = 1.72$ is an initial guess. For the sake of comparison, we conducted the analysis of numerical findings for the van der waals problem, as shown in Table III.

Table III shows the results of our developed approach $AMK-1$, and the existing iterative schemes $MK_{8b}, MK_{8a}, MFN, SJ8$. The comparison of these schemes depicts that our newly developed scheme performs better when compared to COC, and its consecutive iterations.

Example 3 Probability of a Shutout in a Racquetball (see [13]):

In a game of racquetball, the probability of achieving a shutout is determined by various factors. Primarily, it hinges on the winning probability of each rally. Scoring a point necessitates successful serving, and victory is attained upon reaching 21 points. At the game's conclusion, if the loser has not scored any points, they are considered shut out. Let's consider player A, whose winning probability for each rally is denoted as u_r. We define the probability of player A shutting out player B, denoted as U_r based on the number of points A wins while B has none. This relationship is mathematically expressed as:

$$U_r(n) = U_r(n-1)U_r(1). \tag{26}$$

Further application of equation (26) leads to

$$U_r(n) = [U_r(1)]^n. \tag{27}$$

Calculation of $U_r(1)$ involves two scenarios: either A wins the first rally and gains a point, or A loses the first rally but regains the serve after winning the second rally. This leads to the equation:

$$U_r(1) = u_r + (1-u_r)U_r(1). \tag{28}$$

On solving equation (27) the result will be as follows:

$$U_r(n) = \frac{u_r}{1-(1-u_r)u_r}. \tag{29}$$

From equations (27) and (28), we get:

$$U_r(n) = (\frac{u_r}{1-u_r+u_r^2})^n. \tag{30}$$

The probability of A shuttin out B is represented by $U_r(21)$. When A serves initially, $U_r(n)$ is given by (30).If a fair coin toss determines the initial serve, the probability that player A will serve first is $1/2$. In the scenario where player B serves initially, player A must win the serve and subsequently secure 21 points to achieve a shutout. Therefore, the probability of a shutout, denoted as U_r can be expressed as:

$$U_r = \frac{1}{2}U_r(21) + \frac{1}{2}u_r U_r(21). \tag{31}$$

Combining equations (30) and (31) we obtain the final result which is as follows:

$$U_r = \frac{1+u_r}{2}(\frac{u_r}{1-u_r+u_r^2})^{21}. \tag{32}$$

The above equation (32) is nonlinear in u. and can be written in the following expression:

$$G_3(t) = \frac{1}{2}\frac{(1+t)t^{21}}{(1-t+t^2)^{21}}. \tag{33}$$

By using equation (32), we find the roots i.e. -1 and 0 (with multiplicity 20). We choose the desired root is-1 with suitable initial guess is $\varsigma_0 = -0.95$.

The results are displayed in Table IV, highlighting the superior performance of method $AMK-1$ in terms of accuracy and COC. It converges faster compared to schemes MK_{8b}, MK_{8a},$SJ8$,MFN.

Example 4 The Parachute Problem [Ethan Retherford] A parachutist exits an aircraft from a particular height and descends due to the force of gravity. To model the motion of the parachutist, we must consider the forces acting on the body. The net force on the parachutist is described by Newton's second law of motion:

$$F = ma.$$

Where F is the net force, m is the mass of the parachutist, a is the acceleration.

The net force is the sum of two opposing forces: the downward gravitational force F_D, and the upward air resistance force F_U. The gravitational force is given by:

$$F_D = mg,$$

where g is the acceleration due to gravity. The air resistance force can be modeled as a function of the velocity v of the parachutist. For simplicity, assume the air resistance is proportional to the velocity:

$$F_U = cv,$$

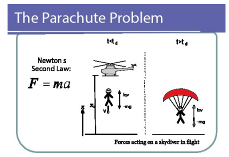


FIGURE 2. Parachute Problem

where, c is a positive constant related to the drag coefficient. The equation of motion for the parachutist can be written as:

$$m\frac{ds}{dt} = mg - cv.$$

To find the velocity $s(t)$ as a function of time t, we need to solve the following first-order linear differential equation:

$$\frac{ds}{dt} + \frac{c}{m}v = g. \tag{34}$$

The relationship above models the acceleration of a falling object considering the forces acting on it. It is expressed as a differential equation because it involves the rate of change of the velocity ($\frac{dv}{dt}$). This equation cannot be solved for the parachutist's velocity using simple algebraic methods. Solving the final equation (34) with the initial condition ($s = 0\ at\ t = 0$) we arrive at the final result given by 35:

$$s(t) = \frac{gm}{c}(1 - e^{-(\frac{c}{m})t}), \tag{35}$$

A parachutist weighing 68.1 kg leaps from a stationary hot air balloon. The drag coefficient is 12.5 kg/s, so v(t) will be in the form of equation (36):

$$s(t) = 53.44(1 - e^{-(0.18355)t}). \tag{36}$$

The Eq. 36 is the nonlinear. Using the proposed scheme, we obtain the numerical outcome to the

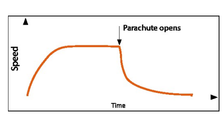


FIGURE 3. Speed time graph

above problem to determine the roots.

$$G_4(t) = 53.44 - 53.44e^{-0.18355t}, \ \varpi = -0.209899. \tag{37}$$

The parachutes problem showcases the effectiveness and efficiency of the newly developed iterative methods via numerical comparison. The results displayed in Table V confirm the exceptional performance of our proposed method within the manuscript. For the function G_5, the root is $\varpi = -.209899$ with the initial point $\varsigma_0 = 0$.

Example 5 To determine the performance of the iterative scheme, we examine a specific 3^{rd} degree nonlinear polynomial:

$$G_5(t) = t^3 - 1.$$

This example ensures the consistency and stability of the results. The exact root of the nonlinear function $G_5(t)$ is $\varpi = 1$,and we start with an initial guess of $\varsigma_0 = 1.01$.

The results presented in Table VI demonstrate that the newly proposed derivative-free method, $AMK-1$, outperforms the other methods, indicating the superior efficiency of $AMK-1$ compared to existing techniques..

Example 6 Our developed scheme surpasses other existing methods for the test function

$$G_6(t) = e^{-t} - 1 + \frac{1}{5}t.$$

The approximate roots are 4.96 and 0.

We have choosen $\varpi = 0$ as root and set initial point to $\varsigma_0 = 0.05$.The results, presented in Table VII, demonstrate that our scheme AMK-1 outperforms the others in terms of consecutive iterations difference.

The data consistently showed that $AMK-1$ not only converges faster but also maintains greater accuracy and stability. These advantages make our method a robust choice for solving similar equations.

Problem 7 We chose a non-linear test function:

$$G_7(t) = e^{t^2-3t}\sin(t) + \ln(t^2+1), \tag{38}$$

that contains the transcendental function. The approximate root for (38) is $\varpi = 0$. Let's consider $\varsigma_0 = 0.5$ as an initial root to get the numerical results, as presented in Table VIII.

The findings showcased in Table VIII highlight the superior performance of the newly proposed methods $AMK-1$ over previously established methods like MK_{8b}, MK_{8a},$SJ8$, and MFN in both computational efficiency and accuracy when applied to the function $G_7(t)$.

TABLE I

TEST FUNCTIONS WITH THE ROOTS

Test Function	Exact root
$G_1(t) = t^4 + 11.50t^3 + 47.49t^2 + 83.06325t + 51.232668$	$\varpi = -1.45$
$G_2(t) = t^3 - 5.22t^2 + 9.0825t - 5.2675$	$\varpi = 1.7199$
$G_3(t) = \frac{1}{2}\frac{(1+t)t^{21}}{(1-t+t^2)^{21}}$	$\varpi = -1$
$G_4(t) = 53.44 - 53.44e^{-0.18355t}$	$\varpi = -0.209899$
$G_5(t) = t^3 - 1$	$\varpi = 1$
$G_6(t) = e^{-t} - 1 + \frac{1}{5}t$	$\varpi = 0$
$G_7(t) = e^{t^2-3t}\sin(t) + \ln(t^2+1)$	$\varpi = 0$

TABLE II

COMPARISON TABLE FOR $G_1(t)$

$G_1(t)$, $\varpi = -1.45$, $\varsigma_0 = -1.451$					
Schemes	$\|t_2 - t_1\|$	$\|t_3 - t_2\|$	$\|t_4 - t_3\|$	$\|G(t_n)\|$	COC
$AMK-1$	$4.08(-23)$	$3.09(-178)$	$3.41(-1419)$	$1.94(-1418)$	8
MK_{8b}	$1.00(-3)$	$3.04(-17)$	$3.04(-17)$	$5.75(-990)$	8
$SJ8$	$5.96(-21)$	$8.90(-159)$	$2.20(-1261)$	$1.25(-1260)$	8
MK_{8a}	$1.00(-3)$	$7.32(-19)$	$7.32(-19)$	$4.80(-1108)$	8
MFN	$1.00(-3)$	$6.14(-21)$	$1.16(-158)$	$1.07(-1259)$	8

TABLE III

COMPARISON TABLE FOR $G_2(t)$

$G_2(t)$, $\varpi = 1.72$, $\varsigma_0 = 1.7120$					
Schemes	$\|t_2 - t_1\|$	$\|t_3 - t_2\|$	$\|t_4 - t_3\|$	$\|G(t_n)\|$	COC
$AMK-1$	$5.36(-6)$	$3.74(-30)$	$2.08(-223)$	$1.87(-226)$	7.99
MK_{8b}	$7.96(-3)$	$2.50(-5)$	$2.50(-5)$	$2.37(-171)$	7.23
$SJ8$	$1.73(-5)$	$1.71(-25)$	$1.48(-185)$	$1.33(-188)$	7.99
MK_{8a}	$7.99(-3)$	$7.11(-6)$	$7.11(-6)$	$5.94(-215)$	7.50
MFN	$7.98(-3)$	$1.77(-5)$	$3.73(-25)$	$1.30(-185)$	7.41

TABLE IV

COMPARISON TABLE FOR $G_3(t)$

$G_3(t)$, $\varpi = -1$, $\varsigma_0 = -0.95$					
Schemes	$\lvert t_2 - t_1\rvert$	$\lvert t_3 - t_2\rvert$	$\lvert t_4 - t_3\rvert$	$\lvert G(t_n)\rvert$	COC
$AMK-1$	$2.53(-9)$	$1.06(-82)$	$1.68(-816)$	$8.07(-827)$	9.99
MK_{8b}	$4.99(-2)$	$1.79(-8)$	$1.79(-8)$	$1.23(-601)$	9.04
$SJ8$	$3.20(-9)$	$1.09(-81)$	$2.25(-806)$	$1.08(-816)$	9.99
MK_{8a}	$5.00(-2)$	$2.63(-9)$	$2.63(-9)$	$2.54(-825)$	10.06
MFN	$5.00(-2)$	$4.64(-9)$	$4.51(-80)$	$1.58(-800)$	10.09

TABLE V

COMPARISON TABLE FOR $G_4(t)$

$G_4(t)$, $\varpi = 0$, $\varsigma_0 = -0.209899$					
Schemes	$\lvert t_2 - t_1\rvert$	$\lvert t_3 - t_2\rvert$	$\lvert t_4 - t_3\rvert$	$\lvert G(t_n)\rvert$	COC
$AMK-1$	$2.01(-7)$	$3.49(-62)$	$1.36(-500)$	$1.33(-499)$	8
MK_{8b}	$2.09(-1)$	$7.35(-7)$	$7.35(-7)$	$1.39(-395)$	7.93
$SJ8$	$3.21(-1)$	19.84	D	D	D
MK_{8a}	$2.09(-1)$	$2.27(-10)$	$2.27(-10)$	$1.97(-814)$	8
MFN	$3.12(-1)$	$1.03(-1)$	$8.73(-6)$	$2.54(-42)$	8

D stands for Divergence.

TABLE VI

COMPARISON TABLE FOR $G_5(t)$

$G_5(t)$, $\varpi = 1$, $\varsigma_0 = 1.01$					
Schemes	$\lvert t_2 - t_1\rvert$	$\lvert t_3 - t_2\rvert$	$\lvert t_4 - t_3\rvert$	$\lvert G(t_n)\rvert$	COC
$AMK-1$	$6.36(-17)$	$1.80(-130)$	$7.44(-1039)$	$2.23(-1038)$	7.99
MK_{8b}	$9.99(-3)$	$2.23(-12)$	$2.23(-12)$	$9.17(-706)$	7.98
$SJ8$	$3.86(-15)$	$2.35(-114)$	$4.48(-908)$	$1.35(-907)$	7.99
MK_{8a}	$9.99e-3$	$7.74(-14)$	$7.74(-14)$	$9.84(-813)$	7.99
MFN	$9.99(-3)$	$4.03(-15)$	$3.53(-114)$	$3.66(-906)$	7.99

TABLE VII

COMPARISON TABLE FOR $G_6(t)$

$G_6\,(t)\,,\,\varpi=0,\,\varsigma_0=0.016$					
Schemes	$\lvert t_2-t_1\rvert$	$\lvert t_3-t_2\rvert$	$\lvert t_4-t_3\rvert$	$\lvert G(t_n)\rvert$	COC
$AMK-1$	$4.74(-17)$	$2.72(-133)$	$3.19(-1063)$	$2.55(-1063)$	8
MK_{8b}	$1.6(-2)$	$2.52(-18)$	$2.52(-18)$	$6.06(-1157)$	8
$SJ8$	$6.23(-17)$	$2.87(-132)$	$5.78(-1055)$	$4.63(-1055)$	8
MK_{8a}	$1.6(-2)$	$1.30(-19)$	$1.30(-19)$	$7.11(-1251)$	8
MFN	$1.6(-2)$	$3.1(-16)$	$5.81(-126)$	$7.01(-1004)$	8

TABLE VIII

COMPARISON TABLE FOR $G_7(t)$

$G_7\,(t)\,,\,\varpi=0,\,\varsigma_0=0.5$					
Schemes	$\lvert t_2-t_1\rvert$	$\lvert t_3-t_2\rvert$	$\lvert t_4-t_3\rvert$	$\lvert G(t_n)\rvert$	COC
$AMK-1$	$3.54(-5)$	$1.12(-34)$	$1.09(-270)$	$1.09(-270)$	8.00
MK_{8b}	$4.95(-1)$	$4.75(-3)$	$4.75(-3)$	$9.91(-107)$	5.71
$SJ8$	$7.22(-4)$	$7.25(-24)$	$7.46(-184)$	$7.46(-184)$	7.99
MK_{8a}	$4.98(-1)$	$1.43(-3)$	$1.43(-3)$	$1.49(-147)$	6.28
MFN	$5.00(-1)$	$3.61(-4)$	$1.17(-26)$	$1.41(-206)$	7.15

5. Stereographic Projection

Stereographic projection is a method for portraying the sphere into a flat surface. This technique is used in various areas of mathematics. This is an excellent resource that is used for the polar areas in conjunction with small-scale maps to determine the relationship between those frameworks and crystal planes as well as to address orientation issues in structural geology. To do this, we often use a special type of graph paper called a stereographic net, or Wulff net. Stereography helps us the three dimensional images.

If we envision connecting a point on the surface of the sphere, denoted as P, to the South Pole, S, then drawing a line from S to P that intersects the equatorial plane at a point designated as p, p represents the stereographic projection of P. The concept of stereographic projection was first introduced by the ancient

Figure 4. Stereographic Projection

Egyptian scientists Hipparchus and Ptolemy. Initially, it was known as the plano-sphere projection.

Today, computer applications designed for stereographic projections are compatible with modern computing systems. These projections have the capability to visualize any pole or orientation on any projection with any layout. Additionally, they can create a spherical shell in the normal direction to the plane.

We utilize a variety of non-linear complex functions to visualize the convergence regions of both the developed scheme and existing schemes of the same domain through stereographic projections. The functions are listed below:

$$\begin{aligned} v_1(\gamma) &= \gamma^3 - 1, \\ v_2(\gamma) &= \gamma^5 - 1, \\ v_3(\gamma) &= \gamma^6 - 1. \end{aligned}$$

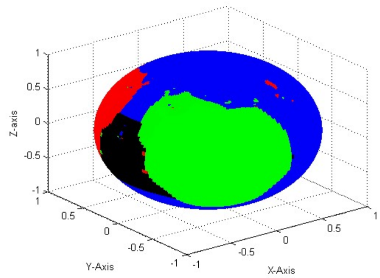
We used MATLAB R2014a on a computer program to create stereographic images, with a maximum of 60 iterations and a resolution of 200. The roots of each polynomial are colored differently. For example, the stereographic image of $v_1(\gamma)$ displays three colors representing its three different roots, with black indicating divergence. Similarly, $v_2(\gamma)$ exhibits five distinguishable colors, and $v_3(\gamma)$ shows six root colors in its stereographic images.

The intensity of the color indicates the convergence behavior of the iterative root-finding sequence. Vibrant colors indicate convergence within a limited number of iterations, whereas black represents the divergence. In the figures, we used a lower value of -2 and a higher value of 2 with a 0.1 increment for polynomials of degree 3^{rd}, 5^{th}, and 6^{th}.

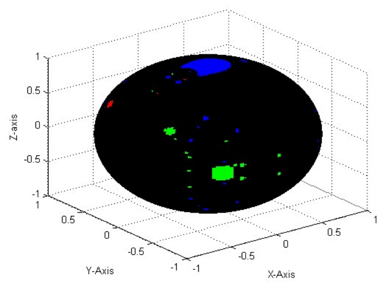
Figures $5-7$, provide visual representations of the dynamic behavior observed in the derivative-free iterative scheme $(AMK-1)$ alongside the already existing eighth order schemes by Obadah et al. [25] referred to as (MK_{8a}) and (MK_{8b}) and by zafar et al. [12] referred to as (MFN).

For the complex polynomial $v_1(\gamma)$, our new scheme $(AMK-1)$ exhibits wider and smoother zones in contrast to (MK_{8a}), (MK_{8b}) and (MFN) , as depicted in figures 5. This trend is consistent for $v_2(\gamma)$, where five roots are symbolized. It is clear that our scheme $(AMK-1)$ consistently shows larger convergence regions in all colors, as shown in Figures 6. Similarly, for complex polynomial $v_3(\gamma)$ with six colors visible. Our scheme $(AMK-1)$ performs better than other schemes, as seen in figures 7. Therefore, we conclude that our newly developed scheme $(AMK-1)$ exhibits greater efficiency, and robustness than (MK_{8a}), (MK_{8b}) . and (MFN), significantly outperforming them in convergence.

Moreover, newly proposed scheme exhibits faster convergence and fewer divergences. This visualization is evident in the polynomials $v_1(\gamma)$, $v_2(\gamma)$, and $v_3(\gamma)$. Considering these observations, it becomes apparent that the developed eighth-order iterative scheme is more consistent, as its dynamical planes are characterized by fewer black regions than the other schemes (MK_{8a}) , (MK_{8b}) and (MFN).

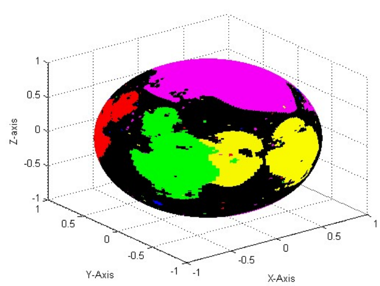


(A) $AMK-1$ (B) MK_{8a}

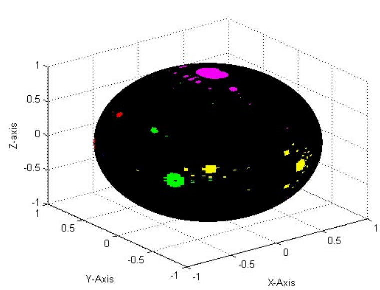


(C) MK_{8b} (D) MFN

FIGURE 5. Stereographic projection on 3rd degree complex function $v_1(\gamma)$

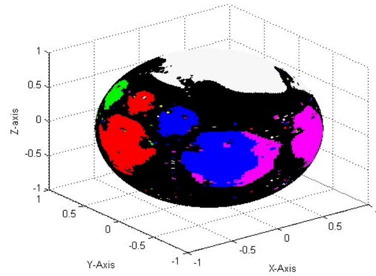


(A) $AMK-1$ (B) MK_{8a}

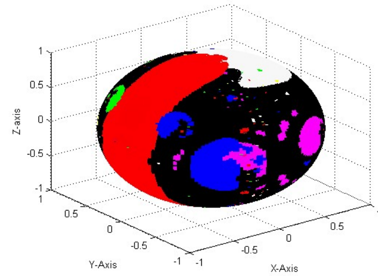


(C) MK_{8b} (D) MFN

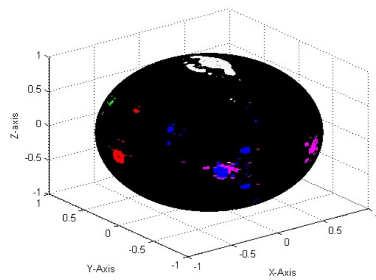
FIGURE 6. Stereographic projection on 3rd degree complex function $v_2(\gamma)$



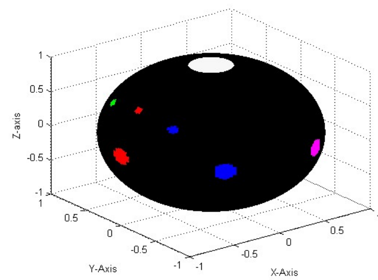
(A) $AMK - 1$



(B) MK_{8a}



(C) MK_{8b}



(D) MFN

FIGURE 7. Stereographic projection on 3rd degree complex function $v_3(\gamma)$

6. Concluding Remarks

This manuscript introduces a new variant of Ostrowski's method for finding the simple root of a non-linear equation. The new method is an eighth-order, three-step, derivative-free iterative approach designed to improve both accuracy and efficiency in solving non-linear equations. The convergence of this method has been carefully analyzed, confirming its optimal performance. All computations were carried out using Maple 18 and MATLAB 2014a, ensuring the reliability of the results. Comparisons with other methods show that the proposed approach is faster and has better convergence properties. The numerical results and visualizations confirm that this method is a strong alternative to existing approaches. Additionally, its ability to handle complex functions with minimal effort makes it a valuable tool for both theoretical and practical applications.

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Methodology, S.A. (Saima Akram), F.A. ; Software, S.A.; Formal analysis, S.A., M.K.; S.H.; Investigation, M.K. S.A., F.A.; Resources, S.A.; Data curation, S.A.; Writing original draft, M.K. and S.M. F.A.;

Writing review and editing, S.A and S.M. F.A.; Visualization, M.K.; Supervision, S.A. All authors have read and agreed to the published version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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