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PARAMETRIC OPERATIONS FOR TWO 3-DIMENSIONAL TRAPEZOIDAL FUZZY SETS

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Abstract. We extend the parametric operator for trapezoidal fuzzy sets to three dimensions. We define and calculate parametric operators between 3D trapezoidal fuzzy sets and present the results in a graph. Since the graph is defined in 3D space and drawn in 4D, it cannot be represented in 3D. The presented graph is drawn in three dimensions using a special definition of a fuzzy number in which the membership function's value ranges from 0 to 1. The membership function's value at each point is expressed as the color intensity at that point. If you cut the graph into a plane passing through the longest axis, you can observe that different function values on the plane are represented by colors of varying intensities. As it is a trapezoidal fuzzy set, a certain portion of the center shares the same color. By presenting this graph, the results will be cited and applied in various areas, similar to the one-dimensional and two-dimensional results.

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1. Introduction

In fuzzy theory, research on various types of fuzzy numbers is important. Although there has been a lot of research on triangular fuzzy numbers, there is not much research on trapezoidal fuzzy sets or pentagonal fuzzy numbers. We have studied research on extension operators for generalized trapezoidal fuzzy sets [1]. The research has been cited in control tools [2], applications to circuit analysis [3], fuzzy logic [4], decision-making technique [5], type-2 trapezoidal fuzzy number [6], decomposition theorem [7], and fuzzy numbers [8]. Research on pentagonal fuzzy numbers [9] has also been cited in forecasting and decision-making [10], optimization [11], order quantity model [12], pentagonal fuzzy numbers [13]. heptagonal fuzzy numbers [14], algorithms [15, 16], and fuzzy mathematical model [17].

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The results obtained in one dimension were extended to two dimensions. The study involved the calculation of one-dimensional expansion operators for fuzzy numbers using an alpha cut. For the one-dimensional case, we computed the boundary of the alpha cut and derived the alpha cut of the fuzzy number through algebraic operations. Subsequently, the fuzzy number was determined through the inversion of the obtained alpha cut. However, in the case of 2D, the alpha cut manifests as a convex set of planes, making it impractical to perform algebraic computations on the boundary. Consequently, we introduced a new operation called the parameter operator. In the one-dimensional scenario, it was demonstrated that the results obtained by defining parameter operators were in line with the outcomes achieved through the utilization of existing algebraic operators. Research on parametric operators between 2D trapezoidal fuzzy sets have also been published [18] and are widely cited and applied.

The 2D triangular fuzzy number can be represented as a cone-shaped graph in 3D space. When a vertical plane passes through the vertex and cuts the cone, the section on the plane forms a one-dimensional triangular fuzzy number. Similarly, two-dimensional trapezoidal fuzzy sets can be illustrated as truncated cone-shaped graphs in three-dimensional space. By cutting it with a plane perpendicular to the upper plane, a one-dimensional trapezoidal fuzzy set can be derived from the intersecting surface. It can be observed that if the results of calculating the parametric operator on a 2-dimensional truncated cone are constrained to 1 dimension, they are consistent with the 1-dimensional results. Various findings were obtained for the 3-dimensional triangular fuzzy number [19, 20], and akin to the one and two-dimensional findings, these will be referenced and utilized across various fields.

In this paper, we present parametric operators for trapezoidal fuzzy sets extended to three dimensions. This enables operations between fuzzy sets in three-dimensional space. We visualize the results of these operations through graphs, using a specific definition of fuzzy numbers to clearly represent them in three-dimensional space. These findings can be utilized in the realms of fuzzy set theory and applications, and the visualization of fuzzy set characteristics in three-dimensional graphs can aid in understanding.

2. The generalized trapezoidal fuzzy sets on $\mathbb{R}$

We define α-cut and α-set of the fuzzy set A with the membership function $\mu_A(x)$.

Definition 2.1. [21] An *α-cut* of the fuzzy number A is defined by $A_\alpha = \{x \in \mathbb{R} \mid \mu_A(x) \geq \alpha\}$ if $\alpha \in (0,1]$ and $A_\alpha = \mathrm{cl}\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ if $\alpha = 0$. For $\alpha \in (0,1)$, the set $A^\alpha = \{x \in X \mid \mu_A(x) = \alpha\}$ is said to be the *α-set* of the fuzzy set A, A^0 is the boundary of $\{x \in \mathbb{R} \mid \mu_A(x) > \alpha\}$ and $A^1 = A_1$.

Definition 2.2. [21] The extended addition $A(+)B$, extended subtraction $A(-)B$, extended multiplication $A(\cdot)B$, and extended division $A(/)B$ are fuzzy sets with membership functions as follows: For all $x \in A$ and $y \in B$,

$$\mu_{A(*)B}(z) = \sup_{z=x*y} \min\{\mu_A(x), \mu_B(y)\},\ * = +, -, \cdot, /$$

We generalized the results of four operations for two generalized trapezoidal fuzzy sets. We employed four operations: addition $A(+)B$, subtraction $A(-)B$, multiplication $A(\cdot)B$, and division $A(/)B$ for generalized trapezoidal fuzzy sets A and B. These operations for two fuzzy numbers (A, μ_A) and (B, μ_B) are defined in Definition 2.2 and are based on the Zadeh's extension principle [22-24]. Addition $A(+)B$ and subtraction $A(-)B$ result generalized trapezoidal fuzzy sets. However, multiplication $A(\cdot)B$ and division $A(/)B$ are not necessarily generalized trapezoidal fuzzy sets.

Definition 2.3. [1] A fuzzy set A having membership function

$$\mu_A(x) = \begin{cases} 0, & x < a_1, a_4 \le x \\ \frac{m(x-a_1)}{a_2-a_1}, & a_1 \le x < a_2 \\ m, & a_2 \le x < a_3 \\ \frac{m(a_4-x)}{a_4-a_3}, & a_3 \le x < a_4 \end{cases}$$

where $a_i \in \mathbb{R}, i = 1, 2, 3, 4$ and $0 < m < 1$, is called *a generalized trapezoidal fuzzy set* and will be denoted by $A = (a_1, a_2, m, a_3, a_4)$.

Theorem 2.4. [1] *Let* $A = (a_1,\ a_2,\ m_1,\ a_3,\ a_4)$ *and* $B = (b_1,\ b_2,\ m_2,\ b_3,\ b_4)$ *be two generalized trapezoidal fuzzy sets. Then we have the followings.*
(1) *The membership function* $\mu_{A(+)B}(z)$ *is*

$$\begin{cases} 0, & z < a_1 + b_1, a_4 + b_4 \le z \\ \frac{m_1 m_2 (z-a_1-b_1)}{m_2(a_2-a_1)+m_1(b_2-b_1)}, & a_1 + b_1 \le z < a_2 + b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2} \\ m_1, & a_2 + b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2} \le z \\ & \qquad < a_3 + b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2} \\ \frac{m_1 m_2 (a_4+b_4-z)}{m_2(a_4-a_3)+m_1(b_4-b_3)}, & a_3 + b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2} \le z < a_4 + b_4 \end{cases}$$

i.e. $A(+)B$ *is a generalized trapezoidal fuzzy set.*
(2) *The membership function* $\mu_{A(-)B}(z)$ *is*

$$\begin{cases} 0, & z < a_1 - b_4, a_4 - b_1 \le z \\ \frac{m_1 m_2 (z+b_4-a_1)}{m_2(a_2-a_1)+m_1(b_4-b_3)}, & a_1 - b_4 \le z < a_2 - \left(b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2}\right) \\ m_1, & a_2 - \left(b_4 - (b_4 - b_3) \cdot \frac{m_1}{m_2}\right) \le z \\ & \qquad < a_3 - \left(b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2}\right) \\ \frac{m_1 m_2 (a_4-b_1-z)}{m_2(a_4-a_3)+m_1(b_2-b_1)}, & a_3 - \left(b_1 + (b_2 - b_1) \cdot \frac{m_1}{m_2}\right) \le z < a_4 - b_1 \end{cases}$$

i.e. $A(-)B$ *is a generalized trapezoidal fuzzy set.*

(3) *The membership function* $\mu_{A(\cdot)B}(z)$ *is*

$$\begin{cases} 0, & z < a_1b_1, a_4b_4 \le z \\ \frac{-D_1+\sqrt{D^2+4m_1m_2(b_2-b_1)(a_2-a_1)z}}{2(b_2-b_1)(a_2-a_1)}, & a_1b_1 \le z < a_2\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big) \\ m_1, & a_2\big(b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}\big) \le z \\ & \quad < a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big) \\ \frac{\widetilde{D}_1-\sqrt{\widetilde{D}^2+4m_1m_2(b_4-b_3)(a_4-a_3)z}}{2m_1(b_4-b_3)}, & a_3\big(b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}\big) \le z < a_4b_4 \end{cases}$$

where

$$\begin{aligned} D &= b_1m_2(a_2-a_1) - a_1m_1(b_2-b_1) \\ D_1 &= b_1m_2(a_2-a_1) + a_1m_1(b_2-b_1) \\ \widetilde{D} &= a_4m_1(b_4-b_3) - b_4m_2(a_4-a_3) \\ \widetilde{D}_1 &= a_4m_1(b_4-b_3) + b_4m_2(a_4-a_3) \end{aligned}$$

i.e. $A(\cdot)B$ *is a fuzzy set on* $\big(a_1b_1, a_4b_4\big)$*, but need not to be a generalized trapezoidal fuzzy set.*

(4) *The membership function* $\mu_{A(/)B}(z)$ *is*

$$\mu_{A(/)B}(z) = \begin{cases} 0, & z < \frac{a_1}{b_4}, \frac{a_4}{b_1} \le z \\ \frac{m_1m_2(b_4z-a_1)}{m_1(b_4-b_3)z+m_2(a_2-a_1)}, & \frac{a_1}{b_4} \le z < \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \\ m_1, & \frac{a_2}{b_4-(b_4-b_3)\cdot\frac{m_1}{m_2}} \le z < \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \\ \frac{m_1m_2(a_4-b_1z)}{m_1(b_2-b_1)z+m_2(a_4-a_3)}, & \frac{a_3}{b_1+(b_2-b_1)\cdot\frac{m_1}{m_2}} \le z < \frac{a_4}{b_1} \end{cases}$$

i.e. $A(/)B$ *is a fuzzy set on* $\left(\frac{a_1}{b_4}, \frac{a_4}{b_1}\right)$*, but need not to be a generalized trapezoidal fuzzy set.*

We provided graphs and examples of the results in [1].

3. Parametric operations for two 2-dimensional trapezoidal fuzzy sets

In this section, we generalize the trapezoidal fuzzy sets on $\mathbb{R}$ to $\mathbb{R}^2$. We calculate the parametric operations for two 2-dimensional trapezoidal fuzzy sets and provide an example, illustrating the results of the example.

Definition 3.1. [19] A fuzzy set A with a membership function

$$\mu_A(x,y) = \begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}}, & h-1 \le \sqrt{\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}} \le h, \\ 1, & 0 \le \sqrt{\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}} \le h-1, \\ 0, & \text{otherwise}, \end{cases}$$

where $a, b > 0$ and $1 < h$ is called *the 2-dimensional trapezoidal fuzzy set* and denoted by $A = ((a,\ x_1,\ h,\ b,\ y_1))^2$.

$\mu_A(x, y)$ forms a truncated cone. The intersections of $\mu_A(x, y)$ with the horizontal planes $z = \alpha\ \ (0 < \alpha < 1)$ result in ellipses. The intersections of $\mu_A(x, y)$ with the vertical planes $y - y_1 = k(x - x_1)\ \ (k \in \mathbb{R})$ are symmetric trapezoidal fuzzy sets in those planes. If $a = b$, the ellipses become circles. The α-cut A_α of a 2-dimensional trapezoidal fuzzy number $A = ((a,\ x_1,\ h,\ b,\ y_1))^2$ is the interior of an ellipse in the xy-plane including its boundary

$$\begin{aligned} A_\alpha &= \left\{ (x, y) \in \mathbb{R}^2 \;\middle|\; b^2(x - x_1)^2 + a^2(y - y_1)^2 \le a^2 b^2 (h - \alpha)^2 \right\} \\ &= \left\{ (x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1}{a(h - \alpha)} \right)^2 + \left(\frac{y - y_1}{b(h - \alpha)} \right)^2 \le 1 \right\}. \end{aligned}$$

Theorem 3.2. [18] *Let $A = ((a_1,\ x_1,\ h_1,\ b_1,\ y_1))^2$ and $B = ((a_2,\ x_2,\ h_2,\ b_2, y_2))^2$ be two 2-dimensional trapezoidal fuzzy sets. Then we have the followings.*

(1) *For $0 < \alpha < 1$, the α-set of $A(+)_p B$ is*

$$(A(+)_p B)^\alpha = \left\{ (x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)} \right)^2 + \left(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)} \right)^2 = 1 \right\}.$$

(2) *For $0 < \alpha < 1$, the α-set of $A(-)_p B$ is*

$$(A(-)_p B)^\alpha = \left\{ (x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)} \right)^2 + \left(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)} \right)^2 = 1 \right\}.$$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = x_1 x_2 + (x_1 a_2(h_2 - \alpha) + x_2 a_1(h_1 - \alpha)) \cos t + a_1 a_2 (h_1 - \alpha)(h_2 - \alpha) \cos^2 t, \quad 0 < \alpha < 1,$$

$$y_\alpha(t) = y_1 y_2 + (y_1 b_2(h_2 - \alpha) + y_2 b_1(h_1 - \alpha)) \sin t + b_1 b_2 (h_1 - \alpha)(h_2 - \alpha) \sin^2 t, \quad 0 < \alpha < 1.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \le t \le 2\pi\}$, *where*

$$x_\alpha(t) = \frac{x_1 + a_1(h_1 - \alpha)\cos t}{x_2 - a_2(h_2 - \alpha)\cos t}, \quad y_\alpha(t) = \frac{y_1 + b_1(h_1 - \alpha)\sin t}{y_2 - b_2(h_2 - \alpha)\sin t}, \quad 0 < \alpha < 1.$$

Example 3.3. [18] For two 2-dimensional trapezoidal fuzzy sets $A = ((2,\ 13,\ 3,\ 3,\ 11))^2$ and $B = ((4,\ 9,\ 4,\ 3,\ 8))^2$, we have the followings.

(1) For $0 < \alpha < 1$, the α-set of $A(+)_p B$ is

$$(A(+)_p B)^\alpha = \left\{ (x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - 22}{22 - 6\alpha} \right)^2 + \left(\frac{y - 19}{21 - 6\alpha} \right)^2 = 1 \right\}.$$

(2) For $0 < \alpha < 1$, the α-set of $A(-)_p B$ is

$$(A(-)_p B)^\alpha = \left\{ (x, y) \in \mathbb{R}^2 \middle| \left(\frac{x - 4}{22 - 6\alpha} \right)^2 + \left(\frac{y - 3}{21 - 6\alpha} \right)^2 = 1 \right\}.$$

(3) $(A(\cdot)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = 117 + (262 - 70\alpha)\cos t + 8(3-\alpha)(4-\alpha)\cos^2 t, \quad 0 < \alpha < 1,$$

$$y_\alpha(t) = 88 + (204 - 57\alpha)\sin t + 9(3-\alpha)(4-\alpha)\sin^2 t, \quad 0 < \alpha < 1.$$

(4) $(A(/)_p B)^\alpha = \{(x_\alpha(t), y_\alpha(t)) \mid 0 \leq t \leq 2\pi\}$, where

$$x_\alpha(t) = \frac{13 + 2(3-\alpha)\cos t}{9 - 4(4-\alpha)\cos t}, \quad y_\alpha(t) = \frac{11 + 3(3-\alpha)\sin t}{8 - 3(4-\alpha)\sin t}, \quad 0 < \alpha < 1.$$

Since we cannot find the concrete forms of membership functions of $\mu_{A(+)B}(x)$, $\mu_{A(-)B}(x)$, $\mu_{A(\cdot)B}(x)$, and $\mu_{A(/)B}(x)$, we are unable to draw the graphs of their membership functions. However, we have proven that the functions f_i and $g_i (i = +, -, \cdot, /)$ are one to one correspondence. Thus we ascertain the unique existence of membership functions. The Mathematica commands to obtain the graphs below are as follows.

A1 := Plot3D[1, {x, 5, 25}, {y, 0, 20}, RegionFunction → Function[{x, y, z}, 0 ≤ Sqrt[((x - 13)^2)/4 + ((y - 11)^2)/9] ≤ 2]]

A2 := Plot3D[3 - Sqrt[((x - 13)^2)/4 + ((y - 11)^2)/9], {x, 5, 25}, {y, 0, 20}, RegionFunction → Function[{x, y, z}, 2 ≤ Sqrt[((x - 13)^2)/4 + ((y - 11)^2)/9] ≤ 3]]

Show[A1, A2]

B1 := Plot3D[1, {x, -10, 25}, {y, -5, 20}, RegionFunction → Function[{x, y, z}, 0 ≤ Sqrt[((x - 9)^2)/16 + ((y - 8)^2)/9] ≤ 3]]

B2 := Plot3D[4 - Sqrt[((x - 9)^2)/16 + ((y - 8)^2)/9], {x, -10, 25}, {y, -5, 20}, RegionFunction → Function[{x, y, z}, 3 ≤ Sqrt[((x - 9)^2)/16 + ((y - 8)^2)/9] ≤ 4]]

Show[B1, B2]

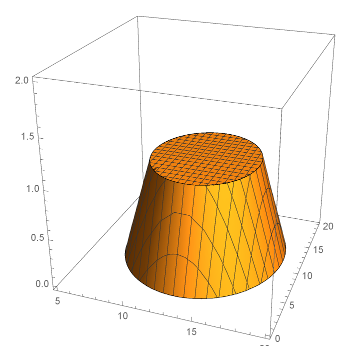

FIGURE 1. $\mu_A(x)$

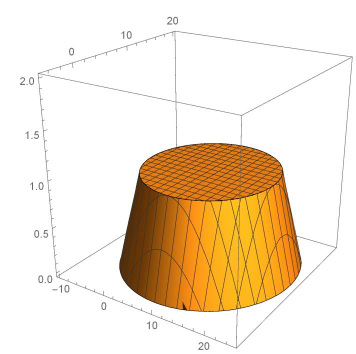

FIGURE 2. $\mu_B(x)$

ContourPlot3D[((x - 22)/(22 - 6 a))^2 + ((y - 19)/(21 - 6 a))^2 == 1, {x, 0, 50}, {y, -5, 50}, {a, 0, 1}]

ContourPlot3D[((x - 4)/(22 - 6 a))^2 + ((y - 3)/(21 - 6 a))^2 == 1, {x, -30, 30}, {y, -30, 30}, {a, 0, 1}]

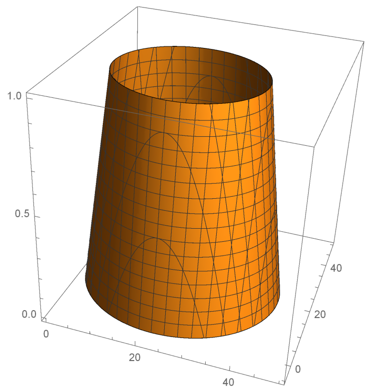

FIGURE 3. $\mu_{A(+)B}(x)$

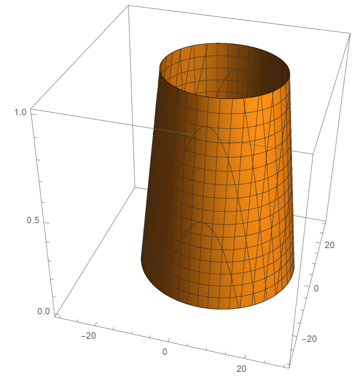

FIGURE 4. $\mu_{A(-)B}(x)$

ParametricPlot3D[{117 + (262 - 70 u) Cos[t] + 8 (3 - u) (4 - u) Cos[t]^2, 88 + (204 - 57 u) Sin[t] + 9 (3 - u) (4 - u) Sin[t]^2, u}, {t, 0, 2 Pi}, {u, 0, 1}, BoxRatios → {1, 1, 1}]

ParametricPlot3D[{(13 + (6 - 2 u) Cos[t])/(9 - (16 - 4 u) Cos[t]), (11 + (9 - 3 u) Sin[t])/(8 - (12 - 3 u) Sin[t]), u}, {t, 0, 2 Pi}, {u, 0, 1}, BoxRatios → {1, 1, 1}]

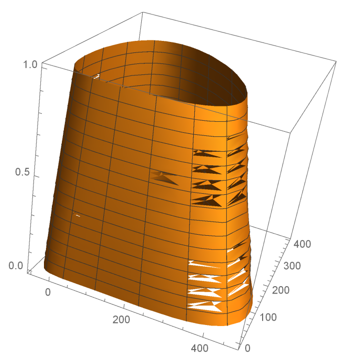

FIGURE 5. $\mu_{A(\cdot)B}(x)$

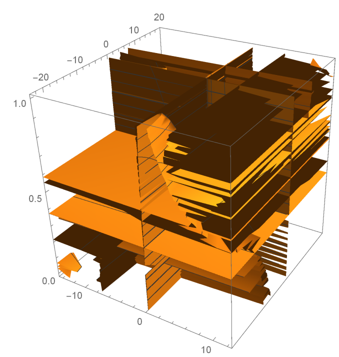

FIGURE 6. $\mu_{A(/)B}(x)$

4. Parametric operations for two 3-dimensional trapezoidal fuzzy sets

In this section, we define 3-dimensional triangular fuzzy numbers on $\mathbb{R}^3$ as a generalization of triangular fuzzy numbers on $\mathbb{R}^2$. We then define parametric operations between two 3-dimensional fuzzy numbers. To achieve this, we have to calculate operations between α-cuts in $\mathbb{R}^3$. While the α-cuts are regions in $\mathbb{R}^2$, in $\mathbb{R}^3$, they are subsets of $\mathbb{R}^3$, rendering the existing method of calculations between α-cuts inapplicable. We reinterpret the existing method from a different perspective and apply it to the subset valued α-cuts on $\mathbb{R}^3$. Additionally, we define 3-dimensional trapezoidal fuzzy sets on $\mathbb{R}^3$ as a generalization of trapezoidal fuzzy sets on $\mathbb{R}^2$ and calculate parametric operations for two 3-dimensional trapezoidal fuzzy sets on $\mathbb{R}^3$, providing an example.

Definition 4.1. [19] A fuzzy set A with a membership function

$$\mu_A(x,y,z)=\begin{cases}1-\sqrt{\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}+\frac{(z-z_1)^2}{c^2}}, & \text{if } b^2c^2(x-x_1)^2+c^2a^2(y-y_1)^2\\ & \quad +a^2b^2(z-z_1)^2\leq a^2b^2c^2,\\ 0, & \text{otherwise,}\end{cases}$$

where $a,b,c>0$ is called the *3-dimensional triangular fuzzy number* and denoted by $(a,\ x_1,\ b,\ y_1,\ c,\ z_1)^3$.

Note that $\mu_A(x,y)$ is a cone in $\mathbb{R}^2$, but it is not possible to determine the shape of $\mu_A(x,y,z)$ in $\mathbb{R}^3$. The α-cut A_α of a 3-dimensional triangular fuzzy number $A=(a,x_1,b,y_1,c,z_1)^3$ is the following set:

$$\begin{aligned}A_\alpha&=\Big\{(x,y,z)\in\mathbb{R}^3\Big|\frac{(x-x_1)^2}{a^2}+\frac{(y-y_1)^2}{b^2}+\frac{(z-z_1)^2}{c^2}\leq(1-\alpha)^2\Big\}\\&=\Big\{(x,y,z)\in\mathbb{R}^3\Big|\Big(\frac{x-x_1}{a(1-\alpha)}\Big)^2+\Big(\frac{y-y_1}{b(1-\alpha)}\Big)^2+\Big(\frac{z-z_1}{c(1-\alpha)}\Big)^2\leq 1\Big\}.\end{aligned}$$

Definition 4.2. [19] A 3-dimensional fuzzy number A defined on $\mathbb{R}^3$ is called a *convex* fuzzy number if, for all $\alpha\in(0,1)$, the α-cuts

$$A_\alpha=\{(x,y,z)\in\mathbb{R}^3|\mu_A(x,y,z)\geq\alpha\}$$

are convex subsets in $\mathbb{R}^3$.

Theorem 4.3. [20] *Let A be a continuous convex fuzzy number defined on $\mathbb{R}^3$ and let $A^\alpha=\{(x,y,z)\in\mathbb{R}^3|\mu_A(x,y,z)=\alpha\}$ be the α-set of A. Then, for all $\alpha\in(0,1)$, there exist continuous functions $f_1^\alpha(s)$, $f_2^\alpha(s,t)$, and $f_3^\alpha(s,t)(0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2})$, such that*

$$A^\alpha=\{(f_1^\alpha(s),f_2^\alpha(s,t),f_3^\alpha(s,t))\in\mathbb{R}^3|0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\}.$$

Definition 4.4. [20] Let A and B be two continuous convex fuzzy numbers defined on $\mathbb{R}^3$, and

$$\begin{aligned}A^\alpha&=\{(x,y,z)\in\mathbb{R}^3|\mu_A(x,y,z)=\alpha\}\\&=\{(f_1^\alpha(s),f_2^\alpha(s,t),f_3^\alpha(s,t))\in\mathbb{R}^3|0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\},\\B^\alpha&=\{(x,y,z)\in\mathbb{R}^3|\mu_B(x,y,z)=\alpha\}\\&=\{(g_1^\alpha(s),g_2^\alpha(s,t),g_3^\alpha(s,t))\in\mathbb{R}^3|0\leq s\leq 2\pi,-\frac{\pi}{2}\leq t\leq\frac{\pi}{2}\}\end{aligned}$$

be the α-sets of A and B, respectively. For $\alpha\in(0,1)$, we define the parametric addition, parametric subtraction, parametric multiplication, and parametric division of two fuzzy numbers A and B as fuzzy numbers with α-sets as follows:

(1) parametric addition $A(+)_pB$:

$$(A(+)_pB)^\alpha = \{(f_1^\alpha(s) + g_1^\alpha(s),\ f_2^\alpha(s,t) + g_2^\alpha(s,t), f_3^\alpha(s,t) + g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(2) parametric subtraction $A(-)_pB$:

$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s+\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t)) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$

$$(A(-)_pB)^\alpha = \{(f_1^\alpha(s) - g_1^\alpha(s-\pi),\ f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t), \\ f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t)) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(3) parametric multiplication $A(\cdot)_pB$:

$$(A(\cdot)_pB)^\alpha = \{(f_1^\alpha(s) \cdot g_1^\alpha(s),\ \ f_2^\alpha(s,t) \cdot g_2^\alpha(s,t),\ \ f_3^\alpha(s,t) \cdot g_3^\alpha(s,t)) \in \mathbb{R}^3 | \\ 0 \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

(4) parametric division $A(/)_pB$:

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s+\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s+\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s+\pi,t)}) \in \mathbb{R}^3 | 0 \le s \le \pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\},$$

$$(A(/)_pB)^\alpha = \{(\frac{f_1^\alpha(s)}{g_1^\alpha(s-\pi)}, \frac{f_2^\alpha(s,t)}{g_2^\alpha(s-\pi,t)}, \frac{f_3^\alpha(s,t)}{g_3^\alpha(s-\pi,t)}) \in \mathbb{R}^3 | \pi \le s \le 2\pi, -\frac{\pi}{2} \le t \le \frac{\pi}{2}\}$$

For $\alpha = 0$ and $\alpha = 1$, $(A(*)_pB)^0 = \lim_{\alpha\to 0^+}(A(*)_pB)^\alpha$ and $(A(*)_pB)^1 = \lim_{\alpha\to 1^-}(A(*)_pB)^\alpha$, where $* = +,\ -,\ \cdot,\ /$.

Definition 4.5. A fuzzy set A with a membership function

$$\mu_A(x,y,z) = \begin{cases} h - \sqrt{\frac{(x-x_1)^2}{a_1^2} + \frac{(y-y_1)^2}{b_1^2} + \frac{(z-z_1)^2}{c_1^2}}, \\ \qquad \text{if} \quad h-1 \le \sqrt{\frac{(x-x_1)^2}{a_1^2} + \frac{(y-y_1)^2}{b_1^2} + \frac{(z-z_1)^2}{c_1^2}} \le h, \\ 1, \quad \text{if} \quad 0 \le \sqrt{\frac{(x-x_1)^2}{a_1^2} + \frac{(y-y_1)^2}{b_1^2} + \frac{(z-z_1)^2}{c_1^2}} \le h-1, \\ 0, \quad \text{otherwise}, \end{cases}$$

where $a_1, b_1, c_1 > 0$ and $1 < h$ is called the *3-dimensional trapezoidal fuzzy set* and denoted by $A = ((h,\ a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1))^3$.

The α-cut A_α of a 3-dimensional trapezoidal fuzzy set $A = ((h,\ a_1,\ x_1,\ b_1, y_1,\ c_1,\ z_1))^3$ is the following set:

$$A_\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \frac{(x-x_1)^2}{a_1^2} + \frac{(y-y_1)^2}{b_1^2} + \frac{(z-z_1)^2}{c_1^2} \le (h-\alpha)^2\Big\} \\ = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x-x_1}{a_1(h-\alpha)}\Big)^2 + \Big(\frac{y-y_1}{b_1(h-\alpha)}\Big)^2 + \Big(\frac{z-z_1}{c_1(h-\alpha)}\Big)^2 \le 1\Big\}.$$

Theorem 4.6. *Let $A = ((h_1,\ a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1))^3$ and $B = ((h_2,\ a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$ be two 3-dimensional trapezoidal fuzzy sets. Then we have the followings.*

(1) *For $0 < \alpha < 1$, the α-set of $A(+)_pB$ is*

$$\begin{aligned}(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x - x_1 - x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 \\ &+ \Big(\frac{y - y_1 - y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 = 1\Big\}.\end{aligned}$$

(2) *For $0 < \alpha < 1$, the α-set of $A(-)_pB$ is*

$$\begin{aligned}(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x - x_1 + x_2}{a_1(h_1-\alpha) + a_2(h_2-\alpha)}\Big)^2 \\ &+ \Big(\frac{y - y_1 + y_2}{b_1(h_1-\alpha) + b_2(h_2-\alpha)}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1(h_1-\alpha) + c_2(h_2-\alpha)}\Big)^2 = 1\Big\}.\end{aligned}$$

(3) *For $0 < \alpha < 1$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where*

$$\begin{aligned}x_\alpha(s) &= x_1x_2 + (x_1a_2(h_2-\alpha) + x_2a_1(h_1-\alpha))\cos s + a_1a_2(h_2-\alpha)(h_1-\alpha)\cos^2 s,\\ y_\alpha(s,t) &= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin s\cos t + b_1b_2(h_2-\alpha)(h_1-\alpha)\sin^2 s\cos^2 t,\\ z_\alpha(s,t) &= z_1z_2 + (z_1c_2(h_2-\alpha) + z_2c_1(h_1-\alpha))\sin s\sin t + c_1c_2(h_2-\alpha)(h_1-\alpha)\sin^2 s\sin^2 t.\end{aligned}$$

(4) *For $0 < \alpha < 1$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where*

$$\begin{aligned}x_\alpha(s) &= \frac{x_1 + a_1(h_1-\alpha)\cos s}{x_2 - a_2(h_2-\alpha)\cos s}, \quad y_\alpha(s,t) = \frac{y_1 + b_1(h_1-\alpha)\sin s\cos t}{y_2 - b_2(h_2-\alpha)\sin s\cos t},\\ z_\alpha(s,t) &= \frac{z_1 + c_1(h_1-\alpha)\sin s\sin t}{z_2 - c_2(h_2-\alpha)\sin s\sin t}.\end{aligned}$$

Proof. Since A and B are continuous convex fuzzy numbers defined on $\mathbb{R}^3$, by Theorem 4.3, there exists $f_1^\alpha(s), g_1^\alpha(s), f_i^\alpha(s,t), g_i^\alpha(s,t)$ $(i = 2,3)$ such that

$$A^\alpha = \{(f_1^\alpha(s), f_2^\alpha(s,t), f_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\},$$

and

$$B^\alpha = \{(g_1^\alpha(s), g_2^\alpha(s,t), g_3^\alpha(s,t)) \in \mathbb{R}^3 | 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}.$$

Since $A = ((h_1, a_1,\ x_1,\ b_1,\ y_1,\ c_1,\ z_1))^3$, $B = ((h_2, a_2,\ x_2,\ b_2,\ y_2,\ c_2,\ z_2))^3$, we have

$$\begin{aligned}f_1^\alpha(s) &= x_1 + a_1(h_1-\alpha)\cos s, \quad f_2^\alpha(s,t) = y_1 + b_1(h_1-\alpha)\sin s\cos t,\\ f_3^\alpha(s,t) &= z_1 + c_1(h_1-\alpha)\sin s\sin t,\end{aligned}$$

and

$$g_1^\alpha(s) = x_2 + a_2(h_2 - \alpha)\cos s, \quad g_2^\alpha(s,t) = y_2 + b_2(h_2 - \alpha)\sin s \cos t,$$
$$g_3^\alpha(s,t) = z_2 + c_2(h_2 - \alpha)\sin s \sin t.$$

(1) Since

$$f_1^\alpha(s) + g_1^\alpha(s) = x_1 + x_2 + (a_1(h_1 - \alpha) + a_2(h_2 - \alpha))\cos s,$$

$$f_2^\alpha(s,t) + g_2^\alpha(s,t) = y_1 + y_2 + (b_1(h_1 - \alpha) + b_2(h_2 - \alpha))\sin s \cos t,$$

and

$$f_3^\alpha(s,t) + g_3^\alpha(s,t) = z_1 + z_2 + (c_1(h_1 - \alpha) + c_2(h_2 - \alpha))\sin s \sin t,$$

we have

$$(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 - x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\Big)^2 + \Big(\frac{y - y_1 - y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\Big)^2 + \Big(\frac{z - z_1 - z_2}{c_1(h_1 - \alpha) + c_2(h_2 - \alpha)}\Big)^2 = 1\Big\}.$$

(2) If $0 \leq s \leq \pi$, we have

$$f_1^\alpha(s) - g_1^\alpha(s+\pi) = x_1 - x_2 + (a_1(h_1 - \alpha) - a_2(h_2 - \alpha))\cos s,$$

$$f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t) = y_1 - y_2 + (b_1(h_1 - \alpha) - b_2(h_2 - \alpha))\sin s \cos t,$$

and

$$f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t) = z_1 - z_2 + (c_1(h_1 - \alpha) - c_2(h_2 - \alpha))\sin s \sin t.$$

In the case of $\pi \leq s \leq 2\pi$, we have

$$f_1^\alpha(s) - g_1^\alpha(s-\pi) = f_1^\alpha(s) - g_1^\alpha(s+\pi),$$

$$f_2^\alpha(s,t) - g_2^\alpha(s-\pi,t) = f_2^\alpha(s,t) - g_2^\alpha(s+\pi,t),$$

and

$$f_3^\alpha(s,t) - g_3^\alpha(s-\pi,t) = f_3^\alpha(s,t) - g_3^\alpha(s+\pi,t).$$

Thus

$$(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| \Big(\frac{x - x_1 + x_2}{a_1(h_1 - \alpha) + a_2(h_2 - \alpha)}\Big)^2 + \Big(\frac{y - y_1 + y_2}{b_1(h_1 - \alpha) + b_2(h_2 - \alpha)}\Big)^2 + \Big(\frac{z - z_1 + z_2}{c_1(h_1 - \alpha) + c_2(h_2 - \alpha)}\Big)^2 = 1\Big\}.$$

(3) Let $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \leq s \leq 2\pi, 0 \leq t \leq \frac{\pi}{2}\}$, then

$$\begin{aligned} x_\alpha(s) &= f_1^\alpha(s) \cdot g_1^\alpha(s) \\ &= x_1x_2 + (x_1a_2(h_2 - \alpha) + x_2a_1(h_1 - \alpha))\cos s + a_1a_2(h_2 - \alpha)(h_1 - \alpha)\cos^2 s, \end{aligned}$$

$$
\begin{aligned}
y_\alpha(s,t) &= f_2^\alpha(s,t)\cdot g_2^\alpha(s,t)\\
&= y_1y_2 + (y_1b_2(h_2-\alpha) + y_2b_1(h_1-\alpha))\sin s\cos t + b_1b_2(h_2-\alpha)(h_1-\alpha)\sin^2 s\cos^2 t,\\
z_\alpha(s,t) &= f_3^\alpha(s,t)\cdot g_3^\alpha(s,t)\\
&= z_1z_2 + (z_1c_2(h_2-\alpha) + z_2c_1(h_1-\alpha))\sin s\sin t + c_1c_2(h_2-\alpha)(h_1-\alpha)\sin^2 s\sin^2 t.
\end{aligned}
$$

(4) Let $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, then

$$
\begin{aligned}
x_\alpha(s) &= \frac{x_1 + a_1(h_1-\alpha)\cos s}{x_2 - a_2(h_2-\alpha)\cos s}, \quad y_\alpha(s,t) = \frac{y_1 + b_1(h_1-\alpha)\sin s\cos t}{y_2 - b_2(h_2-\alpha)\sin s\cos t},\\
z_\alpha(s,t) &= \frac{z_1 + c_1(h_1-\alpha)\sin s\sin t}{z_2 - c_2(h_2-\alpha)\sin s\sin t}.
\end{aligned}
$$

The proof is complete. □

Example 4.7. Let $A = ((2,6,\ 3,\ 8,\ 5,\ 4,\ 7))^3$, $B = ((3,\ 4,\ 2,\ 5,\ 3,\ 6,\ 4))^3$. Then by Theorem 4.6, we have the followings.

(1) For $0 < \alpha < 1$, the α-set of $A(+)_pB$ is

$$
\begin{aligned}
(A(+)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x-5}{6(2-\alpha)+4(3-\alpha)}\Big)^2\\
&+ \Big(\frac{y-8}{8(2-\alpha)+5(3-\alpha)}\Big)^2 + \Big(\frac{z-11}{4(2-\alpha)+6(3-\alpha)}\Big)^2 = 1\Big\}.
\end{aligned}
$$

(2) For $0 < \alpha < 1$, the α-set of $A(-)_pB$ is

$$
\begin{aligned}
(A(-)_pB)^\alpha = \Big\{(x,y,z) \in \mathbb{R}^3 \Big| &\Big(\frac{x-1}{6(2-\alpha)+4(3-\alpha)}\Big)^2\\
&+ \Big(\frac{y-2}{8(2-\alpha)+5(3-\alpha)}\Big)^2 + \Big(\frac{z-3}{4(2-\alpha)+6(3-\alpha)}\Big)^2 = 1\Big\}.
\end{aligned}
$$

(3) For $0 < \alpha < 1$, $(A(\cdot)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where

$$
\begin{aligned}
x_\alpha(s) &= 6 + (12(3-\alpha) + 12(2-\alpha))\cos s + 24(3-\alpha)(2-\alpha)\cos^2 s,\\
y_\alpha(s,t) &= 15 + (25(3-\alpha) + 24(2-\alpha))\sin s\cos t + 40(3-\alpha)(2-\alpha)\sin^2 s\cos^2 t,\\
z_\alpha(s,t) &= 28 + (42(3-\alpha) + 16(2-\alpha))\sin s\sin t + 24(3-\alpha)(2-\alpha)\sin^2 s\sin^2 t.
\end{aligned}
$$

(4) For $0 < \alpha < 1$, $(A(/)_pB)^\alpha = \{(x_\alpha(s), y_\alpha(s,t), z_\alpha(s,t)) \in \mathbb{R}^3 \mid 0 \le s \le 2\pi, 0 \le t \le \frac{\pi}{2}\}$, where

$$
\begin{aligned}
x_\alpha(s) &= \frac{3 + 6(2-\alpha)\cos s}{2 - 4(3-\alpha)\cos s}, \quad y_\alpha(s,t) = \frac{5 + 8(2-\alpha)\sin s\cos t}{3 - 5(3-\alpha)\sin s\cos t},\\
z_\alpha(s,t) &= \frac{7 + 4(2-\alpha)\sin s\sin t}{4 - 6(3-\alpha)\sin s\sin t}.
\end{aligned}
$$

The membership function of the 3-dimensional trapezoidal fuzzy set is a function defined on $\mathbb{R}^3$ with values in $[0, 1]$. In case of the 3-dimensional trapezoidal fuzzy set $A = ((2, 6,\ 3,\ 8,\ 5,\ 4,\ 7))^3$, we represent the values of the membership function with colors, as shown in Figure 7. Cutting the graph of A with plane $z = 7$, we obtain Figure 8. In Figure 9, we restrict the domain to the cutting plane and then represent the membership function as the graph on $\mathbb{R}^2$. Then we observe that the 3-dimensional trapezoidal fuzzy set is an extension of the 2-dimensional trapezoidal fuzzy set. In the case of the 3-dimensional trapezoidal fuzzy set $B = ((3,\ 4,\ 2,\ 5,\ 3,\ 6,\ 4))^3$, we represent the values of membership function with colors, as shown in Figure 10. Cutting the graph of A with plane $z = 4$, we obtain Figure 11. In Figure 12, we restrict the domain to the cutting plane and then represent the membership function as the graph on $\mathbb{R}^2$. The graphs of $A(+)_pB$ and $A(-)_pB$ are shown in Figure 13 and 14, respectively. In Figure 15 and 16, we present two types of the graph for $A(\cdot)_pB$ and six types of the graph for $A(/)_pB$, respectively.

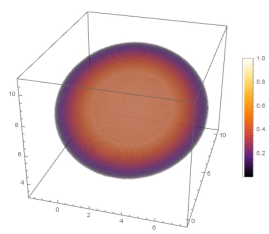

FIGURE 7. A^3

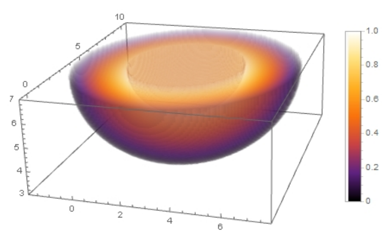

FIGURE 8. $A^3/2$

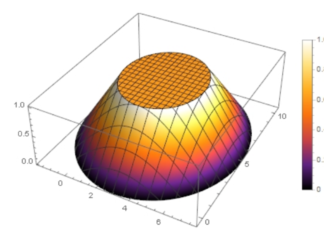

FIGURE 9. A^3_2

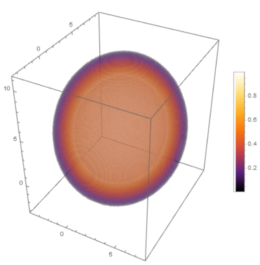

FIGURE 10. B^3

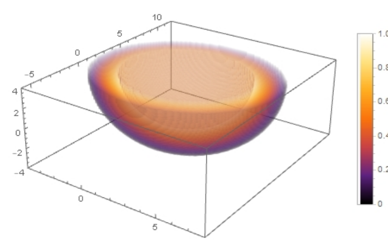

FIGURE 11. $B^3/2$

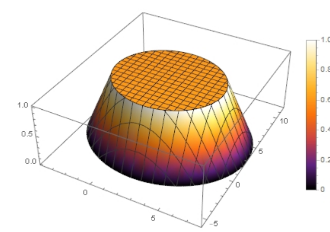

FIGURE 12. B^3_2

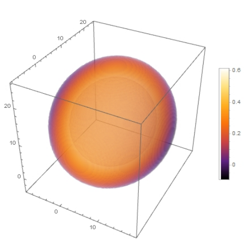

FIGURE 13. $A^3(+)B^3$

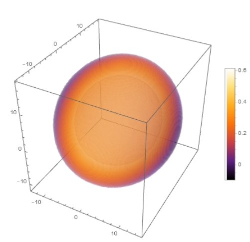

FIGURE 14. $A^3(-)B^3$

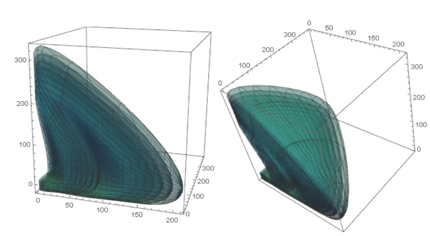

FIGURE 15. $A^3(\cdot)B^3$

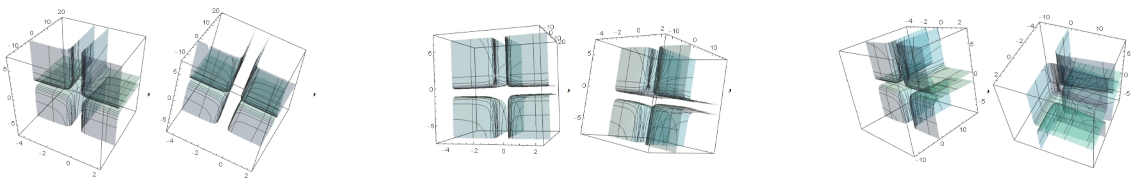

Figure 16. $A^3(/)B^3$

The Mathematica commands to obtain the above graphs are as follows.

(Figure 7)

reg2 = ImplicitRegion[1 ≤ Sqrt[(x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4] ≤ 2, {x, y, z}];

DensityPlot3D[2 - Sqrt[(x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4], {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, BoxRatios → {Sqrt[6], Sqrt[8], 2}, PlotLegends → Automatic]

(Figure 8)

reg2 = ImplicitRegion[1 ≤ Sqrt[(x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4] ≤ 2, {x, y, z}];

DensityPlot3D[2 - Sqrt[(x - 3)^2/6 + (y - 5)^2/8 + (z - 7)^2/4], {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, PlotRange → {{3 - 2 Sqrt[6], 3 + 2 Sqrt[6]}, {5 - 2 Sqrt[8], 5 + 2 Sqrt[8]}, {3, 7}, {0, 1}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotLegends → Automatic]

(Figure 9)

reg4 = ImplicitRegion[1 ≤ Sqrt[(x - 3)^2/6 + (y - 5)^2/8] ≤ 2, {x, y}];

A1 := Plot3D[2 - Sqrt[(x - 3)^2/6 + (y - 5)^2/8], {x, y} ∈ reg4, ColorFunction → "SunsetColors", PlotRange → {{3 - 2 Sqrt[6], 3 + 2 Sqrt[6]}, {5 - 2 Sqrt[8], 5 + 2 Sqrt[8]}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotPoints → 100, PlotLegends → Automatic]

reg5 = ImplicitRegion[0 ≤ Sqrt[(x - 3)^2/6 + (y - 5)^2/8] ≤ 1, {x, y}];

A2 := Plot3D[1, {x, y} ∈ reg5, PlotRange → {{3 - 2 Sqrt[6], 3 + 2 Sqrt[6]}, {5 - 2 Sqrt[8], 5 + 2 Sqrt[8]}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotPoints → 100, PlotLegends → Automatic]

Show[A1, A2]

(Figure 10)

reg2 = ImplicitRegion[2 ≤ Sqrt[(x - 2)^2/4 + (y - 3)^2/5 + (z - 4)^2/6] ≤ 3, {x, y, z}];

DensityPlot3D[3 - Sqrt[(x - 2)^2/4 + (y - 3)^2/5 + (z - 4)^2/6], {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, BoxRatios → {Sqrt[4], Sqrt[5], Sqrt[6]}, PlotLegends → Automatic]

(Figure 11)

reg2 = ImplicitRegion[2 ≤ Sqrt[(x - 2)^2/4 + (y - 3)^2/5 + (z - 4)^2/6] ≤ 3, {x, y, z}];

DensityPlot3D[3 - Sqrt[(x - 2)^2/4 + (y - 3)^2/5 + (z - 4)^2/6], {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, PlotRange → {{-4, 8}, {3 - 4 Sqrt[5], 3 + 4 Sqrt[5]}, {-4, 4}, {0, 1}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotLegends → Automatic]

(Figure 12)

reg4 = ImplicitRegion[2 ≤ Sqrt[(x - 2)^2/4 + (y - 3)^2/5] ≤ 3, {x, y}];

A1 := Plot3D[3 - Sqrt[(x - 2)^2/4 + (y - 3)^2/5], x, y ∈ reg4, ColorFunction → "SunsetColors", PlotRange → {{-4, 8}, {3 - 4 Sqrt[5], 3 + 4 Sqrt[5]}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotLegends → Automatic]

reg5 = ImplicitRegion[0 ≤ Sqrt[(x - 2)^2/4 + (y - 3)^2/5] ≤ 2, {x, y}];

A2 := Plot3D[1, {x, y} ∈ reg5, PlotRange → {{-4, 8}, {3 - 4 Sqrt[5], 3 + 4 Sqrt[5]}}, BoxRatios → {Sqrt[6], Sqrt[8], 1}, PlotLegends → Automatic]

Show[A1, A2]

(Figure 13)

reg2 = ImplicitRegion[2 ≤ Sqrt[(x - 5)^2/24 + (y - 8)^2/31 + (z - 11)^2/26] ≤ 3, {x, y, z}];

h[a_] := DensityPlot3D[1 - ((x - 5)/(6 (2 - a) + 4 (3 - a)))^2 - ((y - 8)/(8 (2 - a) + 5 (3 - a)))^2 - ((z - 11)/(4 (2 - a) + 6 (3 - a)))^2, {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, BoxRatios → {Sqrt[24], Sqrt[31], Sqrt[26]}, PlotLegends → Automatic];

Show[h[1]]

(Figure 14)

reg2 = ImplicitRegion[2 ≤ Sqrt[(x - 1)^2/24 + (y - 2)^2/31 + (z - 3)^2/26] ≤ 3, {x, y, z}];

h[a_] := DensityPlot3D[1 - ((x - 1)/(6 (2 - a) + 4 (3 - a)))^2 - ((y - 2)/(8 (2 - a) + 5 (3 - a)))^2 - ((z - 3)/(4 (2 - a) + 6 (3 - a)))^2, {x, y, z} ∈ reg2, PlotPoints → 100, ColorFunction → "SunsetColors", OpacityFunction → 0.05, BoxRatios → {Sqrt[24], Sqrt[31], Sqrt[26]}, PlotLegends → Automatic];

Show[h[1]]

(Figure 15)

g[a_] := ParametricPlot3D[{6 + (12 (3 - a) + 12 (2 - a)) Cos[s] + 24 (3 - a) (2 - a) (Cos[s])^2, 15 + (25 (3 - a) + 24 (2 - a)) Sin[s] Cos[t] + 40 (3 - a) (2 - a) (Sin[s])^2 (Cos[t])^2, 28 + (42 (3 - a) + 16 (2 - a)) Sin[s] Sin[t] + 24 (3 - a) (2 - a) (Sin[s])^2 (Sin[t])^2}, {s, 0, 2 Pi}, {t, - Pi/2, Pi/2}, PlotStyle → Directive[RGBColor[0.2, 0.5 + a/2, 0.5 + a/2], Opacity[0.3]], BoxRatios → {1, 1, 1}];

```
tg = Table[g[i], i, 0, 0.5, 0.1];

Show[tg]
```

(Figure 16)

```
g[a_] := ParametricPlot3D[{(3 + 6 (2 - a) Cos[s])/(2 - 4 (3 - a) Cos[s]), (5 + 8 (2 - a) Sin[s] Cos[t])/(3 - 5 (3 - a)/4 Sin[s] Cos[t]), (7 + 4 (2 - a) Sin[s] Sin[t])/(4 - 6 (3 - a) Sin[s] Sin[t]) }, {s, 0, 2 Pi}, {t, - Pi/2, Pi/2}, PlotStyle → Directive[RGBColor[0.2, 0.5 + a/2, 0.5 + a/2], Opacity[0.3]], BoxRatios → {1, 1, 1}];

tg = Table[g[i], {i, 0, 0.5, 0.1}];

Show[tg]
```

5. Conclusion

Research on various types of fuzzy numbers is crucial in fuzzy theory. While triangular fuzzy numbers have received significant attention in research, trapezoidal fuzzy sets and pentagonal fuzzy numbers have been relatively understudied. Through previous research, we conducted a study on extension operators for generalized trapezoidal fuzzy sets in one dimension [1]. This research has been cited in various fields. Additionally, we extended the results obtained in one dimension to two dimensions by introducing parameter operators. This study, too, has been widely cited and applied for research on parametric operators between 2D trapezoidal fuzzy sets. Thus, in this study, we aimed to extend the parametric operators for trapezoidal fuzzy sets to three dimensions, deriving new results to be utilized across various fields similarly to the findings in one and two dimensions. In conclusion, we extended the parametric operator for trapezoidal fuzzy sets to three dimensions. Since it was defined in 3D space, the domain became a convex set in the 3D space including the interior, thereby making the alpha cut the boundary surface of a 3D elliptic curve. We defined and computed parametric operators between 3D trapezoidal fuzzy sets, presenting the results in a graph. Although the graph was defined in 3D space and rendered in 4D, it could not be depicted in 3D. The presented graph was illustrated in three dimensions using a specific definition of a fuzzy number, where the membership function values ranged between 0 and 1. The membership function value at each point was represented as the color intensity at that point. Slicing the graph with a plane passing through the longest axis revealed different function values expressed through varying color intensities on the plane. Notably, a certain portion at the center, being a trapezoidal fuzzy set, shared the same color. By presenting the graph, this result would be referenced and utilized in various fields, similar to the one-dimensional and two-dimensional findings. These results suggest significant patterns within trapezoidal fuzzy sets, indicating diverse applications across fields. Additionally, our study would pave the way for future research, offering new avenues for deeper understanding and advancements in fuzzy set theory and its applications.

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Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] B. Lee, Y.S. Yun, The generalized trapezoidal fuzzy sets, J. Chungcheong Math. Soc. 24 (2011), 253–266. https://doi.org/10.14403/JCMS.2011.24.2.11.

[2] E. Povolotskaya, Enhancement of the selected management and planning quality control tools, Thesis, Czech Technical University, (2017).

[3] S. Banerjee, T.K. Roy, Arithmetic behaviors of p-norm generalized trapezoidal intuitionistic fuzzy numbers with application to circuit analysis, Int. J. Fuzzy Syst. Appl. 6 (2017), 6–58. https://doi.org/10.4018/IJFSA.2017070102.

[4] E. Povolotskaya, P. Mach, Affinity diagram as a risk analysis tool with support of fuzzy logic, Adv. Sci. Lett. 19 (2013), 2391–2394. https://doi.org/10.1166/asl.2013.4923.

[5] A.F. Momena, S. Mandal, K.H. Gazi, et al. Prediagnosis of disease based on symptoms by generalized dual hesitant hexagonal fuzzy multi-criteria decision-making techniques, Systems 11 (2023), 231. https://doi.org/10.3390/systems11050231.

[6] M.C.J. Anand, M. Lathamaheswari, A new generalized normal type-2 trapezoidal fuzzy number, Glob. J. Pure Appl. Math. 12 (2016), 565–574.

[7] M.C.J. Anand, J. Bharatraj, A new $(\alpha\ \beta\ \gamma\ \delta)$-cuts in trapezoidal fuzzy number and its approach in decomposition theorem, Glob. J. Pure Appl. Math. 12 (2016), 875–877.

[8] M.A. Alam, A report on "fuzzy numbers and their applications", Thesis, Jahangirnagar University, (2020).

[9] B.J. Lee, Y.S. Yun, The pentagonal fuzzy numbers, J. Chungcheong Math. Soc. 27 (2014), 277–286.

[10] M.G. Alharbi, H.A.E.-W. Khalifa, Enhanced fuzzy delphi method in forecasting and decision-making, Adv. Fuzzy Syst. 2021 (2021), 2459573. https://doi.org/10.1155/2021/2459573.

[11] S. ElMorsy, Optimization of fuzzy zero- base budgeting, Comput. Algorithms Numer. Dimens. 1 (2022), 147–154. https://doi.org/10.22105/cand.2022.155548.

[12] R. Arora, A.P. Singh, R. Sharma, et al. A fuzzy economic order quantity model with credibility induced demand and shortages, Int. J. Serv. Oper. Inf. 11 (2021), 13–26. https://doi.org/10.1504/IJSOI.2021.114112.

[13] A.J. Kamble, Some notes on pentagonal fuzzy numbers, Int. J. Fuzzy Math. Arch. 13 (2017), 113–121. https://doi.org/10.22457/ijfma.v13n2a2.

[14] A. Shapique, Arithmetic operations on heptagonal fuzzy numbers, Asian Res. J. Math. 2 (2017), 1–25. https://doi.org/10.9734/ARJOM/2017/31578.

[15] V. Goyal, N. Rani, D. Gupta, An algorithm for quadratically constrained multi-objective quadratic fractional programming with pentagonal fuzzy numbers, Oper. Res. Decis. 32 (2022), 49–71.

[16] Namarta, Developing algorithms in fuzzy environment, Report, Khalsa College, 2018.

[17] A. Venkatesh, A.B. Manoj, A fuzzy mathematical model for vegetable diet plan using ranking of pentagonal fuzzy number, Malaya J. Mat. 8 (2020), 1995–1999. https://doi.org/10.26637/MJM0804/0113.

[18] Y.S. Yun, Parametric operations for two 2-dimensional trapezoidal fuzzy sets, J. Algebra Appl. Math. 18 (2020), 27–41.

[19] Y.S. Yun, A zadeh's max-min composition operator for 3-dimensional triangular fuzzy number, J. Intell. Fuzzy Syst. 39 (2020), 3783–3793. https://doi.org/10.3233/JIFS-192095.

[20] Y.S. Yun, Graphic representation of a dimensional expansion of triangular fuzzy number, J. Math. 2021 (2021), 1970553. https://doi.org/10.1155/2021/1970553.

[21] H.J. Zimmermann, Fuzzy set theory–and its applications, Kluwer-Nijhoff Publishing, Boston, 1985.

[22] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning–I, Inf. Sci. 8 (1975), 199–249. https://doi.org/10.1016/0020-0255(75)90036-5.

[23] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning–II, Inf. Sci. 8 (1975), 301–357. https://doi.org/10.1016/0020-0255(75)90046-8.

[24] L.A. Zadeh, The concept of a linguistic variable and its application to approximate reasoning-III, Inf. Sci. 9 (1975), 43–80. https://doi.org/10.1016/0020-0255(75)90017-1.