Asia Pac. J. Math. 2024 **11**:11

ON CONVERGENCE AND COMPLETENESS IN GENERALIZED METRIC SPACES

YASSIN ALZUBAIDI

Mathematics Department, Faculty of Sciences, Umm Al-Qura University, Makkah, Saudi Arabia

yazubaidi@uqu.edu.sa

Received Nov. 25, 2023

Abstract. We introduce a generalization to the concept of metric where we define a metric function that can assign a value to the (distance) among any finite number of points. In contrast to the other ways of generalization where the number of points ($n \geq 2$) is fixed, we allow n to vary under the same axiomatic system. We highlight the superiority of the introduced metric and its properties. Then, we build the related theory and define the corresponding notions, such as convergence and Cauchy-type conditions. We prove some significant results concerning the convergence in the introduced metric space and establish a particular form of completeness for some spaces of importance. In addition , we utilize the approach presented here to directly extend the concept of the limit of a sequence from a single point to a set (compact set). This limit generalization enables us to obtain a general form of convergence to some sequences that are not convergent in the standard metric spaces.

2020 Mathematics Subject Classification. 11J83; 40A05.

Key words and phrases. generalized metric; convergence; Cauchy-type condition; complete spaces.

1. Introduction

There have been many attempts and approaches to generalize the notion of metric. These attempts take mainly three different directions. One direction is through modifying or relaxing the axioms. For instance, in [1], S. Czerwik introduced the b-metric via relaxing the triangle inequality: $d(a,b) \leq t[d(a,c)+d(c,b)]$ for some $t \geq 1$. Another attempt in that direction appears in the notion of the dislocated metric (see [2] and [3]). The idea of the dislocated metric is to allow the distance between a point and itself to be positive ($d(a,b)=0 \Rightarrow a=b$). On the other hand, allowing the distance between two distinct points to be equal to zero gives the well-known pseudo-metric ($a=b \Rightarrow d(a,b)=0$) [4]. The second direction for generalizing metrics is via changing the range of the metric values. This direction relies on replacing the ordered space $(\mathbb{R}, \leq)$ with another general ordered space. W. S. Du in [5] introduced the so-called tvs-cone metric $d: X^2 \to P$, where P is a specified set of cones in an

DOI: 10.28924/APJM/11-11

ordered topological vector space. For more on cone metrics, see [5]. The third major approach to generalizing the metric concept is achieved by enabling the metric function to assign a value for the distance among more than two points. An early attempt in that direction was carried out by S. Gähler in 1966 in the papers [6] and [7]. He introduced the concept of 2-metric $d(a,b,c)$, where the domain of a 2-metric on X is X^3. He regarded the value of the 2-metric $d(a,b,c)$ as the triangle's area with vertices a, b, and c. Later in 1992, B.C. Dhage tried to improve the work of S. Gähler, and he defined the D-metric ([8] and [9]). B.C. Dhage attempted to switch the geometric interpretation of the metric $d(a,b,c)$ from the area of the associated triangle to the perimeter of that triangle. However, early this century, Z. Mustafa and B. Sims remarked on the D-metric [10], and they formulated a new system of axioms to ensure the continuity of their introduced metric. In [11], Z. Mustafa and B. Sims defined the so-called G-metric. Since the G-metric was introduced, it has attracted much attention and interest. Many results related to the G-metric were obtained, especially in the area of the fixed point theory. Also, there have been many attempts to take the notion of the G-metric to higher dimensions. For example, K. A. Khan extended the idea in [11] for $n \geq 3$, and he defined the generalized n-metric (see [12] and [13]).

2. Construction

In this section, we introduce the definition of a multiple entries metric. We define a metric that is not restricted to a certain number of points but rather can deal with any finite number of points. In the coming, X will be a non-empty set, and $P^*(X)$ will denote the set of all non-empty finite subsets of X, while $|U|$ is the number of the distinct elements of a set $U \in P^*(X)$.

Definition 2.1. *Let X be a non-empty set, then we say that a non negative set function*

$$d : P^*(X) \longrightarrow [0, \infty)$$

is a multiple points metric and denote it by MP-metric if it satisfies for all $A, B \in P^*(X)$ the following:

- *(M1)* $d(A) = 0$ *if and only if* $|A| = 1$.
- *(M2)* $d(A) \leq d(B)$ *if* $A \subseteq B$.
- *(M3)* $d(A \cup B) \leq d(A) + d(B)$ *if* $A \cap B \neq \phi$.
- *(M4)* $d(A \cup B) = d(A)$ *if* $d(A \cup \{b\}) = d(A)$ *for all* $b \in B$.

Then, the pair (X, d) is called MP-metric space.

Remark 2.2. *In axiom M4, we can replace "if" with "iff," where the other direction of axiom M4 is already satisfied by axiom M2, and it can be obtained as follows.*

$$d(A) \leq d(A \cup \{b\}) \leq d(A \cup B) = d(A) \text{ for all } b \in B$$

Remark 2.3. *The axioms M1, M2, M3, and M4 are independent from each other. In fact, to show the independence of this axiomatic system, it is sufficient to construct set functions d_1, d_2, d_3 and d_4 such that d_i does not satisfy that axiom Mi, but at the same time, d_i satisfies the rest of the axioms. Let X be a non-empty set and fix $a \in X$. Define $d_i : P^*(X) \longrightarrow [0, \infty);\ i = 1, 2, 3, 4$ in the following way.*

$d_1(A) = 1$, *for all* $A \in P^*(X)$.

$$d_2(A) = \begin{cases} 0 & if \quad |A| = 1 \\ 1 & if \quad |A| > 1, \text{ and } a \in A, \\ 2 & if \quad |A| > 1, \text{ and } a \notin A. \end{cases}$$

$$d_3(A) = \begin{cases} 0 & if \quad |A| = 1, \\ 1 & if \quad |A| = 2, \\ 3 & if \quad |A| > 2. \end{cases}$$

$d_4(A) = Max\{d(a, b) : a, b \in A\}$, *where* $(X, d) = (\mathbb{R}^2, d)$ *is the usual Euclidean metric.*

Then, we can check that the axiom Mj holds for d_i if and only if $j \neq i$. For example, to see that d_4 does not satisfy M4 we may consider the sets $A = \{(0, 0), (4, 0)\}$ and $B = \{(3, 2), (1, -2)\}$.

Proposition 2.4. *Let $d : P^*(X) \longrightarrow [0, \infty)$ satisfy the axioms M2 and M3 on X and let $A, B, C \in P^*(X)$, then*

(i)- $d(A) = d(B)$ if $A = B$. (Symmetry)

(ii)- $d(A \cup B) \leq d(A \cup C) + d(C \cup B)$. (Triangle Inequality)

Proof. (i) follows from $M2$ since $A \subseteq B$ and $B \subseteq A$, while (ii) follows from a sequential use of $M2$ and $M3$ in the following way: $d(A \cup B) \leq d(A \cup B \cup C) \leq d(A \cup C) + d(C \cup B)$. □

The following theorem reflects a side of the strength of the introduced MP-metric. It indicates that once we have an MP-metric on X, we can restrict it to a specific subset of $P^*(X)$ to get a metric, D-metric, G-metric, or a generalized n-metric. Therefore, we can import most of the theory of these metrics and have it embedded into the MP-metric.

Theorem 2.5. *Let $d : P^*(X) \longrightarrow [0, \infty)$ be a set function satisfying the axioms M1, M2 and M3 on X, and let $d_{(k)}$ denote the restriction of d to X^k, that is $d_{(k)}(a_1, ..., a_k) = d(\{a_1, ..., a_k\})$. Then*

(i)- $d_{(2)} : X^2 \longrightarrow [0, \infty)$ is a metric.

(ii)- $d_{(3)} : X^3 \longrightarrow [0, \infty)$ is a symmetric D-metric.

(iii)- $d_{(3)} : X^3 \longrightarrow [0, \infty)$ is a symmetric G-metric.

(iv)- $d_{(n)} : X^n \longrightarrow [0, \infty)$ is a generalized n-metric.

Proof. Direct calculations can lead to the desired results. However, for convenience, we will highlight (iii), where we will state the axioms of the symmetric G-metric and indicate how they can be obtained. The rest can be proven in the same way, where the reader may refer to the corresponding references for more on the definitions and properties of these metrics.

(G1) $d_{(3)}(a,b,c) = 0$ if $a = b = c$ (by $M1$)

(G2) $d_{(3)}(a,a,b) > 0$ if $a \neq b$ (by $M1$)

(G3) $d_{(3)}(a,a,b) \leq d_{(3)}(a,b,c)$ (by $M2$)

(G4) $d_{(3)}(a,b,c) = d_{(3)}(b,a,c) = d_{(3)}(a,c,b) = \ldots$ (by Proposition 2.4)

(G5) $d_{(3)}(a,b,c) \leq d(\{a,b,c,x\})$ (by $M2$)
$\leq d_{(3)}(a,x,x) + d_{(3)}(x,b,c)$ (by $M3$ and Proposition 2.4)

(G6) $d_{(3)}(a,b,b) = d_{(3)}(a,a,b)$ (by Proposition 2.4) □

Example 2.6. *$d : P^*(\mathbb{R}) \longrightarrow [0,\infty)$ defined as $d(A) = Max(A) - Min(A)$ is an MP-metric. It is a natural MP-metric on $\mathbb{R}$, as its restriction to two points is the standard metric on $\mathbb{R}$. Indeed, we have that $d_{(2)}(a,b) = d(\{a,b\}) = Max(\{a,b\}) - Min(\{a,b\}) = |a-b|$.*

Example 2.7. *Let X be a non-empty set and $d : P^*(X) \longrightarrow [0,\infty)$. Then, each of the following defines an MP-metric on X.*

(i) $d(A) = |A| - 1$.

(ii) $d(A) = 0$ if $|A| = 1$ and $d(A) = 1$ if $|A| > 1$.

3. Convergence on MP-metrics

We start this chapter by introducing a concept of dependency related to the MP-metric.

Definition 3.1. *Let (X,d) be an MP-metric space, then we say that the elements of $A \in P^*(X)$ are d-independent (or simply, we say A is independent) if $|A| = 1$ or*

$$d(A \setminus \{a\}) < d(A), \forall a \in A, \text{ for } |A| > 1.$$

In the above definition, a finite non-empty set A is independent if every element $a \in A$ contributes to the value of $d(A)$. On the other hand, we define the dependent set on A as follows.

Definition 3.2. *Let (X,d) be an MP-metric space and $A \in P^*(X)$. The set of all dependent points on A is denoted by $D(A)$ and defined as*

$$D(A) = \{x \in X : d(A \cup \{x\}) = d(A)\ \}.$$

If needed, we may write $D_d(A)$ to indicate the implemented MP-metric.

Example 3.3. *Let (X,d) be an MP-metric then,*

(i)- $D(\{a\}) = \{a\}$ since $D(\{a\}) = \{x \in X : d(\{a, x\}) = d(\{a\}) = 0\} = \{a\}$ by M1.

(ii)-$D(\{a, b\}) = \{x \in X : d(\{a, b, x\}) = d(\{a, b\})\}$. If (X, d) is the MP-metric spaces defined in Example 2.6 and $a \leq b$ then $D(\{a, b\})$ is the closed interval $[a, b]$.

Remark 3.4. *Note that $D(A) = D(B)$ does not imply that $A = B$; even A and B are both d-independent. Consider the following example: let $X = \{x, y, z, u\}$ where x, y, z, and u are four distinct points, and let d be defined as in Example 2.7-(ii), then we have that $\{x, y\}$ and $\{z, u\}$ are d-independent, and*

$$d(\{x, y\}) = d(\{z, x, y\}) = d(\{u, x, y\}) = d(\{x, z, u\}) = d(\{y, z, u\}) = d(\{z, u\}) = 1,$$

that is $D(\{x, y\}) = D(\{z, u\}) = \{x, y, z, u\}$ but $\{x, y\} \neq \{z, u\}$.

Now, we are ready to introduce a major definition in this section. If no confusion can arise, we may use $d(a_1, ..., a_n)$ to denote $d(\{a_1, ..., a_n\})$, and we will write $\lim\limits_{n_i \to \infty}$ instead of $\lim\limits_{n_1,...,n_k \to \infty}$.

Definition 3.5. *Let (X, d) be an MP-metric space and $D \subseteq X$. Then, we say that a sequence x_n is d-convergent to D and write $\overset{d}{\lim\limits_{n \to \infty}} x_n = D$ if there is $A \in P^*(X)$ such that $D = D(A)$ and:*
(i)- $\lim\limits_{n_i \to \infty} d(\{x_{n_1}, x_{n_2} ..., x_{n_k}\} \cup A) = d(A)$ for all $k \in \mathbb{N}$,
(ii)- $\liminf\limits_{n \longrightarrow \infty} d(x_n, a) = 0$ for all $a \in A$.

For simplicity, we will say x_n converges to $D(A)$, and write $\lim x_n = D(A)$ to indicate that A satisfies the convergence conditions in the above definition.

Remark 3.6. *The second condition in Definition 3.5 is equivalent to that, for all $a \in A$, there exists a subsequence x_{n_m} such that $\lim d(x_{n_m}, a) = 0$. This result follows from the fact that for all $m \in \mathbb{N}$ there is x_{n_m} such that $n_m \geq m$ and*

$$d(x_{n_m}, a) \leq \inf_{n \geq m} d(x_n, a) + \frac{1}{m} \leq d(x_{n_m}, a) + \frac{1}{m}.$$

Proposition 3.7. *(Continuity of the MP-metric) Let (X, d) be an MP-metric space, and let $(X, d_{(2)})$ be the associated metric space (see Theorem 2.5).*
(i)- If $x_n \longrightarrow a$ in $(X, d_{(2)})$ then $\lim\limits_{n \longrightarrow \infty} d(\{x_n, a_1 ..., a_k\}) = d(\{a, a_1 ..., a_k\})$, $\forall a_1, ..., a_k \in X$.
(ii)- If $x_{n_i} \longrightarrow a_i$ in $(X, d_{(2)})$ $\forall i = 1, ..., k$, then $\lim\limits_{n_i \longrightarrow \infty} d(\{x_{n_1}, ..., x_{n_k}\}) = d(\{a_1 ..., a_k\})$.

Proof. Generally, for any $A \in P^*(X)$ and $x, y \in X$, we can use the axioms M2 and M3 to obtain the following inequalities $d(A \cup \{x\}) \leq d(A \cup \{y\}) + d(x, y)$, and $d(A \cup \{y\}) \leq d(A \cup \{x\}) + d(x, y)$, which we combine to get

$$|d(A \cup \{x\}) - d(A \cup \{y\})| \leq d(x, y) \tag{1}$$

(i)- A direct application of (1) yields $|d(\{x_n, a_1 ..., a_k\}) - d(\{a, a_1 ..., a_k\})| \leq d(x_n, a)$. Hence, taking the limit, we get

$$0 \leq \lim_{n \to \infty} |d(\{x_n, a_1 ..., a_k\}) - d(\{a, a_1 ..., a_k\})| \leq \lim_{n \to \infty} d(x_n, a) = 0,$$

which in turn gives the desired result.

(ii)-Repeating the use of (1) we get
$|d(\{x_{n_1},...,x_{n_k}\})-d(\{a_1...,a_k\})|$

$$= |d(\{x_{n_1},...,x_{n_k}\})-d(\{a_1...,a_k\})+d(\{a_1,x_{n_2}...,x_{n_k}\})-d(\{a_1.x_{n_2}..,x_{n_k}\})|$$

$$\leq |d(\{x_{n_1},...,x_{n_k}\})-d(\{a_1,x_2...,x_{n_k}\})|+|d(\{a_1...,a_k\})-d(\{a_1,x_{n_2}...,x_{n_k}\})|$$

$$\leq d(x_{n_1},a_1)+|d(\{a_1,...,a_k\})-d(\{a_1,x_{n_2}...,x_{n_k}\})|$$

$$= d(x_{n_1},a_1)+|d(\{a_1,...,a_k\})-d(\{a_1,x_{n_2}...,x_{n_k}\})+d(\{a_1,a_2,x_{n_3}...,x_{n_k}\})-d(\{a_1,a_2,x_{n_3}...,x_{n_k}\})|$$

$$\leq d(x_{n_1},a_1)+|d(\{a_1,x_{n_2}...,x_{n_k}\})-d(\{a_1,a_2,x_{n_3}...,x_{n_k}\})|+|d(\{a_1,...,a_k\})-d(\{a_1,a_2,x_{n_3}...,x_{n_k}\})|$$

$$\leq d(x_{n_1},a_1)+d(x_{n_2},a_2)+|d(\{a_1,...,a_k\})-d(\{a_1,a_2,x_{n_3}...,x_{n_k}\})|$$

$$\vdots$$

$$\leq \sum_{i=1}^{k} d(x_{n_i},a_i)$$

Therefore, $\lim_{n_i\longrightarrow\infty} |d(\{x_{n_1},...,x_{n_k}\})-d(\{a_1...,a_k\})| \leq \lim_{n_i\longrightarrow\infty} \sum_{i=1}^{k} d(x_{n_i},a_i)=0,$ □

Theorem 3.8. (*Uniqueness of the limit*) *Let (X,d) be an MP-metric space. If x_n is d-convergent in (X,d), then the limit is unique.*

Proof. Let $A,B\in P^*(X)$ and assume $\lim x_n = D(A)$ and $\lim x_n = D(B)$, we will show that $D(A)=D(B)$. First, note that the assumption $\lim x_n = D(A)$ implies that for all $a\in A$, there is a subsequence (x_{n_k}) converging to a. Also, since $\lim x_n = D(B)$ we get

$$\begin{aligned} & \lim_{n_k\longrightarrow\infty} d(\{x_{n_k}\}\cup B)=d(B) \\ \Rightarrow\quad & d(\{a\}\cup B)=d(B) \text{ (by Proposition 3.7)} \\ \Rightarrow\quad & a\in D(B) \text{ (by Definition 3.2)} \\ \Rightarrow\quad & A\subseteq D(B) \end{aligned}$$

Combining this with the axiom M4, we get that $d(A\cup B)=d(B)$. Similarly, we can show that $B\subseteq D(A)$ and $d(A\cup B)=d(A)$. Hence, we obtain the equality

$$d(A)=d(B) \tag{2}$$

Now, let $x \in D(A)$,

$$\begin{aligned} d(B) &\le d(\{x\} \cup B) \\ &\le d(\{x\} \cup B \cup A) \\ &= d(A) \text{ (by M4 since } [\{x\} \cup B] \subseteq D(A) \text{)} \\ &= d(B) \text{ (by equation (2))} \end{aligned}$$

Thus, we have $d(\{x\} \cup B) = d(B)$, which in turn implies $x \in D(B)$. Therefore, we get $D(A) \subseteq D(B)$ and similarly we show $D(B) \subseteq D(A)$ □

Proposition 3.9. *Let (X, d) be an MP-metric space, and let $(X, d_{(2)})$ be the associated metric space. (x_n) converges to a in $(X, d_{(2)})$ if and only if it is d-convergent to $D(\{a\}) = \{a\}$ in (X, d).*

Proof. For the first direction and since $x_n \longrightarrow a$, it remains to check that

$$d(\{x_{n_1}, ..., x_{n_k}, a\}) \longrightarrow d(\{a\}) \text{ for all } k \in \mathbb{N},$$

which can be obtained from the given and the axiom M3 as follows

$$\lim_{n_i \longrightarrow \infty} d(\{x_{n_1}, ..., x_{n_k}, a\}) \le \lim_{n_i \longrightarrow \infty} \sum_{i=1}^{k} d(x_{n_i}, a) = 0 = d(\{a\}).$$

The other direction follows from the axiom M1 by taking $k = 1$ in the definition of convergence in the MP-metric spaces (Definition 3.5). $\lim_{n \to \infty} d(x_n, a) = d(\{a\}) = 0$. □

Corollary 3.10. *Let (X, d) be an MP-metric space and let $(X, d_{(2)})$ be the associated metric on X . If (x_n) converges to $D(A)$ in (X, d) for some $A \in P^*(X)$ with $|A| > 1$ then x_n does not converge in $(X, d_{(2)})$.*

Example 3.11. *Consider the MP-metric space $(\mathbb{R}, d)$ as defined in Example 2.6. The sequence $(x_n) = (\sin n)$ is d-convergent to $D(\{-1, 1\}) = [-1, 1]$. Note that*

$$\begin{aligned} d(\sin n_1, ... \sin n_k, -1, 1) &= \max\{\sin n_1, ... \sin n_k, -1, 1\} - \min\{\sin n_1, ... \sin n_k, -1, 1\} \\ &= 2 = d(-1, 1) \end{aligned}$$

Also, by the density of $\sin n$ in $[-1, 1]$, there are two subsequences: $\sin n_k \longrightarrow -1$ and $\sin n_l \longrightarrow 1$. On the other hand, the restriction of d to $\mathbb{R}^2$ is the usual distance metric $d(a, b) = \max\{a, b\} - \min\{a, b\} = |a - b|$, and it is known that $\sin n$ is not convergent in $(\mathbb{R}, |\cdot|)$.

4. Cauchy Type Condition on MP-metrics

Even though the convergence in MP-metric spaces does not necessarily imply the convergence in the associated metric spaces, the convergence in MP-metric still reserves most of the useful results, such as boundedness and a generalized form of the Cauchy condition (see Corollary 4.4 and Theorem 4.2 below). In this section, we will show some analogous results to those in the standard metric theory.

Definition 4.1. *Let (X, d) be an MP-metric space and $r \geq 0$. Then we say that x_n is r-Cauchy if there is $N \in \mathbb{N}$ such that*

$$\lim_{m \longrightarrow \infty} \sup_{n_i > m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) = r \text{ for all } k \geq N$$

Theorem 4.2. *Let (X, d) be an MP-metric space. If $\lim x_n = D(A)$ for some $A \in P^*(X)$, then (x_n) is r-Cauchy, and moreover $r = d(A)$.*

Proof. We will show that

$$\lim_{m \longrightarrow \infty} \sup_{n_i \geq m} d(\{x_{n_1}, ..., x_{n_k}\}) = d(A) \text{ for all } k \geq |A| = l.$$

First, note that since $\lim x_n = D(A)$, we have

$$\lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_k}\} \cup A) = \lim_{n_i \longrightarrow \infty} d(\{x_{n_1}, ..., x_{n_k}\} \cup A) = d(A) \text{ for all } k \in \mathbb{N}$$

Therefore,

$$\begin{aligned} & d(\{x_{n_1}, ..., x_{n_l}\}) \leq d(\{x_{n_1}, ..., x_{n_l}\} \cup A) \\ \Rightarrow \quad & \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}) \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\} \cup A) \\ \Rightarrow \quad & \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}) \leq d(A). \end{aligned}$$

For the reverse inequality, we proceed as follows. The given $\lim x_n = D(A)$ implies that for each $a_j \in A$, there is a subsequence $x_{n_{j_i}}$ such that $x_{n_{j_i}} \longrightarrow a_j$. Thus, we get

$$\begin{aligned} & \sup_{n_{j_i} > m} d(\{x_{n_{j_1}}, ..., x_{n_{j_k}}\}) \leq \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}), \\ \Rightarrow \quad & \lim_{m \longrightarrow \infty} \sup_{n_{j_i} > m} d(\{x_{n_{j_1}}, ..., x_{n_{j_l}}\}) \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}), \\ \Rightarrow \quad & \lim_{n_{j_i} \longrightarrow \infty} d(\{x_{n_{j_1}}, ..., x_{n_{j_l}}\}) \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}), \\ \Rightarrow \quad & d(\{a_1, ..., a_l\}) \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}), \\ \Rightarrow \quad & d(A) \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}), \end{aligned}$$

where we used Proposition 3.7- (ii). Therefore, we get $\lim_{m \longrightarrow \infty} \sup_{n_i \geq m} d(\{x_{n_1}, ..., x_{n_l}\}) = d(A) \text{ for } l = |A|$. For the case $k \geq |A|$, note that

$$d(\{x_{n_1}, ..., x_{n_l}\}) \leq d(\{x_{n_1}, ..., x_{n_l}, ..., x_{n_k}\}) \leq d(\{x_{n_1}, ..., x_{n_l}, ..., x_{n_k}\} \cup A).$$

Thus, we get

$$\begin{aligned} d(A) &= \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}\}) \\ &\leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(\{x_{n_1}, ..., x_{n_l}, ..., x_{n_k}\}) \end{aligned}$$

$$\leq \lim_{m\longrightarrow\infty} \sup_{n_i>m} d(\{x_{n_1}, ..., x_{n_l}, ..., x_{n_k}\} \cup A)$$

$$= \lim_{n_i\longrightarrow\infty} d(\{x_{n_1}, ..., x_{n_l}, ..., x_{n_k}\} \cup A) = d(A)$$

Therefore, $\lim_{m\longrightarrow\infty} \sup_{n_i\geq m} d(\{x_{n_1}, ..., x_{n_k}\}) = d(A)$ *for all* $k \geq l$. □

Proposition 4.3. *Let* (X, d) *be an MP-metric space. If* x_n *is r-Cauchy for some* $r \geq 0$*, then* x_n *is bounded.*

Proof. A sequence is bounded if there exists $C > 0$ such that $d(x_n, x_m) < C$ for all $n, m \in \mathbb{N}$. Let x_n be r-Cauchy, then there is $N_1 \in \mathbb{N}$ such that

$$\sup_{n_i>m} d(\{x_{n_1}, ..., x_{n_k}\}) < r + 1 \textit{ for all } m \geq N_1.$$

Therefore, $d(x_{n_1}, x_{n_2}) \leq d(\{x_{n_1}, ..., x_{n_k}\}) < r + 1$ *for all* $n_i > N_1$. Hence, choosing

$$C = r + 1 + \max\{d(x_n, x_m) : n, m \in \{1, ..., N_1\}\}$$

gives the wanted bound. □

Corollary 4.4. *Let* (X, d) *be an MP-metric space. If* $\lim x_n = D(A)$*, then* (x_n) *is bounded.*

Proposition 4.5. *Let* (X, d) *be an MP-metric space. A sequence* x_n *is 0-Cauchy if and only if it is Cauchy with respect to the associated metric* $(X, d_{(2)})$*.*

Proof. Let x_n by 0-Cauchy, then there exists $N \in \mathbb{N}$ such that $N > 2$ and

$$\lim_{m\longrightarrow\infty} \sup_{n_i>m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) = 0 \textit{ for all } k \geq N$$

$$\Rightarrow \quad \lim_{n_i\longrightarrow\infty} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) = 0 \textit{ for all } k \geq N$$

$$\Rightarrow \quad 0 \leq \lim_{n_i\longrightarrow\infty} d(x_{n_1}, x_{n_2}) \leq \lim_{n_i\longrightarrow\infty} d(x_{n_1}, x_{n_2}, ..., x_{n_N}) = 0$$

$$\Rightarrow \quad \lim_{n_i\longrightarrow\infty} d(x_{n_1}, x_{n_2}) = 0$$

Thus, x_n is Cauchy sequence in $(X, d_{(2)})$. For the other direction, we have

$$\lim_{n_i,n_j\longrightarrow\infty} d(x_{n_i}, x_{n_j}) = 0,$$

which we combine with the properties of the MP-metric to get

$$0 \leq d(x_{n_1}, x_{n_2}, ..., x_{n_k}) \leq \sum_{i=1}^{k} d(x_{n_1}, x_{n_i}) \textit{ for all } k \geq 1$$

$$\Rightarrow \quad 0 \leq \lim_{m\longrightarrow\infty} \sup_{n_i>m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) \leq \lim_{m\longrightarrow\infty} \sup_{n_i>m} \sum_{j=1}^{k} d(x_{n_1}, x_{n_j})$$

$$\Rightarrow \quad 0 \leq \lim_{m\longrightarrow\infty} \sup_{n_i>m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) \leq \sum_{j=1}^{k} \lim_{n_1,n_j\longrightarrow\infty} d(x_{n_1}, x_{n_j})$$

$$\Rightarrow \quad 0 \leq \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) \leq 0$$

$$\Rightarrow \quad \lim_{m \longrightarrow \infty} \sup_{n_i > m} d(x_{n_1}, x_{n_2}, ..., x_{n_k}) = 0 \textit{ for all } k \geq 1.$$

Therefore, (x_n) is 0-Cauchy in the MP-Metric spaces (X, d).

□

Theorem 4.6. (*Completeness*) *Let* $(\mathbb{R}, d)$ *be the MP-metric space introduced in Example 2.6. If* x_n *is r-Cauchy for some* $r > 0$ *in* $(\mathbb{R}, d)$*, then there are* $a, b \in \mathbb{R}$ *such that* x_n *is d-convergent to* $D(\{a, b\})$.

Proof. The sequence x_n is bounded (by Proposition 4.3); hence, there are $a, b \in \mathbb{R}$ satisfying $a = \liminf_{n\to\infty} x_n$, *and* $b = \limsup_{n\to\infty} x_n$. For all $\epsilon > 0$ there are N_ϵ^1, $N_\epsilon^2 \in \mathbb{N}$ such that $|\inf_{n \geq N_\epsilon^1} x_n - a| < \epsilon$, and $|\sup_{n \geq N_\epsilon^2} x_n - b| < \epsilon$. In the obvious way, choosing $N_\epsilon = \max\{N_\epsilon^1, N_\epsilon^2\}$, gives $a - \epsilon < x_n < b + \epsilon$ *for all* $n \geq N_\epsilon$. Thus, for any $k \in \mathbb{N}$, we get

$$\begin{aligned} d(x_{n_1}, x_{n_2}, ..., x_{n_k}, a, b) &= \max\{x_{n_1}, x_{n_2}, ..., x_{n_k}, a, b\} - \min\{x_{n_1}, x_{n_2}, ..., x_{n_k}, a, b\} \\ &\leq b + \epsilon - (a - \epsilon), \qquad \textit{for all } n_i \geq N_\epsilon \\ &= d(a, b) + 2\epsilon, \qquad \textit{for all } n_i \geq N_\epsilon. \end{aligned}$$

Hence, we obtain

$$\lim_{n_i \longrightarrow \infty} d(x_{n_1}, x_{n_2}, ..., x_{n_k}, a, b) = d(a, b), \qquad \textit{for all } k \in \mathbb{N}.$$

Now, using the properties of a and b, and the fact that $\mathbb{N}$ is a well-ordered set, we inductively define the subsequences x_{n_l} and x_{m_l} in the following way: $n_1 = m_1 = 1$. For $n_l, m_l \geq 2$, we set

$$n_l = \min\{n \in \mathbb{N} : |x_n - a| < \frac{1}{l} \textit{ and } n > n_{l-1}\},$$

$$m_l = \min\{n \in \mathbb{N} : |x_n - b| < \frac{1}{l} \textit{ and } n > m_{l-1}\}.$$

Therefore, $\lim_{n_l \longrightarrow \infty} x_{n_l} = a$ *and* $\lim_{m_l \longrightarrow \infty} x_{m_l} = b$, which in view of Remark (3.6) gives

$$\liminf_{n \longrightarrow \infty} d(x_n, a) = \liminf_{n \longrightarrow \infty} d(x_n, b) = 0,$$

and that completes the proof. □

5. Products of MP-metric Spaces

Suppose (X_i, d_i) are MP-metric spaces for all $i = 1, ..., n$. Then, in the standard way, we can construct an MP-metric on the product space $\prod_{i=1}^{n} X_i$. Let $A = \{a_1, ..., a_m\} \in P^*(\prod_{i=1}^{n} X_i)$, where $a_j = (a_{j1}, ..., a_{jn})$ and $a_{ji} \in X_i$ for all $j = 1, ..., m$ and $i = 1, ..., n$. We define the MP-metrics: $\delta_1, \delta_2 : \prod_{i=1}^{n} X_i \longrightarrow [0, \infty)$ in the following way.

$$\delta_1(A) = \sum_{i=1}^{n} d_i(a_{1i}, a_{2i}..., a_{mi}), \text{ and}$$

$$\delta_2(A) = \sqrt{\sum_{i=1}^{n} [d_i(a_{1i}, a_{2i}..., a_{mi})]^2}.$$

Theorem 5.1. *Let (X_i, d_i) be MP-metric spaces for all $i = 1, ..., n$, then $(\prod_{i=1}^{n} X_i, \delta_1)$ and $(\prod_{i=1}^{n} X_i, \delta_2)$ are MP-metrics spaces.*

Proof. We will show that $(\prod_{i=1}^{n} X_i, \delta_2)$ is an MP-metric space while the result for $(\prod_{i=1}^{n} X_i, \delta_1)$ can be obtained similarly. For any $A, B \in P^*(\prod_{i=1}^{n} X_i)$, we assume $A = \{a_1, ..., a_m\}$, and $B = \{b_1, ..., b_k\}$ where $A = \{(a_{11}, ..., a_{1n}), ..., (a_{m1}, ..., a_{mn})\}$, and $B = \{(b_{11}, ..., b_{1n}), ..., (b_{k1}, ..., b_{kn})$.

(M1) Let $A \in P^*(\prod_{i=1}^{n} X_i)$.

$$\delta_2(A) = \sqrt{\sum_{i=1}^{n} [d_i(a_{1i}, a_{2i}..., a_{mi})]^2} = 0.$$

$$\Leftrightarrow \quad d_i(a_{1i}, a_{2i}..., a_{mi}) = 0 \textit{ for all } i = 1, ..., n \textit{ since } d_i \textit{ is an M-metric }.$$

$$\Leftrightarrow \quad a_{1i} = a_{2i} =, ..., = a_{mi} \textit{ for all } i = 1, ..., n.$$

$$\Leftrightarrow \quad a_1 = a_2 =, ..., = a_m. \quad \Leftrightarrow \quad |A| = 1$$

(M2) Let $A \subset B$. Then

$$\{a_{1i}, a_{2i}..., a_{mi}\} \subseteq \{b_{1i}, b_{2i}..., b_{ki}\} \Rightarrow d_i(a_{1i}, a_{2i}..., a_{mi}) \leq d_i(b_{1i}, b_{2i}..., b_{ki}) \Rightarrow \delta_2(A) \leq \delta_2(B).$$

(M3) Suppose $A \cap B \neq \phi$

$$\Rightarrow \quad \{a_{1i}, a_{2i}..., a_{mi}\} \cap \{b_{1i}, b_{2i}..., b_{ki}\} \neq \phi \textit{ for all } i = 1, ..., n$$

$$\Rightarrow \quad d_i(\{a_{1i}, ..., a_{mi}\} \cup \{b_{1i}, ..., b_{ki}\}) \leq d_i(\{a_{1i}, ..., a_{mi}\}) + d_i(\{b_{1i}, ..., b_{ki}\}) \textit{ for all } i = 1, ..., n.$$

Therefore, setting $\alpha_i = d_i(\{a_{1i}, ..., a_{mi}\})$ and $\beta_i = d_i(\{b_{1i}, ..., b_{ki}\})$ and applying the Minkowski inequality (see for example [14])

$$(\sum_{i=1}^{n} |\alpha_i + \beta_i|^p)^{\frac{1}{p}} \leq (\sum_{i=1}^{n} |\alpha_i|^p)^{\frac{1}{p}} + (\sum_{i=1}^{n} |\beta_i|^p)^{\frac{1}{p}}$$

with $p = 2$, we get $\delta_2(A \cup B) \leq \delta_2(A) + \delta_2(B)$.

(M4) Suppose $d_\Pi(A \cup \{b_l\}) = d_\Pi(A)$ for all $b_l = (b_{l1}, ..., b_{ln}) \in B$.

$$\Rightarrow \quad \sqrt{\sum_{i=1}^{n}[d_i(a_{1i}, a_{2i}..., a_{mi}, b_{li})]^2} = \sqrt{\sum_{i=1}^{n}[d_i(a_{1i}, a_{2i}..., a_{mi})]^2} \textit{ for all } l = 1, ..., k$$

$$\Rightarrow \quad d_i(a_{1i}, a_{2i}..., a_{mi}, b_{li}) = d_i(a_{1i}, a_{2i}..., a_{mi}) \textit{ for all } i = 1, ..., n \textit{ and } l = 1, ..., k$$

$$(\textit{ where we used the facts } d_i \textit{ is nonnegative and } d_i(a_{1i}, a_{2i}..., a_{mi}) \leq d_i(a_{1i}, a_{2i}..., a_{mi}, b_{li}))$$

$$\Rightarrow \quad d_i(a_{1i}, a_{2i}..., a_{mi}, b_{1i}, ..., b_{ki}) = d_i(a_{1i}, a_{2i}..., a_{mi}) \textit{ for all } i = 1, ..., n$$

$$\Rightarrow \quad \sum_{i=1}^{n}[d_i(a_{1i}, a_{2i}..., a_{mi}, b_{1i}, ..., b_{ki})]^2 = \sum_{i=1}^{n}[d_i(a_{1i}, a_{2i}..., a_{mi})]^2$$

$$\Rightarrow \quad \delta_2(A \cup B) = \delta_2(A).$$

Therefore, $(\prod_{i=1}^{n} X_i, \delta_2)$ is an MP-metric space. □

Proposition 5.2. *For all $i = 1, ..., n$, let (X_i, d_i) be MP-metric spaces, and let $A_i \in X_i$. Then, each of the following holds.*

(i) $\delta_1(\prod_{i=1}^{n} A_i) = \sum_{i=1}^{n} d_i(A_i)$,

(ii) $\delta_2(\prod_{i=1}^{n} A_i) = \sqrt{\sum_{i=1}^{n}[d_i(A_i)]^2}$,

(iii) $D_{\delta_1}(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} D_{d_i}(A_i)$,

(iv) $D_{\delta_2}(\prod_{i=1}^{n} A_i) = \prod_{i=1}^{n} D_{d_i}(A_i)$,

(v) $D_{\delta_1}(A) = \prod_{i=1}^{n} D_{d_i}(\{a_{1i}, ..., a_{mi})\}, \forall\, A = \{(a_{11}, ..., a_{1n}), ..., (a_{m1}, ..., a_{mn})\} \subseteq \prod_{i=1}^{n} X_i$,

(vi) $D_{\delta_2}(A) = \prod_{i=1}^{n} D_{d_i}(\{a_{1i}, ..., a_{mi}\}), \forall\, A = \{(a_{11}, ..., a_{1n}), ..., (a_{m1}, ..., a_{mn})\} \subseteq \prod_{i=1}^{n} X_i$.

Proof. (i) Let $(\prod_{i=1}^{n} A_i) = \{a_1, ..., a_m\}$ where $a_j = (a_{j1}, ..., a_{jn})$ and $a_{ji} \in A_i$. Thus, using Proposition 2.4-i with the fact $\{a_{1i}, ..., a_{mi}\} = A_i$ we get the wanted result in the following way

$$\delta_1(\prod_{i=1}^{n} A_i) = \delta_1(\{a_1, ..., a_m\}) = \sum_{i=1}^{n} d_i(a_{1i}, a_{2i}..., a_{mi}) = \sum_{i=1}^{n} d_i(A_i)$$

(iii) Suppose $x = (x_1, ..., x_n) \in \prod_{i=1}^{n} X_i$, then

$$x \in D_{\delta_1}(\prod_{i=1}^{n} A_i)$$

$$\Leftrightarrow \quad \delta_1(\{x\} \cup \prod_{i=1}^{n} A_i)) = \delta_1(\prod_{i=1}^{n} A_i))$$

$$\Leftrightarrow \quad \sum_{i=1}^{n} d_i(x_i, a_{1i}, a_{2i}..., a_{mi}) = \sum_{i=1}^{n} d_i(a_{1i}, a_{2i}..., a_{mi})$$

$$\Leftrightarrow \quad d_i(x_i, a_{1i}, a_{2i}..., a_{mi}) = d_i(a_{1i}, a_{2i}..., a_{mi}) \text{ for all } i = 1, ..., n$$

$$\Leftrightarrow \quad d_i(\{x_i\} \cup A_i) = d_i(A_i) \text{ for all } i = 1, ..., n$$

$$\Leftrightarrow \quad x_i \in D(A_i) \text{ for all } i = 1, ..., n$$

$$\Leftrightarrow \quad x \in \prod_{i=1}^{n} D_{d_i}(A_i).$$

The remaining: $(ii), (iv), (v)$ and (vi) can be shown in the same way. □

Theorem 5.3. *Let (X_i, d_i) be complete MP-metric spaces for all $i = 1, ..., n$, then $(\prod_{i=1}^{n} X_i, \delta_1)$ and $(\prod_{i=1}^{n} X_i, \delta_2)$ are complete MP-metrics spaces.*

Proof. Let $(x_l) = (x_l^1 . x_l^2, ..., x_l^n)$ be an r-Cauchy in $(\prod_{i=1}^{n} X_i, \delta_1)$ for some $r \geq 0$. Thus, there is $N \in \mathbb{N}$ such that

$$\lim_{m \longrightarrow \infty} \sup_{l_i > m} \sum_{i=1}^{n} d_i(x_{l_1}^i, x_{l_2}^i, ..., x_{l_k}^i) = r \text{ for all } k \geq N.$$

Using the nonnegativity and the monotonicity of the MP-metric (M2), we get

$$\lim_{m \longrightarrow \infty} \sup_{l_i > m} d_i(x_{l_1}^i, x_{l_2}^i, ..., x_{l_k}^i) = r_i \text{ for some } 0 \leq r_i \leq r \text{ and for all } k \geq N.$$

Therefore, (x_l^i) is r_i-Cauchy in (X_i, d_i), and hence by completeness of (X_i, d_i), there is $A_i \in P^*(X_i)$ such that $\lim x_l^i = D(A_i)$. Applying Prposition 5.2 we get

$$\lim_{n \to \infty} \delta_1(\{x_{l_1}, x_{l_2}, ..., x_{l_k}\} \cup \prod_{i=1}^{n} A_i) = \delta_1(\prod_{i=1}^{n} A_i)$$

Also, for all $a = (a^1, a^2, ..., a^n) \in \prod_{i=1}^{n} A_i$ we have

$$\liminf_{l \longrightarrow \infty} \delta_1(x_l, a) = \liminf_{l \longrightarrow \infty} \sum_{i=1}^{n} d_i(x_l^i, a^i) = 0.$$

Thus, $\lim x_l = D(\prod_{i=1}^{n} A_i)$, which proves the cmpleteness of $(\prod_{i=1}^{n} X_i, \delta_1)$. Similarlly, we can show that $(\prod_{i=1}^{n} X_i, \delta_2)$ is a complete MP-metric space. □

There are many ways to define an MP-metric on the Euclidean Spaces $\mathbb{R}^n$. One possible way is to consider the product space of the MP-metric space $(\mathbb{R}, d)$ where $d(a_1, ..., a_k) = \max_{1 \leq j \leq k}\{a_j\} - \min_{1 \leq j \leq k}\{a_j\}$ (see Example 2.6). That is for $a_j = (a_{j1}, ..., a_{jn}) \in \mathbb{R}^n$, we get

$$\delta_1(a_1, ..., a_k) = \sum_{i=1}^{n} [\max_{1 \leq j \leq k} \{a_{ji}\} - \min_{1 \leq j \leq k} \{a_{ji}\}] \tag{3}$$

or

$$\delta_2(a_1, ..., a_k) = \sqrt{\sum_{i=1}^{n} [\max_{1 \leq j \leq k} \{a_{ji}\} - \min_{1 \leq j \leq k} \{a_{ji}\}]^2} \tag{4}$$

The MP-metric δ_2 is natural on $\mathbb{R}^n$ in the sense that once we restrict it to two points, it coincides with the usual metric on the Euclidean space. Geometrically, $\delta_2(A)$ is the length of the diameter of the minimal orthotope (hyperrectangle) containing A, while $\delta_1(A)$ is the sum of the lengths of that orthotope dimensions. However, these two MP-metrics have some equivalence since they satisfy $\delta_2(A) \leq \delta_1(A) \leq \sqrt{n} \cdot \delta_2(A)$.

Theorem 5.4. (*Completeness*) $(\mathbb{R}^n, \delta_1)$ *and* $(\mathbb{R}^n, \delta_2)$ *are complete MP-metric spaces where* δ_2 *and* δ_2 *are defined as in* (3) *and* (4).

Proof. The result follows directly from Theorem 4.6 and Theorem 5.3. □

Remark 5.5. *In the MP-metric spaces* $(\mathbb{R}^n, \delta_2)$ *and* $(\mathbb{R}^n, \delta_1)$*,* $D(A)$ *is the solid minimal orthotope containing* A (*see Example 3.3-*ii *and Proposition 5.2-*v, vi). *That is, in the convergence theory in MP-metric spaces, we have directly extended the concept of the limit of a sequence from a single point to a compact set* (*a closed orthotope in* $\mathbb{R}^n$).

6. Discussion

This paper has potential interests in different fields, such as related theories, graph theory, and practical applications.

There are many open areas in the related theories. For example, there is the study of the emerging topologies and their properties. Also, we can investigate different results concerning the fixed point theory. One may study the functions and their behavior in the MP-metrics spaces.

On the other hand, the fact that the MP-metric can tackle any finite number of points motivates us to consider exploring the graph theory from a metric perspective. The starting point here is to treat the vertices of a graph as points in MP-metric spaces. Then, we can assign values to graphs, which means we can compare them and go further in that direction.

Moreover, extending the concept of the limit from a single point to a set (an interval in $\mathbb{R}$) is suitable for dealing with different real-life problems. In many practical applications, it is not necessary to get an exact value, but rather, it is sufficient to get a reasonable approximation. This way of thinking is compatible with the introduced notion of convergence in MP-metrics.

Conflicts of Interest

The author declares that there are no conflicts of interest regarding the publication of this paper.

References

[1] S. Czerwik, Contraction mappings in b-metric spaces, Acta Math. Inf. Univ. Ostrav. 1 (1993), 5–11. https://eudml.org/doc/23748.

[2] S.G. Matthews, Metric domains for completeness, Ph.D. Thesis, University of Warwick, Coventry, 1985. http://go.warwick.ac.uk/wrap/60767.

[3] P. Hitzler, A.K. Seda, Dislocated topologies, J. Electr. Eng. 51 (2000), 3–7.

[4] D.R. Kurepa, Tableaux ramifiés d'ensembles. Espaces pseudo-distanciés. C. R. Acad. Sci. Paris. 198 (1934), 1563–1565.

[5] W.S. Du, A note on cone metric fixed point theory and its equivalence, Nonlinear Anal.: Theory Methods Appl. 72 (2010), 2259–2261. https://doi.org/10.1016/j.na.2009.10.026.

[6] S. Gähler, 2-metrische Räume und ihre topologische Struktur, Math. Nachr. 26 (1963), 115–148. https://doi.org/10.1002/mana.19630260109.

[7] S. Gähler, Zur geometric 2-metriche raume, Rev. Roum. Math. Pures Appl. XI (1966), 664–669.

[8] B.C. Dhage, Generalized metric space and mapping with fixed point, Bull. Cal. Math. Soc. 84 (1992), 329–336.

[9] B.C. Dhage, Generalized metric spaces and topological structure, I. An. Ştiinţ. Univ. Al. I. Cuza Iaşi. Mat. (N.S.) 46 (2000), 3–24.

[10] Z. Mustafa, B. Sims, Some remarks concerning D-metric spaces, in: Proceedings of the International Conferences on Fixed Point Theory and Applications, Valencia, Spain, July 2003, pp. 189–198.

[11] Z. Mustafa, B. Sims, A new approach to generalized metric spaces, J. Nonlinear Convex Anal. 2 (2006), 289–297.

[12] K.A. Khan, On the possibility of N-topological spaces, Int. J. Math. Arch. 3 (2012), 2520–2523.

[13] K.A. Khan, Generalized n-metric spaces and fixed point theorems, J. Nonlinear Convex Anal. 15 (2014), 1221–1229.

[14] J. Muscat, Functional analysis: an introduction to metric spaces, Hilbert spaces, and Banach algebras, Springer, Cham, 2014. https://doi.org/10.1007/978-3-319-06728-5.