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# TRANSIENT NUMERICAL ANALYSIS FOR MARKOVIAN HETEROGENEOUS ARRIVALS QUEUE WITH WORKING VACATION, TWO-PHASE SERVICE AND IMPATIENT CUSTOMERS

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Abstract. This paper discusses a $M/M/1$ heterogeneous arrivals queueing system with working vacation (WV), two-phase service and impatient customers. Every time a customer enters the system, a random timer is started. If the service is not finished before the impatient timer expires, the customer may leave the system. The server gives service in two phases, once the first phase service is completed and the customer will go for second phase of service. The server goes for a WV if the system is empty. There is a chance that the arriving customers may balk with a certain probability. The time-dependent behavior of the model is studied using Runge-Kutta method. Some performance measures and cost analysis are also discussed.

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Key words and phrases. two-phase of services; steady state; impatient customer; balking; performance measures.

## 1. Introduction

Queueing system with vacation, impatient customers and second optional service are occur in our routine life. when the system has no customers then the server takes a vacation. In queueing theory, there are so many authors worked under the second optional service, various types of vacation and impatient customers. Ramanath and Kalidass [11] analyse a $M/G/1$ two-phase service and vacation queue with general retrial times. Sudesh et al. [13] study an single server vacation queueing system with a system disaster. Yue et al. [18] consider an $M/M/c$ queueing system with impatient customers and synchronous vacations. Perel and Yechiali [9] analyse Markovian c-server queues in a 2-phase random environment with impatient customers. Laxmi et al. [15] analyse an $M/M/1$ queue with infinite capacity, WV and customer's impatience during WV period. Zhang and Shan [19] study the

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$M/M/1$ queue with disasters and impatient customers. Kalidass and Kasturi [6] consider a $M/G/1$ two phase queueing system with a finite number of immediate Bernoulli feedbacks. Yohapriyadharsini and Suvitha [17] analse $M/M/1$ heterogeneous arrival with working vacation and impatient customers.

Rao et al. [12] deals a two-phase queue with customer impatience and server breakdowns. Bagyam et al. [1] analyse a 2-phase queueing system with batch arrival, impatient customers, breakdowns and delayed repair. Tadj et al. [14] study a queueing system with Bernoulli vacation and N-policy for an unreliable server with two-phase. Lakshmi and Ramanath [8] analyse a single server Markovian two-phase queue with impatient customers, Bernoulli feedback, breakdown and repair. Dahmane and Aissani [4] deal with optimal routing control of a retrial 2-phase queueing system. Rajendiran and Kandaiyan [10] analyse a $M^X/G(a,b)/1$ two phase queue with feedback, balking and server failure under Bernoulli vacation.

[3] analyse a two-phase retrial queue evaluation using Decision Making Trial and Evaluation Laboratory. Karthick and Suvitha [7] considers $M/M/3$ queueing system with multiple vacation and breakdown. Wu et al. [16] study a $Geo/G/1$ retrial queueing system with preferred and impatient customers. Bhagat and Jain [2] consider a $M^X/G/1$ retrial queueing system with customer impatience and multi-optional services. Devi et al. [5] analyse $M/H_k/1$ two-phase queueing system with server start-UP, N-Policy, server unreliable and customer balking.

This model can be applied to a number of real-world queueing systems. Here we consider an example of an two phase services in a mechanical shop. The function of this shop is to solve the customer's vehicle issues. Customers who encounter some issues with the mechanic, where the mechanic available to handle the vehicles. In this type, a mechanic is responsible for addressing customer inquiries. Once the vehicle's problem is solved, the customer can choose the water wash. After the mechanic is provided in this type, the vehicle's issue is resolved, and they leave the system. When no customer is waiting for service, the servers may go on WV.

## 2. Model Description

We consider a $M/M/1$ queuing system with impatient customers and multiple WV, in which the customers receives two phase of services (First phase and second phase).

- The arrival of customers according to a Poisson process with heterogeneous arrival rate $\lambda_i$, where

$$\lambda_i = \begin{cases} \lambda_0, & \text{arrival rate during WV} \\ \lambda_1, & \text{arrival rate during first phase} \\ \lambda_2, & \text{arrival rate during second phase} \end{cases}$$

- The customers served on a first-come first-served (FCFS) basis. Arriving customers are initially served in first phase. After completing the first phase of service the customers are served in second phase. The service times of the both phases have exponential distributions with parameters $\mu_1$ and $\mu_2$, respectively.

- When the server completes serving a customer and finds the system empty, the server leaves for a WV of some random period. On return from this WV if the server finds at least one customer, the server serves the customer for its first phase and continues to serve in this manner until the system is empty. Otherwise, the server immediately goes for another WV. The WV period has an exponential distribution with parameter $\phi$.
- A customer who arrives and finds at least one customer (i.e c customers) in the system, when the server is on WV period (first or second phase) either decides to enter the queue with probabilities $\beta_0$ ($\beta_1$ or $\beta_2$) or balk with probabilities $\beta_0'$ ($\beta_1'$ or $\beta_2'$) respectively.
- when a customer enters the system and realizes that the servers are on WV period, then the server activate an impatience timer, which is exponentially distributed with parameters $\epsilon_0$.

## 3. Mathematical model

Let $\mathcal{N}(t)$ be the number of customers in the system at time $t$, and $\mathcal{J}(t)$ represents the servers state at time $t$, where

$$\mathcal{J}(t) = \begin{cases} 0, \text{all the servers are in WV} \\ 1, \text{all the servers are in busy with first phase} \\ 2, \text{all the servers are in busy with second phase} \end{cases}$$

The process $\{\mathcal{N}(t), \mathcal{J}(t); t \geq 0\}$ is defined as a continuous-time Markov process with a state space $\Omega = \{(0,0)\} \bigcup \{(n,j), j = 0,1,2; n \geq 1\}$.

Defining the probabilities:

- $P_{0,0}(t)$ denotes probability that there is no customers in the system and the server is in WV period at time $t$.
- $P_{n,0}(t)$ denotes probability that there are $n$ customers in the system and the server is in WV period at time $t$, $n \geq 1$.
- $P_{n,1}(t)$ denotes probability that there are $n$ customers in the system and the server is in first phase at time $t$, $n \geq 1$.
- $P_{n,2}(t)$ denotes probability that there are $n$ customers in the system and the server is in second phase at time $t$, $n \geq 1$.

The state-transition diagram is given below in figure 1.

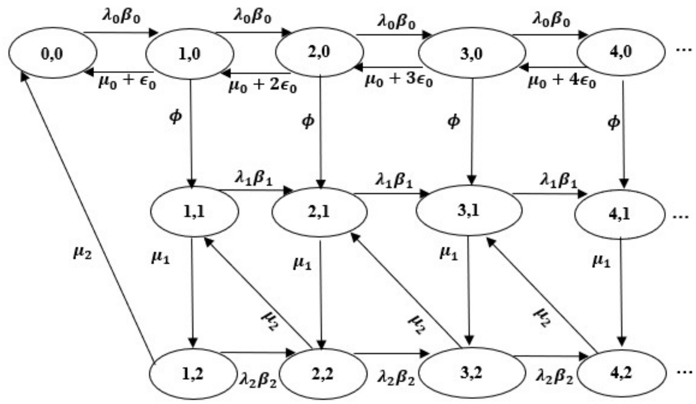


FIGURE 1. State-transition diagram

The transient equations of the model are:

$$\frac{d}{dt}P_{0,0}(t) = -\lambda_0\beta_0P_{0,0}(t) + +(\mu_0+\epsilon_0)P_{1,0}(t) + \mu_2P_{1,2} \tag{1}$$

$$\frac{d}{dt}P_{n,0}(t) = -(\lambda_0\beta_0+\mu_0+n\epsilon_0+\phi)P_{n,0}(t) + (\mu_0+(n+1)\epsilon_0)P_{n+1,0}(t) + \lambda_0\beta_0P_{n-1,0}(t), n \geq 1 \tag{2}$$

$$\frac{d}{dt}P_{1,1}(t) = -(\lambda_1\beta_1+\mu_1)P_{1,1}(t) + \phi P_{1,0}(t) + \mu_2P_{2,2}(t) \tag{3}$$

$$\frac{d}{dt}P_{n,1}(t) = -(\lambda_1\beta_1+\mu_1)P_{n,1}(t) + \lambda_1\beta_1P_{n-1,1}(t) + \phi P_{n,0}(t) + \mu_2P_{n+1,2}(t), n \geq 2 \tag{4}$$

$$\frac{d}{dt}P_{1,2}(t) = -(\lambda_2\beta_2+\mu_2)P_{1,2}(t) + \mu_1P_{1,1}(t) \tag{5}$$

$$\frac{d}{dt}P_{n,2}(t) = -(\lambda_2\beta_2+\mu_2)P_{n,2}(t) + \lambda_2\beta_2P_{n-1,2}(t) + \mu_1P_{n,1}(t), n \geq 2 \tag{6}$$

## 4. Numerical Analysis of the Model

We perform the model's transient analysis in this section. Since it is difficult to get an explicit analytical solution, we use the fourth-order Runge-Kutta method to get the transient solution. The transient numerical results are computed using MATLAB software's $ode45$ function.

## 5. Performance measures

Let us denote $E(t)$ be the expected size of the system and $W(t)$ be the expected waiting time in the system.

- $E(t) = \sum_{n=1}^{N} n[P_{n,0}(t) + P_{n,1}(t) + P_{n,2}(t)]$
- $W(t) = \dfrac{L(t)}{\mu_0(1-P_{0,0}(t))}$

In this section we find performance measures numerically by using MATLAB software. We fix the parameters as $\lambda = 1, \lambda_0 = 6\lambda, \lambda_1 = 8\lambda, \lambda_2 = 7\lambda, \mu = 1, \mu_0 = 9\mu, \mu_1 = 8\mu, \mu_2 = 7\mu, \beta = 1, \beta_0 = 0.6\beta, \beta_1 = 0.8\beta, \beta_2 = 0.7\beta, \epsilon_0 = 3, \phi = 2, N = 9$.

From figure 2, we increases the time and the probabilities are shown in this figure. Here, it is observed that all the probabilities increase to a certain period and after that it attains steady state. From figure 3 and 4, we increases the time and the probabilities are shown in this figures. Here, it is observed that all the probabilities increase to a certain period and after that it attains steady state. In figures 5, 6 and 7, we can increases the time and the expected size of the system gradually increases by varying lowering the values of $\mu$ and increasing the values of $\lambda, \beta$.

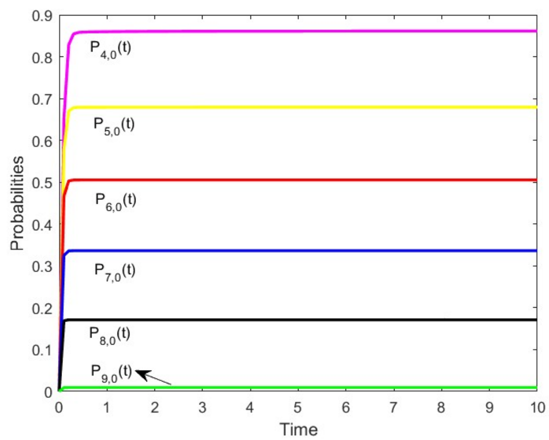


FIGURE 2. Time Vs probabilities for WV

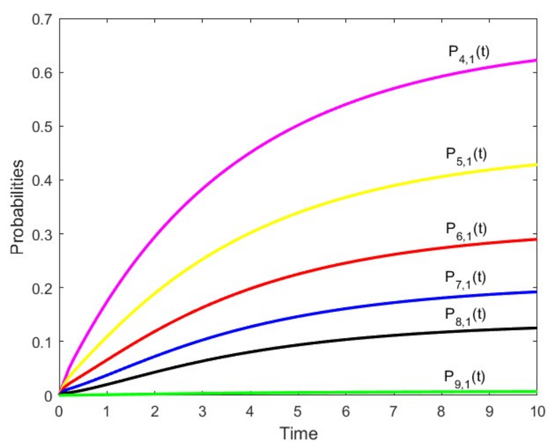


FIGURE 3. Time Vs probabilities of first phase

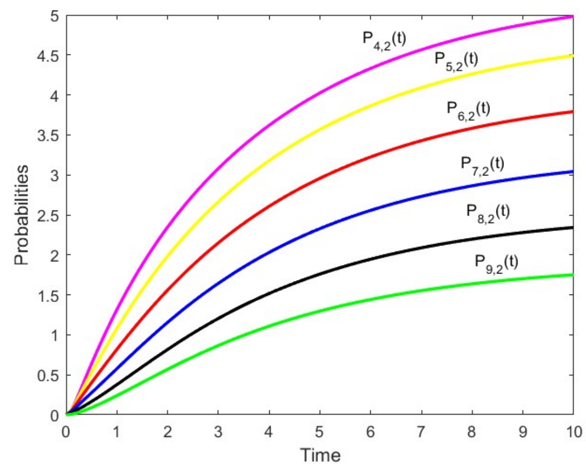


FIGURE 4. Time Vs probabilities of second phase

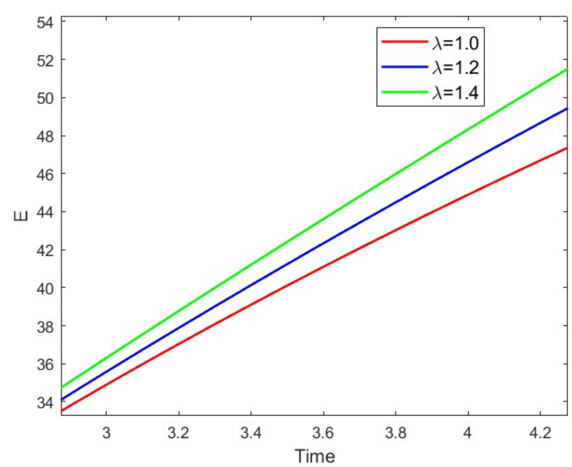


FIGURE 5. Time Vs E by varying $\lambda$

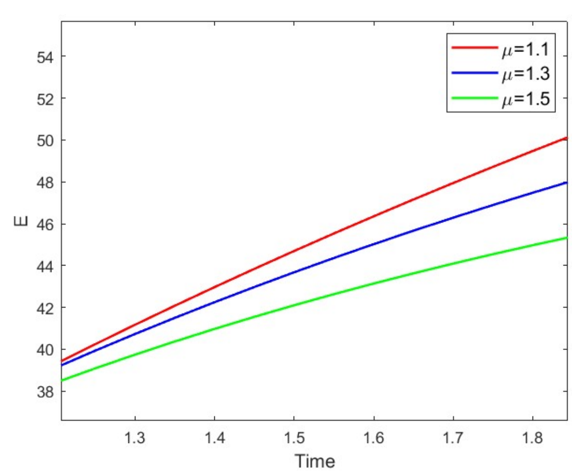


FIGURE 6. Time Vs E by varying $\mu$

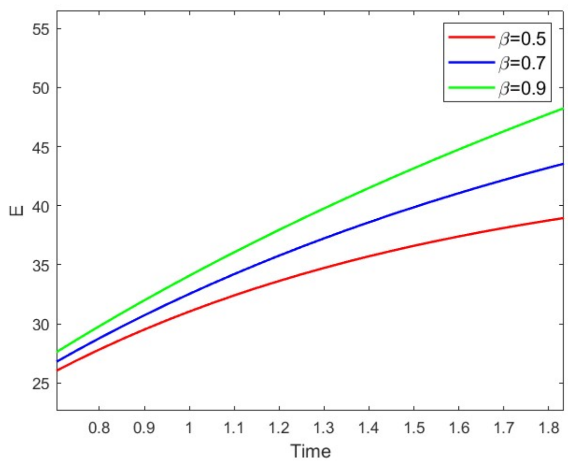


FIGURE 7. Time Vs E by varying $\beta$

## 6. Cost Analysis

In this section, we develop a model for the costs obtained in this queueing system. Let us consider the below notations.

- $C_0$ - Cost per unit period whenever the servers are busy.
- $C_1$ - Cost per unit period whenever the servers are on FES.
- $C_2$ - Cost per unit period whenever the servers are on SOS.
- $C_q$ - Cost per unit period whenever a customer joins the queue and waits for service.
- $C_b$ - Cost per unit period whenever a customer balks.
- $C_s$ - Cost per service per unit period.
- $C_F$ - Cost per unit to fixed server purchase.
- $TC$ - Expected total cost per unit period.
- $R_b$ - Average rate of balking.
- $E_s$ - Expected number of customers served per unit period.

$TC = C_0P_{.,0} + C_1P_{.,1} + C_2P_{.,2} + C_qW(t) + C_bR_b + (\mu_0 + \mu_1 + \mu_2)C_s + C_F$
where,
$R_b = (\lambda_0(1-\beta_0) + \lambda_1(1-\beta_1) + \lambda_2(1-\beta_2))\,(1 - P_{0,0}(t) - P_{1,1}(t) - P_{1,2}(t))\,.$
From table 1, we can increase the time values and find the variations of $\mu$, the total cost values are increase for certain extent after that it attains the steady state.

Table 1. Effect of $\mu$ on cost function by varying $t$

| Time($t$) | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|
| $\mu$=1.1 | 0.830 | 3.848 | 5.312 | 6.457 | 7.269 | 7.832 | 8.219 | 8.485 |
| $\mu$=1.2 | 0.890 | 3.750 | 4.849 | 5.620 | 6.106 | 6.405 | 6.587 | 6.698 |
| $\mu$=1.3 | 0.950 | 3.639 | 4.408 | 4.888 | 5.154 | 5.297 | 5.373 | 5.414 |
| $\mu$=1.4 | 1.010 | 3.699 | 4.468 | 4.948 | 5.214 | 5.357 | 5.433 | 5.474 |
| $\mu$=1.5 | 1.070 | 3.759 | 4.528 | 5.008 | 5.274 | 5.417 | 5.493 | 5.534 |
| $\mu$=1.6 | 1.130 | 3.819 | 4.588 | 5.068 | 5.334 | 5.477 | 5.553 | 5.594 |
| $\mu$=1.7 | 1.190 | 3.879 | 4.648 | 5.128 | 5.394 | 5.537 | 5.613 | 5.654 |
| $\mu$=1.8 | 1.250 | 3.939 | 4.708 | 5.188 | 5.454 | 5.597 | 5.673 | 5.714 |

## 7. Conclusion

In this paper, we have analyzed the single server queueing system with two phase of services with WV and impatient customers. Our queueing model approach and cost function are analysed using the $ode45$ function MATLAB software. The transient state probabilities, various performance measures and cost analyses are presented in this paper. In future, this model can be develop by various types of vacations, customer feedback, breakdown and impatience customers.

## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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