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ALGEBRAIC PROPERTIES OF INFINITELY DIVISIBLE MATRICES, SEPARABLE MATRICES AND HADAMARD POWERS OF A MATRIX

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Abstract. This research paper has provided a glimpse into the algebraic properties of various classes of matrices like infinitely divisible matrix, separable matrix and the set of hadamard powers of any given matrix. The study begins with a comprehensive exploration of the fundamental concepts and properties, including the binary operations, identity element, inverses and closure properties. A python program is included for finding the r^{th} hadamard power where $r \in R$, of a given matrix of any order along with its eigenvalues and eigenvectors. Isomorphisms are also established for the newly obtained algebraic structures with the pre-existing groups like $(R, +)$, $(Z, +)$, $(Z_n, +_n)$.

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1. Introduction

The hadamard product, often called the element-wise or entry-wise product is a fundamental operation in linear algebra and matrix theory [3], [6]. This simple yet powerful operation finds application in various fields such as statistics, signal processing, optimization etc. Consider the $m \times n$ matrices $A = [a_{ij}]$ and $B = [b_{ij}]$. Then the Hadamard product(or the entry wise product) of A and B is the matrix $A \circ B = [a_{ij}b_{ij}]$.

Example:

$$\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix} \circ \begin{bmatrix} 1 & 8 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 4 & -8 \\ 0 & 6 \end{bmatrix}$$

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The hadamard power, a concept rooted in linear algebra and matrix theory, represents an operation that elevates each element of a matrix to a specified power [4], [2]. For each non-negative integer m, the m^{th} Hadamard power of A is defined as $A^{\circ m} = [a_{ij}^m]$. For example:

$$\begin{bmatrix} 4 & -1 \\ 0 & 2 \end{bmatrix}^{\circ 3} = \begin{bmatrix} 64 & -1 \\ 0 & 8 \end{bmatrix}$$

For $a_{ij} \geq 0$ and for any non-negative real number r, the fractional Hadamard power is defined as $A^{\circ r} = [a_{ij}^r]$.

A matrix $A = [a_{ij}] \in M_n$ where $a_{ij} \geq 0$ is said to be infinitely divisible, if every fractional Hadamard power of A, defined as $A^{or} = [a_{ij}^r] \,\forall\, r \geq 0$, is positive semidefinite [1], [5].

The tensor product(Kronecker product) of matrices is a mathematical operation which combines two or more matrices to produce a new higher-dimensional matrix. The tensor product of $A = [a_{ij}] \in M_n(F)$ and $B = [b_{ij}] \in M_m(F)$ is denoted by $X = A \otimes B \in M_n(M_m(F))$ and defined to be the block matrix

$$X = A \otimes B = [a_{ij}B] = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix}$$

A separable matrix is one that can be decomposed into simpler independent submatrices. This decomposition can simplify complex matrix operations and aid in solving various mathematical problems, making separable matrices a subject of interest in linear algebra and its applications [7]. A matrix $X \in M_n(M_m)$ is said to be separable, if there exists positive semidefinite matrices $A_i \in M_n$ and $B_i \in M_m$ such that $X = \sum_{i=1}^{k} A_i \otimes B_i$.

2. Algebraic Properties of Various Classes of Matrix

In this section, we will be seeing the algebraic structure of different classes of matrix under different binary operations.

The following theorem gives a basic property of tensor product which will be used in the proof of Theorem 2.2.

Theorem 2.1. *Let A and B be two matrices of order $m \times n$ and $r \times s$ and let k be a scalar. Then $k(A \otimes B) = (kA) \otimes B = A \otimes (kB)$.*

Proof:

$k(A \otimes B) = k(a_{ij}B) = ka_{ij}B$

$(kA) \otimes B = (ka_{ij}) \otimes B = ka_{ij}B$

$A \otimes (kB) = a_{ij}kB = ka_{ij}B$

Therefore, $k(A \otimes B) = (kA) \otimes B = A \otimes (kB)$. □

In the following theorem, we will be considering the set of all infinitely divisible matrices under the operation tensor product.

Theorem 2.2. *Let I be the set of all infinitely divisible matrix under the operation $\otimes$. Then $(I, \otimes)$ forms a monoid.*

Proof:

Closure property

If $A,\ B \in I$, then $A \otimes B \in I$: Since $(A \otimes B)^{or} = [a_{ij}B]^{or} = [a_{ij}^{r}B^{or}] = A^{or} \otimes B^{or}\, \forall\, r \geq 0$; due to the fact that A and B are infinitely divisible, A^{or} and B^{or} are positive semidefinite $\forall\ r \geq 0$. Therefore, $A^{or} \otimes B^{or}$ is positive semidefinite $\forall\, r \geq 0$. Therefore, by above relation $(A \otimes B)^{or}$ is positive semidefinite $\forall\, r \geq 0$. Thus $A \otimes B$ is infinitely divisible. Therefore I is closed under $\otimes$.

Associative property

Consider the infinitely divisible matrices A, B and C of order $n \times n$, $m \times m$ and $r \times r$ respectively. Then order of both the matrices $(A \otimes B) \otimes C$ and $A \otimes (B \otimes C)$ will be $nmr \times nmr$.

$(A \otimes B) \otimes C = [a_{ij}B] \otimes C$

$$= \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}B & a_{n2}B & \cdots & a_{nn}B \end{bmatrix} \otimes C$$

$$= \begin{bmatrix} (a_{11}B) \otimes C & (a_{12}B) \otimes C & \cdots & (a_{1n}B) \otimes C \\ (a_{21}B) \otimes C & (a_{22}B) \otimes C & \cdots & (a_{2n}B) \otimes C \\ \cdots & \cdots & \cdots & \cdots \\ (a_{n1}B) \otimes C & (a_{n2}B) \otimes C & \cdots & (a_{nn}B) \otimes C \end{bmatrix}$$

By Theorem 2.1,

$$= \begin{bmatrix} a_{11}(B \otimes C) & a_{12}(B \otimes C) & \cdots & a_{1n}(B \otimes C) \\ a_{21}(B \otimes C) & a_{22}(B \otimes C) & \cdots & a_{2n}(B \otimes C) \\ \cdots & \cdots & \cdots & \cdots \\ a_{n1}(B \otimes C) & a_{n2}(B \otimes C) & \cdots & a_{nn}(B \otimes C) \end{bmatrix}$$

$= A \otimes (B \otimes C)$

Existence of identity element

Consider the 1×1 matrix $[1]$. Clearly $[1] \in I$. Then $A \otimes [1] = [1] \otimes A = A$, for any infinitely divisible matrix A.

Thus, I forms a monoid under $\otimes$. $\square$

Since $(I, \otimes)$ is a monoid, it is also a semigroup. But $(I, \otimes)$ does not form a group, since inverse does not exist.

For the next theorem, we will be considering the set of all $n \times n$ infinitely divisible matrices under the operation addition.

Theorem 2.3. *Let ρ be the set of all $n \times n$ infinitely divisible matrix under the operation $+$. Then $(\rho, +)$ forms a monoid.*

Proof:

Closure property

Let ρ be the set of all $n \times n$ infinitely divisible matrix. Let $A,\ B \in \rho$, then $A + B \in \rho$: Suppose $A, B \in \rho$. $(A+B)^{or}$, for any $r \geq 0$, can be decomposed as $A^{or} + B^{or} + S_1 + S_2 + \cdots$, where $S_i = \alpha_i A^{or_1} \circ B^{or_2}$ and $\alpha_i, r_1, r_2 \geq 0$. Now, since A and B are infinitely divisible matrices, A^{or} and B^{or} are positive semidefinite for any $r \geq 0$. And also by Schur's theorem, Hadamard product of positive semidefinite matrices is again positive semidefinite. Therefore, $S_i = \alpha_i A^{or_1} \circ B^{or_2}$, $\alpha_i, r_1, r_2 \geq 0$ is positive semidefinite. i.e., Every term in the decomposition of $(A + B)^{or}$ is positive semidefinite for any $r \geq 0$. Since the sum of positive semidefinite matrix is again positive semidefinite, $(A + B)^{or}$ is positive semidefinite $\forall\ r \geq 0$. Thus $A + B$ is infinitely divisible.

Associative property

Consider the infinitely divisible matrices A, B and C of order $n \times n$.

$A + (B + C) = (A + B) + C$ is true, since matrix addition is associative.

Existence of identity element

Consider the zero matrix of order $n \times n$ denoted by $[0]$, which is clearly infinitely divisible. Then $A + [0] = [0] + A = A$, for any infinitely divisible matrix A.

Thus, ρ forms a monoid under $+$. $\square$

Since $(\rho, +)$ is a monoid, it is also a semigroup. But $(\rho, +)$ does not form a group, since inverse does not exist, as the inverse matrices will be negative semidefinite.

For the following theorem, we will be examining the set of all $n \times n$ separable matrices under the operation addition.

Theorem 2.4. *Let S be the set of all $n \times n$ separable matrix under the operation $+$. Then $(S, +)$ forms a monoid.*

Proof:

Closure property

Let $A,\ B \in S$, then $A = \sum_{i=1}^{k} P_i \otimes Q_i$ and $B = \sum_{j=1}^{r} U_j \otimes V_j$, where P_i, Q_i, U_j, V_j are all positive semidefinite

$\forall\ i = 1, 2 \dots k\ \&\ j = 1, 2, \dots\ r$. Thus, $A + B = \sum_{i=1}^{k} P_i \otimes Q_i + \sum_{j=1}^{r} U_j \otimes V_j$ is separable, since P_i, Q_i, U_j, V_j are all positive semidefinite $\forall\ i = 1, 2 \dots k\ \&\ j = 1, 2, \dots\ r$. Therefore, S is closed under $+$.

Associative property

Let $A, B, C \in S$. Then, $A + (B + C) = (A + B) + C$ is true, since matrix addition is associative.

Existence of identity element

Consider the zero matrix of order $n \times n$. Clearly $[0]_{n \times n} \in S$. Then $A + [0] = [0] + A = A$, for any separable matrix A.

Thus, S forms a monoid under $+$. □

Since $(S, +)$ is a monoid, it is also a semigroup. But $(S, +)$ does not form a group, since inverse does not exist, as the inverse matrices will be negative semidefinite.

Finding the hadamard powers of a given matrix and checking the positive definiteness of that matrix is not an easy task, especially when the order of the matrix is a higher value. The following is a python program for finding the r^{th} Hadamard power where $r \in R$, of a given matrix of any order and finding their eigenvalues and eigenvectors.

```
import numpy as np
import math

try:
  n = int(input("\nEnter the dimension of matrix:"))
except:
  print("\nInvalid Input, exiting the program")
  exit()

# Initialize matrix
matrix = []
print("\nEnter the entries row wise:")

# For user input
for i in range(n):              # A for loop for row entries
  a =[]
  for j in range(n):        # A for loop for column entries
    try:
      e = int(input())
    except:
      print("\nInvalid Input, exiting the program")
      exit()
```

```
    a.append(e)
  matrix.append(a)



m = np.array(matrix)

print("\nPrinting the input matrix:\n",m)



w, v = np.linalg.eig(m)

# printing eigen values
print("\nEigen values of the given matrix:\n",w)

# printing eigen vectors
print("\nEigenvectors of the given matrix:\n",v)

print("---------------------------------------------------------")

while 1==1:
  s = input("\nEnter the value of r:")
  if s=='.':
    break
  try:
    s=((s.lower()).replace('sqrt','math.sqrt')).replace\
    ('pi','math.pi')
    r = float(eval(s))
  except:
    print("\nInvalid Input, exiting the program")
    exit()

  hadamard = []

  for i in range(n):
    b =[]
    for j in range(n):
      b.append(matrix[i][j]**r)
    hadamard.append(b)

  h = np.array(hadamard)
```

```
print("\nPrinting the hadamard matrix:\n",h)

wh, vh = np.linalg.eig(h)

# printing eigen values
print("\nEigen values of the hadamard matrix:\n",wh)

# printing eigen vectors
print("\nEigenvectors of the hadamard matrix:\n",vh)

print("--------------------------------------------------------")
```

Note that a period$(.)$ can be entered as a value for r to exit the loop.

For the forthcoming theorem, we will be looking into the algebraic properties of the set $H = \{A^{\circ r}, r \in R\}$, where A is an $m \times n$ matrix with strictly positive entries under the operation hadamard product.

Theorem 2.5. *Consider any $m \times n$ matrix A with $a_{ij} > 0$. Then $H = \{A^{\circ r}, r \in R\}$ under the operation $\circ$ forms a group.*

Proof:

Closure property

$A^{\circ r_1} \circ A^{\circ r_1} = [a_{ij}^{r_1}.a_{ij}^{r_2}] = [a_{ij}^{r_1+r_2}] = A^{\circ(r_1+r_2)} \in H$

Therefore, H is closed under $\circ$.

Associative property

$(A^{\circ r_1} \circ A^{\circ r_2}) \circ A^{\circ r_3} = (a_{ij}^{r_1}.a_{ij}^{r_2}).a_{ij}^{r_3} = a_{ij}^{r_1}.(a_{ij}^{r_2}.a_{ij}^{r_3}) = A^{\circ r_1} \circ (A^{\circ r_2} \circ A^{\circ r_3})$. Hence H satisfies associative property under $\circ$.

Existence of identity element

Let $I_{m \times n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \cdots & \cdots & \cdots & \cdots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = A^{\circ(0)} \in H$. Then $A^{\circ r} \circ I = [a_{ij}^{r}.1] = [1.a_{ij}^{r}] = I \circ A^{\circ r} = A^{\circ r}$. Therefore, I forms the identity element for $(H, \circ)$.

Existence of inverse

Let $A^{\circ k} \in H$.

Then $A^{\circ k} \circ A^{\circ(-k)} = [a_{ij}^{k}.a_{ij}^{-k}] = [a_{ij}^{0}] = [1] = I$.

i.e., Inverse exist for each element.

Thus $(H, \circ)$ forms a group. □

For example, consider the $n \times n$ Hilbert matrix, $H = [h_{ij}] = [\frac{1}{(i+j-1)}]$. Here $h_{ij} > 0$ for all i, j. Take $G = \{H^{\circ r},\ r \in R\}$ under the operation $\circ$. Then $(G, \circ)$ forms a group.

Likewise, considering any matrix with positive entries, say $\begin{bmatrix} 10 & 3 \\ 4 & 12 \end{bmatrix}$, then $G' = \{ \begin{bmatrix} 10 & 3 \\ 4 & 12 \end{bmatrix}^{\circ r}, \ r \in R\}$ under the operation $\circ$ forms a group.

Also, note that a matrix of order $m \times n$ with non-zero entries is always invertible under the operation Hadamard product if the identity matrix is taken as flat matrix($m \times n$ matrix with all entries equal to 1). Such a matrix can be termed as Hadamard invertible. Hadamard inverse of a Hadamard invertible matrix $A = [a_{ij}]$ is $A^{\circ(-1)} = \frac{1}{a_{ij}}$.

Theorem 2.6. *Let $H = \{A^{\circ r};\ r \in R\}$, where A is any $m \times n$ matrix with $a_{ij} > 0$ and R be the set of all real numbers. Then $(H, \circ)$ is isomorphic to $(R, +)$.*

Proof:

Define $\phi : H \to R$ by $\phi(A^{\circ r}) = r$, where $r \in R$

ϕ is one-one: Since $\phi(A^{\circ r_1}) = \phi(A^{\circ r_2}) \implies r_1 = r_2$.

ϕ is onto: Since for each $r \in R$, there exist $A^{\circ r}$ such that $\phi(A^{\circ r}) = r$.

ϕ is homomorphic: Since $\phi(A^{\circ r_1} \circ A^{\circ r_2}) = \phi(A^{\circ(r_1+r_2)}) = r_1 + r_2 = \phi(A^{\circ r_1}) + \phi(A^{\circ r_2})$.

ϕ maps identity of $(H, \circ)$ to identity of $(R, +)$:

When $r = 0$; $A^{\circ r} = A^{\circ(0)} = [a_{ij}^0] = [1] = I_H$, identity of $(H, \circ)$.

Now, $\phi(A^{\circ 0}) = 0 = I_R$, identity of $(R, +)$.

Therefore, $\phi(I_H) = I_R$.

Let $k \in R$.

$\phi\left[(A^{\circ k})^{-1}\right] = \phi[(A^{\circ(-k)})] = -k$

$\left[\phi(A^{\circ k})\right]^{-1} = [k]^{-1} = -k$.

Thus, $\phi\left[(A^{\circ r})^{-1}\right] = [\phi(A^{\circ r})]^{-1}$ Thus, $(H, \circ)$ and $(R, +)$ are isomorphic groups. □

Thus, we can say $(H, \circ)$ is isomorphic to all groups to which $(R, +)$ is isomorphic.

Next corollary gives a subgroup of H in Theorem 2.5 which is isomorphic to the infinite cyclic group.

Corollary 2.6.1. *$H_Z = \{A^{\circ z};\ z \in Z\}$ is a subgroup of $H = \{A^{\circ r};\ r \in R\}$, where A is any $m \times n$ matrix with $a_{ij} > 0$ and $(H_Z, \circ)$ is isomorphic to the infinite cyclic group $(Z, +)$.*

Proof:

Consider any $m \times n$ matrix A and let $z_0 \in Z$ and $r_0 \in R$

$A^{\circ z_0} \in H_Z \Rightarrow A^{\circ z_0} \in H$, since $Z \subset R$

Let $A^{\circ z_1}, A^{\circ z_2} \in H_Z$ where $z_1, z_2 \in Z$

By one step subgroup test,

$A^{\circ z_1} \circ (A^{\circ z_2})^{-1} = A^{\circ z_1} \circ A^{\circ(-z_2)} = A^{\circ(z_1 - z_2)} \in H_Z$, since $z_1 - z_2 \in Z$

Therefore, H_Z is a subgroup of H.

Now, define $\phi : H_Z \to Z$ by $\phi(A^{\circ z}) = z$ where $z \in Z$.

Then it can be proved as in Theorem 2.6 that ϕ is bijective and homomorphic. Therefore, $(H_Z, \circ)$ is isomorphic to $(Z, +)$. $\square$

Theorem 2.7. *Consider any $m \times n$ matrix A with $a_{ij} > 0$. Then $H_{Z_n} = \{A^{\circ(z \bmod n)}, z \in Z_n\}$ under the operation $\circ$ forms a group of order n and is isomorphic to the finite cyclic group $(Z_n, +_n)$.*

Proof:

Closure property

$A^{\circ(z_1 \; mod \; n)} \circ A^{\circ(z_2 \; mod \; n)} = [a_{ij}^{(z_1 \; mod \; n)}.a_{ij}^{(z_2 \; mod \; n)}]$

$= [a_{ij}^{(z_1+z_2) \; mod \; n}] = A^{\circ(z_1+z_2) \; mod \; n} \in H_{Z_n}$,

since $(z_1 + z_2) \; mod \; n \in Z_n$

Therefore, H_{Z_n} is closed under $\circ$.

Associative property

$(A^{\circ(z_1 \; mod \; n)} \circ A^{\circ(z_2 \; mod \; n)}) \circ A^{\circ(z_3 \; mod \; n)}$

$= (a_{ij}^{(z_1 \; mod \; n)}.a_{ij}^{(z_2 \; mod \; n)}).a_{ij}^{(z_3 \; mod \; n)}$

$= a_{ij}^{(z_1 \; mod \; n)}.(a_{ij}^{(z_2 \; mod \; n)}.a_{ij}^{(z_3 \; mod \; n)})$

$= A^{\circ(z_1 \; mod \; n)} \circ (A^{\circ(z_2 \; mod \; n)} \circ A^{\circ(z_3 \; mod \; n)})$

Hence H_{Z_n} satisfies associative property under $\circ$.

Existence of identity element

Let $I_{m\times n} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \ldots & \ldots & \ldots & \ldots \\ 1 & 1 & \cdots & 1 \end{bmatrix} = A^{\circ(0)} \in H_{Z_n}$.

Then for $z \in Z_n$,

$A^{\circ(z \; mod \; n)} \circ I = [a_{ij}^{(z \; mod \; n)}.1] = [1.a_{ij}^{(z \; mod \; n)}]$

$= I \circ A^{\circ(z \; mod \; n)} = A^{\circ(z \; mod \; n)}$. Therefore, I forms the identity element for $(H_{Z_n}, \circ)$.

Existence of inverse

Let $A^{\circ(k \; mod \; n)} \in H_{Z_n}$.

Then $A^{\circ(k \; mod \; n)} \circ A^{\circ(n-k) \; mod \; n}$

$= [a_{ij}^{(k \; mod \; n)}.a_{ij}^{(n-k) \; mod \; n}] = [a_{ij}^{(k+n-k) \; mod \; n}]$

$= [a_{ij}^{(n \; mod \; n)}] = [a_{ij}^0] = [1] = I$.

i.e., Inverse exist for each element.

Thus $(H_{Z_n}, \circ)$ forms a group.

Define $\phi : H_{Z_n} \to Z_n$ by $\phi(A^{\circ(z \; mod \; n)}) = z \; mod \; n$ where $z \in Z_n$.

Then it can be proved as in Theorem 2.6 that ϕ is bijective and homomorphic. Therefore $(H_{Z_n}, \circ)$ is isomorphic to $(Z_n, +_n)$. $\square$

3. Conclusion

This research paper has offered an overview of the algebraic properties associated with different classes of matrix, including infinitely divisible matrices, separable matrices and the set of hadamard power of any matrix A with positive entries. The study commenced with an in-depth examination of the fundamental principles and characteristics, encompassing its binary operation, identity element, inverse elements and closure properties. Through rigorous examination, we have demonstrated that the above mentioned classes of matrix satisfies the axioms of various algebraic structures, making it a well-defined and coherent mathematical system.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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