Asia Pac. J. Math. 2024 **11**:21

(τ_1, τ_2)-CONTINUITY FOR FUNCTIONS

CHAWALIT BOONPOK, NAPASSANAN SRISARAKHAM*

Mathematics and Applied Mathematics Research Unit, Department of Mathematics, Faculty of Science, Mahasarakham University, Maha Sarakham, 44150, Thailand

*Corresponding author: napassanan.sri@msu.ac.th

Received Sep. 28, 2023

Abstract. This article deals with the concept of (τ_1, τ_2)-continuous functions. Moreover, some characterizations of (τ_1, τ_2)-continuous functions are established.

2020 Mathematics Subject Classification. 54C08; 54E55.

Key words and phrases. $\tau_1\tau_2$-open set; $\tau_1\tau_2$-closed set; (τ_1, τ_2)-continuous function.

1. Introduction

The field of the mathematical science which goes under the name of topology is concerned with all questions directly or indirectly related to continuity. Semi-open sets, preopen sets, α-open sets and β-open sets play an important role in the researches of generalizations of continuity. Levine [11] introduced and investigated the concept of semi-continuous functions. Mashhour et al. [14] introduced and investigated the notion of precontinuous functions. Mashhour et al. [13] introduced and studied the concept of α-continuous functions. Noiri [16] investigated several characterizations of α-continuous functions. Moreover, Noiri [15] introduced and studied the concept of almost α-continuous functions as a generalization of α-continuity. Abd El-Monsef et al. [1] introduced the notion of β-continuous functions as a generalization of semi-continuity and precontinuity. Marcus [12] introduced and investigated the concept of quasi-continuous functions. Borsík and Doboš [8] introduced the notion of almost quasi-continuity which is weaker than that of quasi-continuity and obtained a decomposition theorem of quasi-continuity. Popa and Noiri [17] investigated some characterizations of β-continuity and showed that almost quasi-continuity is equivalent to β-continuity. Viriyapong and Boonpok [18] introduced and studied the concept of (Λ, sp)-continuous functions. Furthermore, several characterizations of almost (Λ, s)-continuous functions were investigated in [2]. In [3], the present authors introduced and

DOI: 10.28924/APJM/11-21

studied the concept of weakly (Λ, p)-continuous functions. Laprom et al. [10] studied the notion of $\beta(\tau_1, \tau_2)$-continuous multifunctions. Viriyapong and Boonpok [19] introduced and investigated the concept of $(\tau_1, \tau_2)\alpha$-continuous multifunctions. Moreover, some characterizations of almost weakly (τ_1, τ_2)-continuous multifunctions and $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions were established in [4] and [5], respectively. In [7], the author investigated several characterizations of (i, j)-M-continuous functions in biminimal structure spaces. Dungthaisong et al. [9] introduced and studied the notion of $g_{(m,n)}$-continuous functions in bigeneralized topological spaces. In this article, we introduce the concept of (τ_1, τ_2)-continuous functions. In particular, several characterizations of (τ_1, τ_2)-continuous functions are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [6] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. Let A be a subset of a bitopological space (X, τ_1, τ_2). The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [6] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [6] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [6] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) *$A \subseteq \tau_1\tau_2$-Cl(A) and $\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Cl$(A)) = \tau_1\tau_2$-Cl(A).*
(2) *If $A \subseteq B$, then $\tau_1\tau_2$-Cl$(A) \subseteq \tau_1\tau_2$-Cl$(B)$.*
(3) *$\tau_1\tau_2$-Cl(A) is $\tau_1\tau_2$-closed.*
(4) *A is $\tau_1\tau_2$-closed if and only if $A = \tau_1\tau_2$-Cl(A).*
(5) *$\tau_1\tau_2$-Cl$(X - A) = X - \tau_1\tau_2$-Int$(A)$.*

3. On (τ_1, τ_2)-continuous functions

In this section, we introduce the concept of (τ_1, τ_2)-continuous functions. Furthermore, several characterizations of (τ_1, τ_2)-continuous functions are discussed.

Definition 1. *A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous at a point $x \in X$ if for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$, there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. A function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is called (τ_1, τ_2)-continuous if f has this property at each point of X.*

Theorem 1. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (τ_1, τ_2)-continuous at $x \in X$;*

(2) *$x \in \tau_1\tau_2$-Int$(f^{-1}(V))$ for every $\sigma_1\sigma_2$-open set V of Y containing $f(x)$;*

(3) *$x \in f^{-1}(\sigma_1\sigma_2$-Cl$(f(A)))$ for every subset A of X with*

$$x \in \tau_1\tau_2\text{-}Cl(A);$$

(4) *$x \in f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ for every subset B of Y with*

$$x \in \tau_1\tau_2\text{-}Cl(f^{-1}(B));$$

(5) *$x \in \tau_1\tau_2$-Int$(f^{-1}(B))$ for every subset B of Y with*

$$x \in f^{-1}(\sigma_1\sigma_2\text{-}Int(B));$$

(6) *$x \in f^{-1}(K)$ for every $\sigma_1\sigma_2$-closed set K of Y with*

$$x \in \tau_1\tau_2\text{-}Cl(f^{-1}(K)).$$

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (1), there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $U \subseteq f^{-1}(V)$ and hence $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$.

(2) $\Rightarrow$ (3): Let A be any subset of X, $x \in \tau_1\tau_2$-Cl(A) and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$$

and there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Since $x \in \tau_1\tau_2$-Cl(A), we have

$$\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A).$$

Thus, $f(x) \in \sigma_1\sigma_2$-Cl$(f(A))$ and hence $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(f(A)))$.

(3) $\Rightarrow$ (4): Let B be any subset of Y and $x \in \tau_1\tau_2$-Cl$(f^{-1}(B))$. By (3), we have $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(f(f^{-1}(B)))) \subseteq f^{-1}(\sigma_1\sigma_2$-Cl$(B))$ and hence $x \in f^{-1}(\sigma_1\sigma_2$-Cl$(B))$.

(4) $\Rightarrow$ (5): Let B be any subset of Y such that $x \notin \tau_1\tau_2$-Int$(f^{-1}(B))$. Then,

$$\begin{aligned} x \in X - \tau_1\tau_2\text{-Int}(f^{-1}(B)) &= \tau_1\tau_2\text{-Cl}(X - f^{-1}(B)) \\ &= \tau_1\tau_2\text{-Cl}(f^{-1}(Y - B)). \end{aligned}$$

By (4),

$$\begin{aligned} x \in f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)). \end{aligned}$$

Thus, $x \notin f^{-1}(\sigma_1\sigma_2$-Int$(B))$.

(5) $\Rightarrow$ (6): Let K be any $\sigma_1\sigma_2$-closed set of Y and $x \notin f^{-1}(K)$. Then,

$$x \in X - f^{-1}(K) = f^{-1}(Y - K) = f^{-1}(\sigma_1\sigma_2\text{-Int}(Y - K)).$$

By (5), we have

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(Y - K)) = \tau_1\tau_2\text{-Int}(X - f^{-1}(K)) = X - \tau_1\tau_2\text{-Cl}(f^{-1}(K))$$

and hence $x \notin \tau_1\tau_2\text{-Cl}(f^{-1}(K))$.

(6) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Suppose that $x \notin \tau_1\tau_2\text{-Int}(f^{-1}(V))$. Then,

$$x \in X - \tau_1\tau_2\text{-Int}(f^{-1}(V)) = \tau_1\tau_2\text{-Cl}(X - f^{-1}(V)) = \tau_1\tau_2\text{-Cl}(f^{-1}(Y - V)).$$

By (6), $x \in f^{-1}(Y - V) = X - f^{-1}(V)$ and hence $x \notin f^{-1}(V)$. This contraries to the hypothesis.

(2) $\Rightarrow$ (1): Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq f^{-1}(V)$. Thus, $f(U) \subseteq V$ and hence f is (τ_1, τ_2)-continuous at x. □

Theorem 2. *For a function $(X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$, the following properties are equivalent:*

(1) *f is (τ_1, τ_2)-continuous;*
(2) *$f^{-1}(V)$ is $\tau_1\tau_2$-open in X for every $\sigma_1\sigma_2$-open set V of Y;*
(3) *$f(\tau_1\tau_2\text{-}Cl(A)) \subseteq \sigma_1\sigma_2\text{-}Cl(f(A))$ for every subset A of X;*
(4) *$\tau_1\tau_2\text{-}Cl(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-}Cl(B))$ for every subset B of Y;*
(5) *$f^{-1}(\sigma_1\sigma_2\text{-}Int(B)) \subseteq \tau_1\tau_2\text{-}Int(f^{-1}(B))$ for every subset B of Y;*
(6) *$f^{-1}(K)$ is $\tau_1\tau_2$-closed in X for every $\sigma_1\sigma_2$-closed set K of Y.*

Proof. (1) $\Rightarrow$ (2): Let V be any $\sigma_1\sigma_2$-open set of Y and $x \in f^{-1}(V)$. Then, $f(x) \in V$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $f(U) \subseteq V$. Thus, $U \subseteq f^{-1}(V)$ and hence

$$x \in \tau_1\tau_2\text{-Int}(f^{-1}(V)).$$

Therefore, $f^{-1}(V) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(V))$. This shows that $f^{-1}(V)$ is $\tau_1\tau_2$-open in X.

(2) $\Rightarrow$ (3): Let A be any subset of X, $x \in \tau_1\tau_2\text{-Cl}(A)$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then, $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$ and there exists a $\tau_1\tau_2$-open set U of X such that $x \in U \subseteq f^{-1}(V)$. Since $x \in \tau_1\tau_2\text{-Cl}(A)$, we have $U \cap A \neq \emptyset$ and

$$\emptyset \neq f(U \cap A) \subseteq f(U) \cap f(A) \subseteq V \cap f(A).$$

Thus, $f(x) \in \sigma_1\sigma_2\text{-Cl}(f(A))$.

$(3) \Rightarrow (4)$: Let B be any subset of Y. Then by (3),

$$f(\tau_1\tau_2\text{-Cl}(f^{-1}(B))) \subseteq \sigma_1\sigma_2\text{-Cl}(f(f^{-1}(B))).$$

Thus, $\tau_1\tau_2\text{-Cl}(f^{-1}(B)) \subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(B))$.

$(4) \Rightarrow (5)$: Let B be any subset of Y. By (4), we have

$$\begin{aligned} X - \tau_1\tau_2\text{-Int}(f^{-1}(B)) &= \tau_1\tau_2\text{-Cl}(X - f^{-1}(B)) \\ &= \tau_1\tau_2\text{-Cl}(f^{-1}(Y - B)) \\ &\subseteq f^{-1}(\sigma_1\sigma_2\text{-Cl}(Y - B)) \\ &= f^{-1}(Y - \sigma_1\sigma_2\text{-Int}(B)) \\ &= X - f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \end{aligned}$$

and hence $f^{-1}(\sigma_1\sigma_2\text{-Int}(B)) \subseteq \tau_1\tau_2\text{-Int}(f^{-1}(B))$.

$(5) \Rightarrow (6)$: Let K be any $\sigma_1\sigma_2$-closed set of Y. Then, $Y - K$ is $\sigma_1\sigma_2$-open in Y and $Y - K = \sigma_1\sigma_2\text{-Int}(Y - K)$. By (5),

$$\begin{aligned} X - f^{-1}(K) &= f^{-1}(Y - K) \\ &= f^{-1}(\sigma_1\sigma_1\text{-Int}(Y - K)) \\ &\subseteq \tau_1\tau_2\text{-Int}(f^{-1}(Y - K)) \\ &= \tau_1\tau_2\text{-Int}(X - f^{-1}(K)) \\ &= X - \tau_1\tau_2\text{-Cl}(f^{-1}(K)). \end{aligned}$$

Thus,$\tau_1\tau_2\text{-Cl}(f^{-1}(K)) \subseteq f^{-1}(K)$. This shows that $f^{-1}(K)$ is $\tau_1\tau_2$-closed in X.

$(6) \Rightarrow (2)$: The proof is obvious.

$(2) \Rightarrow (1)$: Let $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. By (2), we have $x \in \tau_1\tau_2\text{-Int}(f^{-1}(V))$ and there exists a $\tau_1\tau_2$-open set U of X containing x such that $U \subseteq f^{-1}(V)$, Thus, $f(U) \subseteq V$ and hence f is (τ_1, τ_2)-continuous at x. This shows that f is (τ_1, τ_2)-continuous. □

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*compact* [6] if every cover of X by $\tau_1\tau_2$-open sets of X has a finite subcover.

Theorem 3. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (τ_1, τ_2)-continuous surjection and (X, τ_1, τ_2) is $\tau_1\tau_2$-compact, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-compact.*

Proof. Let $\{V_\gamma \mid \gamma \in \Gamma\}$ be any cover of Y by $\sigma_1\sigma_2$-open sets of Y. Since f is (τ_1, τ_2)-continuous, by Theorem 2, $\{f^{-1}(V_\gamma) \mid \gamma \in \Gamma\}$ is a cover of X by $\tau_1\tau_2$-open sets of X. Thus, there exists a finite subset

Γ_0 of Γ such that $X = \cup_{\gamma\in\Gamma_0} f^{-1}(V_\gamma)$. Since f is surjective, $Y = f(X) = \cup_{\gamma\in\Gamma_0} V_\gamma$. This shows that (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-compact. □

Recall that a bitopological space (X, τ_1, τ_2) is said to be $\tau_1\tau_2$-*connected* [6] if X cannot be written as the union of two nonempty disjoint $\tau_1\tau_2$-open sets.

Theorem 4. *If $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is a (τ_1, τ_2)-continuous surjection and (X, τ_1, τ_2) is $\tau_1\tau_2$-connected, then (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected.*

Proof. Suppose that (Y, σ_1, σ_2) is not $\sigma_1\sigma_2$-connected. There exist nonempty $\sigma_1\sigma_2$-open sets U and V of Y such that $U \cap V = \emptyset$ and $U \cup V = Y$. Then, $f^{-1}(U) \cap f^{-1}(V) = \emptyset$ and

$$f^{-1}(U) \cup f^{-1}(V) = X.$$

Moreover, $f^{-1}(U)$ and $f^{-1}(V)$ are nonempty $\tau_1\tau_2$-open sets of X. Thus, (X, τ_1, τ_2) is not (τ_1, τ_2)-connected. Therefore, (Y, σ_1, σ_2) is $\sigma_1\sigma_2$-connected. □

The $\tau_1\tau_2$-*frontier* of a subset A of a bitopological space (X, τ_1, τ_2), denoted by $\tau_1\tau_2$-fr(A), is defined by

$$\begin{aligned}\tau_1\tau_2\text{-fr}(A) &= \tau_1\tau_2\text{-Cl}(A) \cap \tau_1\tau_2\text{-Cl}(X - A)\\ &= \tau_1\tau_2\text{-Cl}(A) - \tau_1\tau_2\text{-Int}(A).\end{aligned}$$

Theorem 5. *The set of all points $x \in X$ at which a function $f : (X, \tau_1, \tau_2) \to (Y, \sigma_1, \sigma_2)$ is not (τ_1, τ_2)-continuous is identical with the union of the $\tau_1\tau_2$-frontier of the inverse images of $\sigma_1\sigma_2$-open sets containing $f(x)$.*

Proof. Suppose that f is not (τ_1, τ_2)-continuous at $x \in X$. There exists a $\sigma_1\sigma_2$-open set V of Y containing $f(x)$ such that $f(U) \not\subseteq V$ for every $\tau_1\tau_2$-open set U of X containing x. Then, $U \cap (X - f^{-1}(V)) \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x. Thus, $x \in \tau_1\tau_2$-Cl$(X - f^{-1}(V))$. On the other hand, we have $x \in f^{-1}(V) \subseteq \tau_1\tau_2$-Cl$(f^{-1}(V))$ and hence $x \in \tau_1\tau_2$-fr$(f^{-1}(V))$.

Conversely, suppose that f is (τ_1, τ_2)-continuous at $x \in X$ and V be any $\sigma_1\sigma_2$-open set of Y containing $f(x)$. Then by Theorem 1, we have $x \in \tau_1\tau_2$-Int$(f^{-1}(V))$. Thus, $x \notin \tau_1\tau_2$-fr$(f^{-1}(V))$ for each $\sigma_1\sigma_2$-open set V of Y containing $f(x)$. This completes the proof. □

Acknowledgements

This research project was financially supported by Mahasarakham University.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] M.E. Abd El-Monsef, S.N. El-Deeb, R.A. Mahmoud, β-open sets and β-continuous mappings, Bull. Fac. Sci. Assiut Univ. 12 (1983), 77–90.

[2] C. Boonpok, C. Viriyapong, On some forms of closed sets and related topics, Eur. J. Pure Appl. Math. 16 (2023), 336–362.

[3] C. Boonpok, C. Viriyapong, On (Λ, p)-closed sets and the related notions in topological spaces, Eur. J. Pure Appl. Math. 15 (2022), 415–436. https://doi.org/10.29020/nybg.ejpam.v15i2.4274.

[4] C. Boonpok, C. Viriyapong, Upper and lower almost weak (τ_1, τ_2)-continuity, Eur. J. Pure Appl. Math. 14 (2021), 1212–1225. https://doi.org/10.29020/nybg.ejpam.v14i4.4072.

[5] C. Boonpok, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, Heliyon. 6 (2020), e05367. https://doi.org/10.1016/j.heliyon.2020.e05367.

[6] C. Boonpok, C. Viriyapong, M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Comp. Sci. 18 (2018), 282–293. https://doi.org/10.22436/jmcs.018.03.04.

[7] C. Boonpok, M-continuous functions in biminimal structure spaces, Far East J. Math. Sci. 43 (2010), 41–58.

[8] J. Borsík and J. Doboš, On decompositions of quasicontinuity, Real Anal. Exchange, 16 (1990/91), 292–305.

[9] W. Dungthaisong, C. Boonpok, C. Viriyapong, Generalized closed sets in bigeneralized topological spaces, Int. J. Math. Anal. 5 (2011), 11751184.

[10] K. Laprom, C. Boonpok, C. Viriyapong, $\beta(\tau_1, \tau_2)$-continuous multifunctions on bitopological spaces, J. Math. 2020 (2020), 4020971. https://doi.org/10.1155/2020/4020971.

[11] N. Levine, Semi-open sets and semi-continuity in topological spaces, Amer. Math. Mon. 70 (1963), 36–41. https://doi.org/10.1080/00029890.1963.11990039.

[12] S. Marcus, Sur les fonctions quasicontinues au sens de S. Kempisty, Colloq. Math. 8 (1961), 47–53.

[13] A.S. Mashhour, I.A. Hasanein, S.N. El-Deeb, α-continuous and α-open mappings, Acta Math Hung. 41 (1983) 213–218. https://doi.org/10.1007/bf01961309.

[14] A.S. Mashhour, M.E. Abd El-Monsef, S.N. El-Deeb, On precontinuous and weak precontinuous mappings, Proc. Math. Phys. Soc. Egypt. 53 (1982), 47–53. https://cir.nii.ac.jp/crid/1573950400472990592.

[15] T. Noiri, Almost α-continuous functions, Kyungpook Math. J. 28 (1988), 71–77.

[16] T. Noiri, On α-continuous functions, Časopis Pěst. Mat. 109 (1984), 118–126. http://dml.cz/dmlcz/108508.

[17] V. Popa, T. Noiri, On β-continuous functions, Real Anal. Exchange. 18 (1992/93), 544–548.

[18] C. Viriyapong and C. Boonpok, (Λ, sp)-continuous functions, WSEAS Trans. Math. 21 (2022), 380–385.

[19] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, J. Math. 2020 (2020), 6285763. https://doi.org/10.1155/2020/6285763.