UNVEILING THE POTENTIAL OF SIMILARITY MEASURES IN ROUGH LABELING OF GRAPHS

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Abstract. Rough set theory is a mathematical framework developed by Polish computer scientist Zdzislaw Pawlak in the early 1980s. It is a mathematical approach for dealing with uncertainty and vagueness in data. Rough set theory provides a formal method to analyze and extract knowledge for imprecise or incomplete data. Rough graphs are another approach to modeling these types of imprecise data in which it combines the concepts of graph theory in rough set domain. This emerging concept can be applied in social network analysis, biological networks and semantic graph analysis. Tong He introduced Rough graph in 2006 using set approximations. In this graph, objects are represented as vertices(nodes) and the relationship between objects are marked with edges. Rough graphs are specifically used in visualizing complex datasets and understanding the structure and patterns within the data. In this paper we have introduced labeling on rough graph using a similarity measure between vertices (v_i, v_j). Also, we have calculated energy of a rough graph.

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1. Introduction

The extensive study of rough sets [2] made Tong He expand the idea in the context of graphs named as Rough graphs based on approximations followed by Weighted rough graph along with different forms of representation [3–5,35]. In 2012, Chen, Jinkun, and Jinjin Li provided a new method for testing the bipartiteness of graphs from the perspective of the rough set [6]. Bibin Mathew et al. defined vertex rough graph along with vertex and edge precision. In their study, two rough graphs are compared using the degree of similarity measure [7]. Anitha and Arunadevi constructed the rough graph by fixing rough membership values for objects from an information system,developing a framework for rough graphs and they have calculated metric dimension of the rough graph [8,9,36]. Anitha and

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Nithya then established the structure of this rough graph as Rough Path, Rough Cycle and Rough Star using rough approximations, for proving that graceful labeling can be implemented in rough Graph [17].

Diverse kinds of classical and fuzzy graph labeling are discussed in [1, 10] Anitha and Nithya introduced even vertex ζ-graceful labeling on various forms of rough graphs [18]. The graph labeling of classical graphs has a wide range of applications in the areas of network analysis, data compression, Optimization, image processing and cryptography. Whereas labeling of rough graphs and fuzzy graphs will address the data with partial truth and an uncertain knowledge base. Both rough and fuzzy sets approach these types of data with their boundary values and degree of membership, respectively. Theoretical and real time applications of these sets are being implemented by many researchers [11–16].

Graph energy is another milestone in the structure of a graph that represents the structural properties of a graph in numerical quantity. The trace of the adjacency matrix of a graph denotes the energy of the graph. Ivan Gutman introduced graph energy and he demonstrated this energy for specific families of graphs [19–21, 23]. In 2006 [30, 31], Gutman et al. determined the Laplacian energy for a graph as the total absolute deviations of the graph's eigenvalues. K. Fan and W. Fulton described some theorems on the eigen values of linear transformations and invariant factors [32,33]. Nagarani et al. extended the research on energy in fuzzy labeling graphs [22] and Kartheek et al. found the minimum dominating energy value [29]. Alexander et al. resolved four conjectures on the path energy of the graphs and also computed an efficient algorithm for the path matrix [24]. Pirzada and Ganie introduced the Laplacian matrix of the graph derived from the adjacency matrix. The eigen value of this matrix will bring the unique properties of the graph and they called the sum of the absolute values of the eigen as Laplacian energy [25]. Meenakshi and Lavanya brief out the various types of energy of simple graphs and their properties [26]. Jog and Raja Kotambari compute the coalescence of a pair of complete graph's adjacency and Laplacian energies [27]. The mathematical features of energy in a graph are covered by many authors where the vertices are labeled as 0 and 1, following that they proved the results of energy in a star graph [28]. Graph energy has its applications in various fields such as the prediction of molecule properties, analyzing the behavior of networks and machine learning algorithms. In this paper, we have also demonstrated the energy of a rough graph with respect to its labeling. Section 2 provides the preliminary concepts of rough sets and rough graphs while Section 3 gives the methodology for labeling the vertices and edges using similarity measures. Sections 4 and 5 discuss the energy and Laplacian energy of rough labeling graphs. And the last part is Section 6 which describes the relationship between energies and Section 7 which gives the conclusion.

2. Preliminaries

In this section, basic notions of rough set and rough graph are discussed.

2.1. **Information System/Decision System [2,34].** Assuming $\mathcal{U}$ and $\mathcal{A}$ are non-empty finite sets where $\mathcal{U}$ is the Universe of discourse and $\mathcal{A}$ is the set of attributes. An Information system $\mathcal{T} = (\mathcal{U}, \mathcal{A})$ where $\mathfrak{a} : \mathcal{U} \to \mathbb{V}_\mathfrak{a}$ for $\mathfrak{a} \in \mathcal{A}$, $\mathbb{V}_\mathfrak{a}$ is referred as the value set of a, and if $d \notin \mathcal{A}$ is the decision attribute where the elements of $\mathcal{A}$ are named as condition attributes, then the pair $(\mathcal{U}, \mathcal{A} \cup \{d\})$ is termed a decision system.

Example 1. The following Table 1 denotes the decision system with six students as objects, six condition attributes and a decision attribute. The description of attributes are as follows:

$$\textit{Condition Attributes} = \begin{cases} \textit{AD – Anxiety and Depression} \\ \textit{DS – Difficult in studies} \\ \textit{DB – Difficult in behaviour} \\ \textit{DTM – Difficult in time management} \\ \textit{DM – Difficult in memory} \\ \textit{VS – Vision Issues} \end{cases}$$

$$\textit{Decision Attribute} = \{\textit{RG-Result grade}\}$$

Table 1. Decision system

Objects (Names)	Condition Attributes						Decision Attribute
	DS	DB	DTM	DM	VS	AD	Result Grade
C_1	Yes	Yes	Yes	Yes	Yes	Yes	Poor
C_2	Yes	Yes	Yes	Yes	No	Yes	Poor
C_3	No	Yes	No	No	No	No	Good
C_4	No	No	No	No	No	No	Good
C_5	No	Yes	No	No	No	No	Good
C_6	Yes	No	Yes	Yes	Yes	Yes	Poor

2.2. **Rough Set [2–4].** Let $\mathcal{T} = (\mathcal{U}, \mathcal{A})$ be the Information system which consists of universe of discourse $\mathcal{U}$ and the set of attributes $\mathcal{A}$. The Indiscernibility relation is defined by

$$IND_{\mathcal{T}}(\mathcal{R}) = \{(\mathcal{Y}, \mathcal{Y}') \in U^2 |\ a \in \mathcal{R},\ a(\mathcal{Y}) = a(\mathcal{Y}')\}. \tag{1}$$

where $(\mathcal{R} \subset \mathcal{A})$ and $\mathcal{Z} \subset \mathcal{U}$ then the relation is divided into different equivalence classes $[\mathcal{Y}]_\mathcal{R}$. The lower and upper approximation is defined as

$$\underline{\mathcal{R}}\mathcal{Z} = \bigcup_{\mathcal{Y} \in U} \{[\mathcal{Y}]_\mathcal{R} : [\mathcal{Y}]_\mathcal{R} \subseteq \mathcal{Z}\} \tag{2}$$

$$\overline{\mathcal{R}}\mathcal{Z} = \bigcup_{\mathcal{Y} \in U} \{[\mathcal{Y}]_\mathcal{R} : [\mathcal{Y}]_\mathcal{R} \cap \mathcal{Z} \neq \emptyset\} \tag{3}$$

The difference between the upper and the lower approximation of $\mathcal{Z}$ is said to be boundary region of $\mathcal{Z}$.The non-empty intersection of the pair $(\underline{\mathcal{R}}\mathcal{Z}, \overline{\mathcal{R}}\mathcal{Z})$ is said to be Rough Set. It has the following properties:

(1) $\underline{\mathcal{R}}\mathcal{Z} \subset \mathcal{Z} \subset \overline{\mathcal{R}}\mathcal{Z}$
(2) $\underline{\mathcal{R}}\mathcal{U} = \overline{\mathcal{R}}\mathcal{U} = \mathcal{U}$ & $\underline{\mathcal{R}}\emptyset = \overline{\mathcal{R}}\emptyset = \emptyset$
(3) $\underline{B\mathcal{R}}(X \cap Y) = \underline{\mathcal{R}}X \cap \underline{\mathcal{R}}Y$
(4) $\underline{\mathcal{R}}(X \cap Y) \supset \underline{\mathcal{R}}X \cap \underline{\mathcal{R}}Y$
(5) $\overline{\mathcal{R}}(X \cup Y) = \overline{\mathcal{R}}X \cup \overline{\mathcal{R}}Y$
(6) $\underline{\mathcal{R}}(X - Y) = \overline{\mathcal{R}}X - \overline{\mathcal{R}}Y$
(7) $\sim \underline{\mathcal{R}}X = \overline{\mathcal{R}}(\sim X)$

2.3. **Rough Membership Function [8].** Rough membership function is described through the function $f_{\mathcal{R}} : \mathcal{Z} \to [0\ 1]$ and defined by

$$\omega^{\mathcal{R}}_{\mathcal{Z}}(y) = \frac{|[y]_{\mathcal{R}} \cap \mathcal{Z}|}{|[y]_{\mathcal{R}}|}, \ \forall y \in \mathcal{U} \tag{4}$$

It measures the degree of attributes at which degree it belongs to the set $\mathcal{Z}$ with following mathematical qualities,

(1) $\omega^{\mathcal{R}}_{\mathcal{Z}}(y) = 1$ iff $y \in \underline{\mathcal{R}}(\mathcal{Z})$
(2) $\omega^{\mathcal{R}}_{\mathcal{Z}}(y) = 0$ iff $y \in U - \overline{\mathcal{R}}\mathcal{Z}$
(3) $0 < \omega^{\mathcal{R}}_{\mathcal{Z}}(y) < 1$ iff $y \in BN_{\mathcal{R}}(\mathcal{Z})$
(4) If $IND_{\mathcal{T}}(\mathcal{R}) = \{(y, y') \in U^2 | a \in \mathcal{R}, a(y) = a(y')\}$ then $\omega^{\mathcal{R}}_{\mathcal{Z}}(y)$ is the characteristic function of $\mathcal{Z}$.
(5) If $xIND(\mathcal{T})y$ then $\omega^{\mathcal{R}}_{\mathcal{Z}}(x)$=$\omega^{\mathcal{R}}_{\mathcal{Z}}(y)$
(6) $\omega^{\mathcal{R}}_{\mathcal{Z}}(y) - \mathcal{Z}(y) = 1 - \omega^{\mathcal{R}}_{\mathcal{Z}}(y)$ for any $y \in \mathcal{Z}$.
(7) $\omega^{\mathcal{R}}_{X \cup Y}(y) \geq \max(\omega^{\mathcal{R}}_{X}(y), \omega^{\mathcal{R}}_{Y}(y))$ for any $y \in \mathcal{U}$.
(8) $\omega^{\mathcal{R}}_{X \cap Y}(y) \leq \min(\omega^{\mathcal{R}}_{X}(y), \omega^{\mathcal{R}}_{Y}(y))$ for any $y \in \mathcal{U}$
(9) $\omega^{\mathcal{R}}_{\cup \mathcal{Z}}(y) = \sum_{y \in \mathcal{Z}} \omega^{\mathcal{R}}_{\mathcal{Z}}(y)$

2.4. **Rough Graph [8].** Consider the non-empty triplet $\mathfrak{R} = \{V, E, \omega\}$ in which $V = \{v_1, v_2, \dots v_n\} = \mathcal{U}$, where $\mathcal{U}$ is called a universe, $E = \{e_1, e_2, \dots e_n\}$ is a collection of unordered pairs of distinct elements of V and $\omega : V \to [0, 1]$, then Rough graph can be constructed with following considerations:

$$\mathfrak{R}(v_i, v_j) = \begin{cases} max(\omega^V_G(v_i), \omega^V_G(v_j)) > 0, & \textit{edge exists between } (v_i, v_j) \\ max(\omega^V_G(v_i), \omega^V_G(v_j)) = 0, & \textit{edge doesn't exist between } (v_i, v_j). \end{cases}$$

Following the construction of this rough graph, Aruna and Anitha [8] have proved the following prepositions:

- A Rough graph is always a connected and pendent free graph.

- In a rough graph, any $v_1 - v_2$ rough walk contains $v_1 - v_2$ rough path
- A closed rough walk of odd length contains a rough cycle.
- The degree of a vertex $\mathbf{v_i}$ of a rough graph $\mathfrak{R}$ is defined as the number of edges incident to that vertex. It is denoted by ΔR.
- rough Adjacency Matrix: The rough adjacency matrix for the Rough graph is defined as,

$$a_{ij} = \begin{cases} 1 & for\ i \neq j, ij \in E \\ 0 & for\ i \neq j,\ ij \notin E \\ 0 & for\ i = j,\ ij \in E \end{cases}$$

- rough Union: Let $\mathfrak{R}_1(V_1, E_1)$ and $\mathfrak{R}_2(V_2, E_2)$ be two Rough graphs with $V_1 \cap V_2 = \phi$. Then the Rough union of $\mathfrak{R}_1$ and $\mathfrak{R}_2$ is defined as $\mathfrak{R}_1 \cup \mathfrak{R}_2 = (V_1 \cup V_2, E_1 \cup E_2)$, where $V_1 \cup V_2(x) = \begin{cases} \omega_1(x)\ if\ x \in V_1 \\ \omega_2(x)\ if\ x \in V_2 \end{cases}$

2.4.1. *Construction of rough graph.* Let us take a decision system from Example 1 (Table 1). From this decision system the following graph is being constructed [8,18]

Equivalence classes for Table 1

$$R\{C_1\} = \{C_1\}, R\{C_2\} = \{C_2\}, R(C_3) = \{C_3, C_5\} = R\{C_5\},$$

$$R\{C_4\} = \{C_4\}, R\{C_6\} = \{C_6\}$$

Assuming that the outcome evaluation decision is good, we consider the target set as $X = \{C_3, C_4, C_5\}$

Rough Membership values are

$$\omega(C_1) = \frac{|[C_1]_R \cap X|}{|[C_1]_R|} = 0; \quad \omega(C_2) = 0; \quad \omega(C_3) = 2/3 = 0.6;$$

$$\omega(C_4) = \frac{1}{3} = 0.3; \quad \omega(C_5) = \frac{2}{3} = 0.6; \quad \omega(C_6) = 0$$

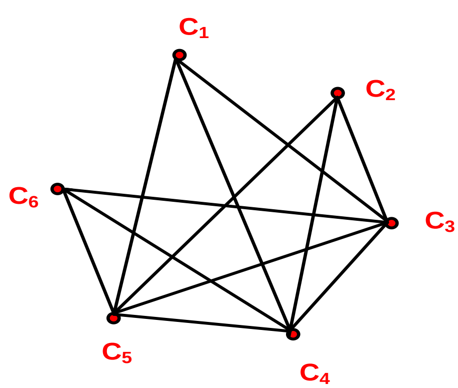

Figure 1. Rough graph

3. Methodology

3.1. **Rough Labeling Graph.** A rough graph $\mathcal{R}_{\mathcal{L}}^{\varphi} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ is said to be a rough labeling graph if V=$\{\rho^{\varphi}(v_i)\}$ for $i = 1, 2, \ldots n$ and E=$\{\sigma^{\varphi}(v_i, v_j)\}$ for $i = 1, 2, \ldots n$ and $\omega : V * V \rightarrow [0, 1]$ is a bijection such that edges and vertices can be labeled using the similarity representation of the membership function if it complies with the following requirements:

(1) if $\mathcal{R}_{\mathcal{L}}^{\varphi} = \max(\omega(v_i^{\varphi}), \omega(v_j^{\varphi})) > 0$ then edge exists for $v_i, v_j \in V$.
(2) Vertex labeling: $\rho^{\varphi}(v_i) = (\omega_G[v_i]_{\mathcal{S}_r})$
(3) Edge labeling: $\sigma^{\varphi}(v_i, v_j) = Sim(v_i, v_j)$ where $Sim(v_i, v_j) = \frac{|[v_i]_{\mathcal{S}_r} \cap [v_j]_{\mathcal{S}_r}|}{|[v_i]_{\mathcal{S}_r} \cup [v_j]_{\mathcal{S}_r}|}$ and $[v_i]_{\mathcal{S}_r} = \{v_j / v_i \mathcal{S}_r v_j\}$

3.2. **Measures of Similarity.**

Definition 1. A mapping $\mathbb{S} : \mathcal{R}_{\mathcal{L}}^{\varphi}(v_i, v_j) \rightarrow [0, 1]$, then $\mathcal{S}_r(v_i)$ is said to be the degree of similarity between v_i and v_j in $\mathcal{R}_{\mathcal{L}}^{\varphi}$ if $\mathcal{S}_r(v_i, v_j)$ satisfies the following properties:

(1) $0 \leq \mathcal{S}_r(v_i, v_j) \leq 1$
(2) $\mathcal{S}_r(v_i, v_j) = \mathcal{S}_r(v_j, v_i)$
(3) $\mathcal{S}_r(v_i, v_k) \leq \mathcal{S}_r(v_i, v_j)$ and $\mathcal{S}_r(v_i, v_k) \leq \mathcal{S}_r(v_j, v_k)$
(4) $[v_i]_{\mathcal{S}_r} = \{v_j / v_i \mathcal{S}_r v_j\}$

From the decision system (Table 1), we have constructed the following Similarity table, which shows the relationship between objects with respect to their attributes. Since we have only six attributes, each value in Table 2 requires seven values (0, 1, 2, 3, 4, 5, and 6). The value 0 represents that there are no remaining attributes that overlap between the two objects and the value 6 denotes that two objects are identical.

Table 2. Similarity table

$\mathcal{S}_r$	C_1	C_2	C_3	C_4	C_5	C_6
C_1	6	5	1	0	1	5
C_2	5	6	2	1	2	4
C_3	1	2	6	5	6	0
C_4	0	1	5	6	5	1
C_5	1	2	6	5	6	0
C_6	5	4	0	1	0	6

From Table 2, the following similarity classes have been identified,

$$[C_1]_{\mathcal{S}_r} = \{C_1, C_2, C_3, C_5, C_6\}$$

$$[C_2]_{\mathcal{S}_r} = \{C_1, C_2, C_3, C_4, C_5, C_6\}$$

$$[C_3]_{\mathcal{S}_r} = \{C_1, C_2, C_3, C_4, C_5\}$$

$$[C_4]_{\mathcal{S}_r} = \{C_2, C_3, C_4, C_5, C_6\}$$

$$[C_5]_{\mathcal{S}_r} = \{C_1, C_2, C_3, C_4, C_5\}$$

$$[C_6]_{\mathcal{S}_r} = \{C_1, C_2, C_4, C_6\}$$

3.2.1. *Vertex and edge labeling.*

Vertex labeling: $\rho^{\varphi}(v_i) = (\omega_G[v_i]_{\mathcal{S}_r})$

We have taken that the target set as $\mathbb{X} = \{C_3, C_4, C_5\}$

$$\omega_X^S([C_i]_{\mathcal{S}_r}) = \frac{|[C_i]_{\mathcal{S}_r} \cap \mathbb{X}|}{|[C_i]_{\mathcal{S}_r}|}$$

$$\omega_X^S([C_1]_{\mathcal{S}_r}) = \frac{2}{5} = 0.4; \quad \omega_X^S([C_2]_{\mathcal{S}_r}) = \frac{3}{6} = 0.5; \quad \omega_X^S([C_3]_{\mathcal{S}_r}) = \frac{3}{5} = 0.6;$$

$$\omega_X^S([C_4]_{\mathcal{S}_r}) = \frac{3}{5} = 0.6; \quad \omega_X^S([C_5]_{\mathcal{S}_r}) = \frac{3}{5} = 0.6; \quad \omega_X^S([C_6]_{\mathcal{S}_r}) = \frac{1}{4} = 0.25$$

Edge labeling: $E_G^{\varphi}(v_i, v_j) = Sim(v_i, v_j)$ where $Sim(v_i, v_j) = \frac{|[v_i]_{\mathcal{S}_r} \cap [v_j]_{\mathcal{S}_r}|}{|[v_i]_{\mathcal{S}_r} \cup [v_j]_{\mathcal{S}_r}|}$

$$Sim(C_1, C_3) = \frac{|[C_1]_{\mathcal{S}_r} \cap [C_3]_{\mathcal{S}_r}|}{|[C_1]_{\mathcal{S}_r} \cup [C_3]_{\mathcal{S}_r}|} = \frac{4}{6} = 0.66;$$

$$Sim(C_1, C_4) = 0.67; \quad Sim(C_1, C_5) = 0.67$$

$$Sim(C_2, C_3) = 0.83; \quad Sim(C_3, C_4) = 0.67;$$

$$Sim(C_3, C_5) = 1; \quad Sim(C_3, C_6) = 0.5;$$

$$Sim(C_4, C_5) = 0.67; \quad Sim(C_4, C_6) = 0.5; \quad Sim(C_5, C_6) = 0.5$$

Here, the following Figure 2 represents the implementation of the labeling for vertices and edges on rough graph.

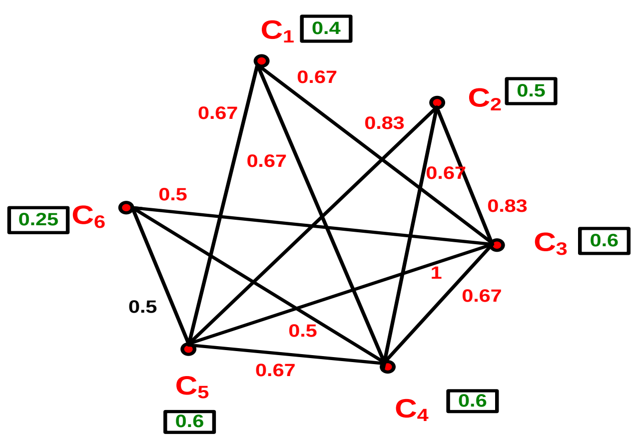

FIGURE 2. Rough labeling graph

4. Proposed Work

The energy of a graph is a measure that provides insights into the structural properties and characteristics of the graph. It is calculated based on the eigenvalues or eigenvalue-related properties of the graph's adjacency matrix or Laplacian matrix. The purpose of finding the energy of a graph is to determine the connectivity, components and clustering properties of an information system.

4.1. **Energy of Rough Labeling Graph.**

Here, rough labeling graphs are represented as RLG. The following are some of the basic definitions:

Definition 2. The adjacency matrix $\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \mathcal{A}(\sigma^{\varphi}(v_iv_j))$ of a rough labeling graph (RLG) $\mathcal{R}^{\varphi}_{\mathcal{L}} = (V^{\varphi}, E^{\varphi}, \sigma^{\varphi}, \omega)$ is defined as a square matrix $\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = [a_{ij}]$ where $a_{ij} = \sigma^{\varphi}(v_iv_j)$ in which $\sigma^{\varphi}(v_iv_j)$ represents the maximum membership value between v_i and v_j respectively.

Definition 3. A matrix that represents a rough labeling relation is defined by $M^{\varphi} = [m^{\varphi}_{ij}]$ where $m^{\varphi}_{ij} = \sigma^{\varphi}(v_iv_j)$

Definition 4. The collection of eigenvalues for $\mathcal{A}(\sigma^{\varphi}(v_iv_j))$ is the spectrum of the adjacency matrix $Spec(\mathcal{R}^{\varphi}_{\mathcal{L}})$.

Definition 5. Let $\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ be a n*n matrix of rough labeling graph. The scalar ψ is called an eigen value of $\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ if there is a non zero vector χ such that $\mathcal{A}\psi = \chi\psi$.

Definition 6. The trace of a matrix of rough labeling graph is the sum of n eigen values of the given matrix and it is denoted by $tr(\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}}))$.

Definition 7 ([20]). The sum of eigen values in absolute terms is called energy of rough labeling $\mathcal{R}^{\varphi}_{\mathcal{L}}$ which is denoted by $\mathfrak{E}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \sum_{i=1}^{n} |\psi_i|$ and also it should satisfies the following criteria:

(1) $\mathfrak{E}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \sum_{i=1}^{n} |\psi_i|$
(2) $0 \leq \omega(v_i) \leq 1$
(3) $\rho^{\varphi}(v_i) = (\omega_G[v_i]_{\mathcal{S}_r})$
(4) $\sigma^{\varphi}(v_i, v_j) = Sim(v_i, v_j)$ where $Sim(v_i, v_j) = \frac{|[v_i]_{\mathcal{S}_r} \cap [v_j]_{\mathcal{S}_r}|}{|[v_i]_{\mathcal{S}_r} \cup [v_j]_{\mathcal{S}_r}|}$ and $[v_i]_{\mathcal{S}_r} = \{v_j / v_i \mathcal{S}_r v_j\}$

The adjacency matrix of RLG is given as follows from Figure 2.

$$\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \begin{pmatrix} 0 & 0 & 0.67 & 0.67 & 0.67 & 0 \\ 0 & 0 & 0.83 & 0.67 & 0.83 & 0 \\ 0.67 & 0.83 & 0 & 0.67 & 1 & 0.5 \\ 0.67 & 0.67 & 0.67 & 0 & 0.67 & 0.5 \\ 0.67 & 0.83 & 1 & 0.67 & 0 & 0.5 \\ 0 & 0 & 0.5 & 0.5 & 0.5 & 0 \end{pmatrix}$$

The Spectral energy and its bounds for Figure 2 is demonstrated as follows,

$$Spec(\mathcal{R}_{\mathcal{L}}^{\varphi}) = \{0, -1, -1.345, -0.585, 0.016, 2.914\}$$

$$\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) = 5.86$$

$$Lower\ bound = 3.412$$

$$Upper\ bound = 8.358$$

Theorem 1. Let $\mathcal{R}_{\mathcal{L}}^{\varphi} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ be a rough labeling graph and $\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})$ be its adjacency matrix. If the eigen values of $\mathcal{A}(\sigma^{\varphi}(v_i v_j))$ are given as $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$ respectively, then

(1) $\sum_{i=1}^{n} \psi_i = 0$

(2) $\sum_{i=1}^{n} {\psi_i}^2 = 2 \sum_{1 \leq i < j \leq n} (\sigma^{\varphi}(v_i v_j))^2$

Proof. From the Definition 6,

we say that the trace of a matrix equals the sum of n eigen values of it.

$$(i.e) tr(\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})) = tr(\sigma^{\varphi}(v_i v_j)) = \psi_1 + \psi_2 + \cdots + \psi_n = 0$$

(1) Proof: Inferred from a matrix's trace characteristics, we have

$$(tr(\sigma^{\varphi}(v_i v_j))^2) = (0 + (\sigma^{\varphi}(v_1 v_2))^2 + \cdots (\sigma^{\varphi}(v_1 v_n))^2 + (\sigma^{\varphi}(v_2 v_1))^2$$

$$+ 0 + \cdots (\sigma^{\varphi}(v_2 v_n))^2 \ldots (\sigma^{\varphi}(v_n v_1))^2 + (\sigma^{\varphi}(v_n v_2))^2 + \cdots 0)$$

$$\sum_{i=1}^{n} {\psi_i}^2 = 2 \sum_{1 \leq i < j \leq n} (\sigma^{\varphi}(v_i v_j))^2$$

□

Theorem 2. Let $\mathcal{R}_{\mathcal{L}}^{\varphi} = (V^{\varphi}, E^{\varphi}, \sigma^{\varphi}, \omega)$ be a rough labeling graph and $\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})$ be the adjacency matrix of $\mathcal{R}_{\mathcal{L}}^{\varphi}$ with n vertices, then

$$\sqrt{2 \sum (\sigma^{\varphi}(v_i v_j))^2 + n(n-1)|\mathcal{A}(\sigma^{\varphi}(v_i v_j))|^{2/n}} \leq E(\sigma^{\varphi}(v_i v_j))$$

$$\leq \sqrt{2n \sum (\sigma^{\varphi}(v_i v_j))^2}$$

Proof. **Upper bound:**

Assume that the eigen values of rough labeling graph are $\psi_1 \geq \psi_2 \geq \cdots \geq \psi_n$. The vertices are (1,1,...1) and $(|\psi_1|, |\psi_2|, \ldots |\psi_n|)$ with n entries are subject to Cauchy-Schwarz inequality and the result is $[\sum_{i=1}^{n} u_i v_i]^2 \leq [\sum_{i=1}^{n} u_i]^2 [\sum_{i=1}^{n} v_i]^2$

Choose $u_i = 1$, $v_i = |v_i|$

$$\left[\sum_{i=1}^{n} |v_i|\right]^2 \leq \left[\sum_{i=1}^{n} 1\right] \left[\sum_{i=1}^{n} |\psi_i|^2\right] = n \sum_{i=1}^{n} \psi_i^2$$

$$\left[\sum_{i=1}^{n}|\psi_i|\right] \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}|\psi_i|^2} \tag{5}$$

$$\left[\sum_{i=1}^{n}\psi_i\right]^2 = \sum_{i=1}^{n}|\psi_i|^2 + 2\sum_{1\leq i<j\leq n}\psi_i\psi_j \tag{6}$$

Comparing the coefficient of ψ^{n-2} in the characteristic polynomial,

$$\prod_{i=1}^{n}(\psi-\psi_i) = |\mathcal{A}(\mathcal{R}^{\varphi}(\mathcal{G}))-\psi I|$$

We have

$$\sum_{1\leq i<j\leq n}\psi_i\psi_j = -\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 \tag{7}$$

Sub (7) in (6), we obtain

$$\left[\sum_{i=1}^{n}\psi_i\right]^2 = 2\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 \tag{8}$$

Sub (8) in (5) , we obtain

$$\left[\sum_{i=1}^{n}|\psi_i|\right] \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}2\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2}$$

$$= \sqrt{2n\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2}$$

$$\therefore \mathfrak{E}(\sigma^{\varphi}(v_iv_j)) = \sqrt{2n\sum(\sigma^{\varphi}(v_iv_j))^2}$$

Lower bound:

$$\mathfrak{E}(\sigma^{\varphi}(v_iv_j))^2 = \left[\sum_{i=1}^{n}|\psi_i|^2\right] = \sum_{i=1}^{n}|\psi_i|^2 + 2\sum_{1\leq i<j\leq n}|\psi_i\psi_j|$$

$$= 2\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 + \frac{2n(n-1)}{2}\mathrm{AM}\{|\psi_i\psi_j|\}$$

Since $\mathrm{AM}\{|\psi_i\psi_j|\} \geq \mathrm{GM}\{|\psi_i\psi_j|\}, 1 \leq i < j \leq n$

$$\mathfrak{E}(\sigma^{\varphi}(v_iv_j)) = \sqrt{2n\sum(\sigma^{\varphi}(v_iv_j))^2 + n(n-1)\ \mathrm{GM}\{|\lambda_i\lambda_j|\}},$$

Also since

$$\mathrm{GM}\{|\psi_i\psi_j|\} = \left(\prod_{1\leq i\leq n}|\psi_i\psi_j|\right)^{\frac{2}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n}|\psi_i|^{n-1}\right)^{\frac{2}{n(n-1)}}$$

$$= \left(\prod_{i=1}^{n} |\psi_i|\right)^{\frac{2}{n}}$$

$$= |\mathcal{A}(\sigma^{\varphi}(v_i v_j))|^{\frac{2}{n}}$$

So $\mathfrak{E}(\sigma^{\varphi}(v_i v_j)) \geq \sqrt{2\sum(\sigma^{\varphi}(v_i v_j))^2 + n(n-1)|\mathcal{A}(\sigma^{\varphi}(v_i v_j))|^{2/n}}$

Thus

$$\sqrt{2\sum(\sigma^{\varphi}(v_i v_j))^2 + n(n-1)|\mathcal{A}(\sigma^{\varphi}(v_i v_j))|^{2/n}} \leq \mathfrak{E}(\sigma^{\varphi}(v_i v_j)) \leq \sqrt{2n\sum(\sigma^{\varphi}(v_i v_j))^2}$$

Hence proved. □

5. Laplacian Energy of Rough Labeling Graph

In this part, the Laplacian energy of RLG and its characteristics are discussed.

Definition 8. The degree matrix $\mathcal{D}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \mathcal{D}(\sigma^{\varphi}(v_i v_j)) = [d_{ij}]$ of rough labeling graph $\mathcal{R}^{\varphi}_{\mathcal{L}}$ is termed as a n*n diagonal matrix has n vertices with the following definition:

$$d_{ij} = \begin{cases} d(\rho^{\varphi}(v_i)) & if\ i = j \\ 0 & otherwise \end{cases} \quad \text{where } d(\rho^{\varphi}(v_i)) = \sum_{v_i v_j \in E(\mathcal{R}^{\varphi}_{\mathcal{L}})} \sigma^{\varphi}(v_i v_j)$$

Definition 9. The Laplacian matrix of a rough labeling graph $\mathcal{R}^{\varphi}_{\mathcal{L}} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ is defined as $\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \mathbb{L}(\sigma^{\varphi}(v_i v_j)) = \mathcal{D}^{\varphi}(\mathcal{R}^{\varphi}_{\mathcal{L}}) - \mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ where $\mathcal{D}^{\varphi}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ is a degree matrix and $\mathcal{A}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ is an adjacency matrix.

Definition 10. The spectrum of Laplacian matrix of rough Laplacian matrix is defined as $\mathcal{S}_{\mathcal{L}}$ where $\mathcal{S}_{\mathcal{L}}$ is the set of Laplacian eigen values of $\mathbb{L}(\sigma^{\varphi}(v_i v_j))$ respectively.

Definition 11. The Laplacian energy of $\mathcal{R}^{\varphi}_{\mathcal{L}}$ is described as $\mathbb{L}\mathfrak{E}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_i v_j)) = \sum_{i=1}^{n} |\psi_i|$ where $\psi_i = \delta_i - 2\frac{\sum_{1\leq i<j\leq n} \sigma^{\varphi}(v_i v_j)}{n}$

$$\mathcal{D}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \begin{pmatrix} 2.01 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2.33 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.67 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3.18 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3.67 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1.5 \end{pmatrix}$$

$$\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = \begin{pmatrix} 2.01 & 0 & -0.67 & -0.67 & -0.67 & 0 \\ 0 & 2.33 & -0.83 & -0.67 & -0.83 & 0 \\ -0.67 & -0.83 & 3.67 & -0.67 & -1 & -0.5 \\ -0.67 & -0.67 & -0.67 & 3.18 & -0.67 & -0.5 \\ -0.67 & -0.83 & -1 & -0.67 & 3.67 & -0.5 \\ 0 & 0 & -0.5 & -0.5 & -0.5 & 1.5 \end{pmatrix}$$

$$\mathbb{L}(spec(\sigma^{\varphi}(v_iv_j)) = \{0, 4.67, 1.629, 2.150, 3.822, 4.089\}$$

$$\mathbb{L}\mathfrak{E}(\mathcal{R}^{\varphi}_{\mathcal{L}}) = 32.73$$

$$\text{Lower bound} = 24.16$$

$$\text{Upper bound} = 41.84$$

Theorem 3. Let $\mathcal{R}^{\varphi}_{\mathcal{L}} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ be rough labeling graph and $\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ be the Laplacian matrix of RLG. If $\delta_1 \geq \delta_2 \geq \cdots \geq \delta_n$ are the eigen values of $\mathbb{L}(\sigma^{\varphi}(v_iv_j))$ respectively, then

(1) $\sum_{i=1}^{n} \psi_i = 2\sum_{1\leq i<j\leq n} \sigma^{\varphi}(v_iv_j)$
(2) $\sum_{i=1}^{n} \psi_i^2 = 2\sum_{1\leq i<j\leq n} (\sigma^{\varphi}(v_iv_j))^2 + \sum_{i=1}^{n} d^2(v_i)$

Proof. (1) Proof:

Given that $\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ is a symmetric matrix with positive eigen values, then

$$\sum_{i=1}^{n} \psi_i = tr(\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}})) = \sum_{i=1}^{n} d(\rho^{\varphi}(v_i) = 2 \sum_{1\leq i<j\leq n} \sigma^{\varphi}(v_iv_j)$$

Therefore

$$\sum_{i=1}^{n} \psi_i = 2 \sum_{1\leq i<j\leq n} \sigma^{\varphi}(v_iv_j)$$

(2) Proof:

$$\mathbb{L}(\sigma^{\varphi}(v_iv_j) = \begin{pmatrix} d_{\sigma^{\varphi}(v_iv_j)}(v_1) & -\sigma^{\varphi}(v_1v_2) & \cdots & -\sigma^{\varphi}(v_1v_n) \\ -\sigma^{\varphi}(v_2v_1) & d_{\sigma^{\varphi}(v_iv_j)}(v_2) & \ldots & -\sigma^{\varphi}(v_2v_n) \\ \vdots & \vdots & \ddots & \vdots \\ -\sigma^{\varphi}(v_nv_1) & -\sigma^{\varphi}(v_nv_2) & \cdots & d_{\sigma^{\varphi}(v_iv_j)}(v_n) \end{pmatrix}$$

The trace characteristics of a matrix provide us

$$tr(\mathbb{L}(\sigma^{\varphi}(v_iv_j))^2) = \sum_{i=1}^{n} |\psi_i|^2$$

where

$$tr(\mathbb{L}(\sigma^{\varphi}(v_iv_j))^2) = (d^2_{\sigma^{\varphi}(v_iv_j)}(v_1) + (\sigma^{\varphi}(v_1v_2))^2 + \cdots (\sigma^{\varphi}(v_1v_n))^2 + (\sigma^{\varphi}(v_2v_1))^2$$

$$+ d^2_{\sigma^{\varphi}(v_iv_j)}(v_2)\ldots(\sigma^{\varphi}(v_2v_n))^2\ldots(\sigma^{\varphi}(v_nv_1))^2$$

$$+ (\sigma^{\varphi}(v_nv_2))^2 + \ldots d^2_{\sigma^{\varphi}(v_iv_j)}(v_n))$$

$$= 2\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 + \sum_{i=1}^{n} d^2(v_i)$$

□

Theorem 4. Let $\mathcal{R}^{\varphi}_{\mathcal{L}} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ be a rough labeling graph with n vertices and if the Laplacian matrix of $\mathcal{R}^{\varphi}_{\mathcal{L}}$ is $\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}})$, then

$$\mathfrak{E}(\sigma^{\varphi}(v_iv_j) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 + n\sum_{i=1}^{n}(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n})^2}$$

Proof. Applying Cauchy Schwarz inequality, we have n integers $1, 1, \ldots 1$ and $(|\psi_1|, |\psi_2|, \ldots |\psi_n|)$ with the give result,

$$\sum_{i=1}^{n}|\psi_i| \leq \sqrt{n}\sqrt{\sum_{i=1}^{n}|\psi_i|^2}$$

$$\mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_iv_j) \leq \sqrt{n}\sqrt{2M} = \sqrt{2nM}$$

Since

$$M = \sum_{1\leq i<\leq n}(\sigma^{\varphi}(v_iv_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n}\right)^2$$

Therefore

$$\mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_iv_j)) \leq \sqrt{2n\sum_{1\leq i<j\leq n}(\sigma^{\varphi}(v_iv_j))^2 + n\sum_{i=1}^{n}\left(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n}\right)^2}$$

□

Theorem 5. Let $\mathcal{R}^{\varphi}_{\mathcal{L}} = (V, E, \rho^{\varphi}, \sigma^{\varphi}, \omega)$ be a rough labeling graph with n vertices and if $\mathbb{L}(\mathcal{R}^{\varphi}_{\mathcal{L}})$ be the laplacian matrix of $\mathcal{R}^{\varphi}_{\mathcal{L}}$ then

$$\mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_iv_j)) \geq 2\sqrt{\sum_{1\leq i\leq j\leq n}(\sigma^{\varphi}(v_iv_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n}\right)^2}$$

Proof.

$$\left(\sum_{i=1}^{n}|\psi_i|\right)^2 = \sum_{i=1}^{n}|\psi_i|^2 + 2\sum|\psi_i\psi_i| \geq 4M$$

$$\mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_iv_j) \geq 2\sqrt{M}$$

Since we have the value for

$$M = \sum_{1\leq i<j\leq n} (\sigma^{\varphi}(v_iv_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n}\right)^2$$

Therefore

$$\mathbb{L}\mathfrak{E}(\sigma^{\varphi}(v_iv_j)) \geq 2\sqrt{\sum_{1\leq i<j\leq n} (\sigma^{\varphi}(v_iv_j))^2 + \frac{1}{2}\sum_{i=1}^{n}\left(d(v_i) - 2\frac{\sum_{1\leq i<j\leq n}\sigma^{\varphi}(v_iv_j)}{n}\right)^2}$$

□

6. Relation Between $\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi})$ and $\mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi})$

The relationship between energy and Laplacian energy of rough labeling graph is written as

$$\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) \leq \mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi})$$

(i.e) we can also write the relation as $\mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) \leq \mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) + 2\sum_{i=1}^{\tau}(d_i - \frac{2m}{n})$

Lemma 1. Let $\mathcal{R}_{\mathcal{L}}^{\varphi}$ be a rough labeling graph with n vertices and m edges, from [15] we have the statement as following:

$$\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) = 2\sum_{i=1}^{\theta+}\psi_i = -2\sum_{i=1}^{\theta-}\psi_{n-i+1} = 2\max_{1\leq t\leq n}\left(\sum_{i=1}^{t}\psi_i\right)$$

$$= 2\max_{1\leq t\leq n}\left(\sum_{i=1}^{t}-\psi_{n-i+1}\right)$$

where $\theta+$ and $\theta-$ are the no. of positive and negative eigen values of $\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})$ respectively.

Lemma 2. Let $\tau(1 \leq \tau \leq n)$ be the largest positive integer such that
${\sigma^{\varphi}}_{\tau} \geq \frac{2m}{n}$ then from [27], we have

$$\mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) = \sum_{i=1}^{n}\left|\psi_i - \frac{2m}{n}\right|$$

$$= 2\sum_{i=1}^{\tau}\psi_i - \frac{4m\tau}{n}$$

Theorem 6. Let $\mathcal{R}_{\mathcal{L}}^{\varphi}$ be a rough graph of n vertices and m edges and vertex degrees d_i for $i = 1, 2, \ldots n$, then $\mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) \leq \mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) + 2\sum_{i=1}^{\tau}(d_i - \frac{2m}{n})$ where τ is the largest positive integers of $\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})$.

Proof. For any $t(1 \leq t \leq n)$, we write

$$\sum_{i=1}^{t}\psi_i(-\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})) = -\sum_{i=1}^{t}\psi_{n-i+1} \tag{9}$$

where $\psi_i(-\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi}))$ is the i^{th} largest eigen value of $-\mathcal{A}(\mathcal{R}_{\mathcal{L}}^{\varphi})$.

Using the result in [28,29], we get

$$\sum_{i=1}^{t}\psi_i \le \sum_{i=1}^{t} d_i - \sum_{i=1}^{t}\psi_{n-i+1} \tag{10}$$

From Lemma 1, we have the result,

$$\begin{aligned}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) &= 2\max_{1\le t\le n}(\sum_{i=1}^{t} -\psi_{n-i+1}) \\ &= -2\sum_{i=1}^{t}\psi_{n-i+1} \quad \text{for any } t,\ 1\le t\le n-1 \end{aligned}\tag{11}$$

Using Lemma 2, we can write the result in (10) as

$$\begin{aligned}\sum_{i=1}^{t}\psi_i &\le \sum_{i=1}^{t} d_i - \sum_{i=1}^{t}\psi_{n-i+1} \\ \sum_{i=1}^{t}\psi_i - \frac{2m}{n} &\le \sum_{i=1}^{t} d_i - \sum_{i=1}^{t}\psi_{n-i+1} - \frac{2m}{n} \\ 2\left(\sum_{i=1}^{\tau}\psi_i - \frac{2m\tau}{n}\right) &\le 2\left(\sum_{i=1}^{\tau} d_i - \sum_{i=1}^{\tau}\psi_{n-i+1} - \frac{2m\tau}{n}\right) \\ 2\sum_{i=1}^{\tau}\psi_i - \frac{4m\tau}{n} &\le 2\sum_{i=1}^{\tau} d_i - 2\sum_{i=1}^{\tau}\psi_{n-i+1} - \frac{4m\tau}{n} \\ &\le -2\sum_{i=1}^{\tau}\psi_{n-i+1} + 2\sum_{i=1}^{\tau} d_i - \frac{4m\tau}{n} \\ &\le -2\sum_{i=1}^{\tau}\psi_{n-i+1} + 2\sum_{i=1}^{\tau}\left(d_i - \frac{2m\tau}{n}\right)\end{aligned}$$

Therefore $\mathbb{L}\mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) \le \mathfrak{E}(\mathcal{R}_{\mathcal{L}}^{\varphi}) + 2\sum_{i=1}^{\tau}(d_i - \frac{2m\tau}{n})$ □

7. Conclusion

This work defines a brand-new style of labeling for rough graphs based on the membership function and similarity measure. We defined energy and found that Laplacian energy had the greatest strength for rough labeling graphs. The benefits of rough labeling using a similarity measure are its adaptability to various data types and applications. Both information theory and image processing depend heavily on Laplacian energy. We discovered unanticipated application for our technology in fields of research and engineering like crystallography, facial recognition, network analysis, satellite communication, etc.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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