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PROPERTIES OF HEMICOMPACT TOPOLOGICAL GROUPS

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Abstract. The hemicompact spaces is an important tool in general topology. Analogous to this concept in topological space, this paper study the hemicompact topological group and discuss its relation with locally compact and σ-compact topological groups.

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1. Introduction

It was discovered in the middle of the 20th century that Hausdorff Abelian groups might be included in the standard Fourier analysis. The subject known to demonstrate how to achieve this is called topological groups (a topological space that is also an abstract group and in which group operations are continuous). Sophus Lie was the first to think about topological groups, although he was only interested in groups having analytic definitions.

A topological group combines the properties of a group and a topological space, one of this property is compactness and it is generalization classes such as hemicompact space or k_ω-space (a topological space with a sequence of compact subsets that cover the space and any compact subset contained in some of it is terms). E. Michael in [15] introduced the definition of hemicompact space and Graev in [11] discussed such spaces in topological group setting. This paper study the introduce the hemicompact group and some of its fundamental characteristics.

2. Fundamentals

Compactness is an important property in the general topology, In 1923, Alexandroff and Urysohn provided the definition of compactness that we use here, The Heine-Borel condition is a generalization of the property of closed and bounded subsets of the real line to topological space.

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Definition 1. *A collection $\mathcal{A}$ of subset of a space X is said to cover X, or to be a covering of X, if the union of the elements of $\mathcal{A}$ is equal to X. It is called an open covering of X if its elements are open subset of X.*

Definition 2. [7]. A topolofical space X is said to be compact if every open covering $\mathcal{A}$ of X contains a finite subcollection (subcover) that also covers X.

Theorem 1. (**Heine-Borel**) Every closed bounded subset of $\mathbb{R}$ is compact.

Lemma 1. Let Y be a subspace of X. Then Y is compact if and only if every covering of Y by sets open in X contains a finite subcollection covering Y.

Theorem 2. [7]. The image of a compact space under a continuous map is compact, provided it is a Hausdorff space.

Definition 3. A space X is σ-compact if and only if X can be written as the union of countably many compact subsets.

Example 1. [13].

(a) Every compact space is σ-compact.

(b) Every countable set with the discrete topology is σ-compact.

(c) It is not true that an uncountable set with the discrete topology is σ-compact. For example, the real line with discrete topology is not a countable union of compact space and cannot be written as one.

Theorem 3. [5]

(a) Let X be a compact space and $F \subseteq X$ a closed subset of X. Then F is compact.

(b) Let X be a compact Hausdorff space. Then $A \subseteq X$ is compact if and only if A is closed.

(c) Every compact Hausdorff space is normal.

Theorem 4. [5]. A compact subset of a Hausdorff space X is closed.

Theorem 5. [7]. Let $f : X \to Y$ be a bijective continuous function. If X is compact and Y is Hausdorff, Then f is a homeomorphism.

Definition 4. [7]. A space X is said to be locally compact at x if there is some compact subspace K of X that contains a neighbourhood of x. If X is locally compact at each of its points, X is said to be locally compact.

Example 2. (a) Every compact space is locally compact.

(b) The real line $\mathbb{R}$ with usual topology is locally compact. The point x lies in some interval (a, b), which in turn is contained in the compact subspace $[a, b]$. The subspace $\mathbb{Q}$ of rational numbers is not locally compact.

(c) Every discrete space is locally compact.

Remark 1. [6]. A topological space X is called a locally compact space if for every $x \in X$ there exists a neighbourhood U of the point x such that $\overline{U}$ is a compact subspace of X.

Proposition 1. [13]. An open subset of locally compact (in particular, compact) space is always locally compact.

Proposition 2. [14]. Every closed subspace in a locally compact space is locally compact.

Proposition 3. [14]. The product of two locally compact spaces is locally compact.

Definition 5. [6]. A Hausdorff space X is a compactly generated space (k-space) if and only if for each $F \subseteq X$, the set F is closed in X provided that the intersection of F with any compact subspace K of the space X is closed in K.

Definition 6. [10]. A Hausdorff topological space X is a hemicompact topological space (k_ω-space) if there exists a sequence of compact sets $K_1 \subseteq K_2 \subseteq ... \subseteq X$, such that $X = \cup_{n\in\mathbb{N}} K_n$ and U is open subset of X if and only if $U \cap K_n$ is open set in K_n for all $n \in \mathbb{N}$.

Remark 2. (a) [13]. From Definition 6, a topological space X is hemicompact topological space if there exists a countable compact subsets K_n of X where $X = \cup_{n=1}^{\infty} K_n$ and $K \subseteq K_n$ for every compact subset of X and in some K_n.
(b) [10]. The sequence $(K_n)_{n\in\mathbb{N}}$ is called a k_ω-sequence for X.
(c) [10]. A Hausdorff topological space X is locally hemicompact
if every point has an open set which is a hemicompact topological space.

Example 3. (a) Compact topological space is hemicompact.
(b) The set of all real numbers equipped with the usual topology is hemicompact topological space.

Remark 3. (a) Every hemicompact topological space is σ-compact space.
The converse is not true. For example the set of rational number equipped with the usual topology is σ-compact topological space which is not hemicompact. (See [12]).
(b) [8]. Any hemicompact topological space is compactly generated topological space.

Now we recall from [10], [9] and [3] the following properties of hemicompact space and locally heicompact space.

Theorem 6. (a) *The product of hemicompact topological space, is a hemicompact topological space.*
(b) *The countable disjoint unions of hemicompact space are k_ω.*

(c) *Let Y be a σ-compact subset of a locally hemicompact space X. Then Y has an open neighbourhood U in X that is a hemicompact topological space.*

Theorem 7. (a) *Locally compact topological space is locally hemicompact.*

(b) *Open sets of locally hemicompact topological space are locally hemicompact.*

Proposition 4. [11].

(a) *A first countable hemicompact topological space is locally compact.*

(b) *A σ-compact and locally compact space is hemicompact.*

3. Topological Group

Definition 7. A topological groups G is a group which is also a topological space, such that the maps

$$\begin{aligned} \mu : G{\times}G &\to G \\ (x,y) &\mapsto xy \end{aligned} \qquad \text{and} \qquad \begin{aligned} \upsilon : G &\to G \\ x &\mapsto x^{-1} \end{aligned}$$

are both continuous. ($G{\times}G$ is provided with the product topology).

If G is a group, and S and T are subsets of G, we let ST and S^{-1}

$$ST = \{st : s \in S, t \in T\} \qquad \textit{and} \qquad S^{-1} = \left\{s^{-1} | s \in S\right\}.$$

The subset S is called symmetric if $S^{-1} = S$. We will let e denote the identity element of a group.

Example 4. (a) The additive groups $\mathbb{R}$ and $\mathbb{C}$ equipped with the usual topology are topological groups.

(b) For any group G equipped With the discrete topology is topological groups.

(c) The indiscrete topology on G, the topology where the collection of open sets is $\{\emptyset, G\}$, makes G into a topological group.

(d) The sets of integers $\mathbb{Z}$, rational numbers $\mathbb{Q}$, and complex numbers $\mathbb{C}$ are all topological groups under addition when they are given their usual topologies.

(e) If n is any positive integers, $\mathbb{Z}^n$. $\mathbb{R}^n$. $\mathbb{Q}^n$. $\mathbb{C}^n$ are all topological groups under addition when given their usual topologies.

Definition 8. A topological group G is said to be homogeneous if for any $g, h \in G$ there is an homeomorphism $h : G \to G$ such that $h(g) = h$.

Proposition 5. [16]. Let G be a group with a topology on it. Then G is a topological group if and only if the following two conditions are satisfied:

(a) For every neighbourhood U of xy, there is a neighbourhood V of x and a neighbourhood W of y such that $VW \subseteq U$.

(b) For every neighbourhood U of x^{-1}, there is a neighbourhood V of x such that $V^{-1} \subseteq U$.

Proof. Suppose first that the multiplication map μ is continuous. Let U be neighbourhood of xy. Since $\mu(x, y) = xy$ and μ is continuous at (x, y), there is a neighbourhood V of x and a neighbourhood W of y such that $VW \subseteq \mu(V \times W) \subseteq U$.

Now suppose that (a) holds. Let $(x, y) \in G \times G$, and let U be any neighbourhood of $\mu(x, y)$=xy. Then choose a neighbourhood V of x and a neighbourhood W of y such that $VW \subseteq U$. But $V \times W$ is a neighbourhood of (x, y) and $\mu(V \times W)$=$VW \subseteq U$. It follows that μ is continuous at (x, y).

Definition 9. Any subgroups H of a topological group G is a topological group with respect to the subspace topology.

Proposition 6. [17]. Let G be a topological group and let $H \subseteq G$ be a subgroup. Then we have the following.

(a) The subgroup H is open if and only if it contains a nonempty open set.

(b) The subgroup H is closed if and only if there exists an open set $U \subseteq G$ such $U \cap H$ is nonempty and closed in U.

Remark 4. A topological subgroup $H \subseteq G$ of a topological group G is called a closed subgroup if as a topological subspace it is a closed subspace.

Proposition 7. [22]. All topological groups are homogeneous spaces.

Proof. Let G be a topological group with product function μ and inversion v. Since the identity map $id : G \to G$ and the constant map $g \to x$ are continuous for any $x \in G$, the map

$$\varphi_x : G \to G \times G$$

$$g \mapsto (g, x)$$

is also continuous and then so is the composition $r_x = \mu \circ \varphi_x$, sending g to gx. Clearly, r_x and $r_{x^{-1}}$ are inverse to each other, both continuous, hence r_x is an homeomorphism. In particular, given $x, y \in G$, $r_{x^{-1}y}$ is an homeomorphism and $r_{x^{-1}y}(x) = xx^{-1}y = y$ for any $x, y \in G$.

The application r_x is called the right translation, and in the same way we can define the left translation l_x, an homeomorphism given by $l_x(g) = xg$.

Proposition 8. [22]. Let G be a topological group, $x \in G$ and $A, B \subseteq G$. Then,

(a) if A is open subset then so are Ax and xA;

(b) if A is open subset then so are AB and BA;

(c) if A is closed subset then so are Ax and xA;

(d) if A is closed subset and B finite then AB and BA are closed.

Proof. Ax and xA are the image of A under the homeomorphisms r_x and l_x respectively. Hence, if A is open then so are Ax and xA, and if A is closed Ax and xA are closed too.

Write $AB = \cup_{b\in B} bA$. If A is open, AB and BA are a union of open subsets, hence open. And if A is closed and B finite, AB and BA are a union of finitely many closed subsets, hence closed.

Proposition 9. If G is a topological group, then every open subgroup of G is also closed.

Proof. Let H be an open subgroup of G. Then any translation xH is also open. So,

$$Y = \cup_{x\in G-H} xH$$

is also open. From elementary group theory, $H = G - Y$, and so H is closed.

Remark 5. Let V be a neighbourhood of x, then $x^{-1}V$ is a neighbourhood of identity. Conversely, let U be a neighbourhood of identity, then $xU = \{xu : u \in U\}$ is a neighbourhood of x.

Proposition 10. [19]. A topological group is metrizable if and only if it is a Hausdorff topological space and the identity element has a countable neighbourhood basis.

4. Compact Topological Group

We call a topological group G a compact group if the topology on G is Hausdorff topological space and compact.

Proposition 11. [18]. If G is a topological group, and H is a subgroup of G, then the topological closure of H, $\overline{H}$, is a subgroup of G.

Corollary 1. (a) A topological group is first-countable if and only if there is a countable neighbourhood base at the identity (or any point).

(b) A topological group G that is both locally compact and Hausdorff is called a locally compact group.

5. Hemicompact Topological Groups (k_ω-Group)

The definition of hemicompact topological group has been introduced in [4] (see [9] and [10]). In this section, we recall From [4] and [10] the definition and the main properties of hemicompact topological group. Also we prove that a locally compact topological group is locally hemicompact topological group and a σ-compact and locally compact topological group is hemicompact topological group.

Definition 10. A Hausdorff topological group is called a hemicompact topological group if its underlying topological space a heimcompact space.

Example 5. (a) The rational numbers $\mathbb{Q}$ equipped with the topology induced by the real numbers $\mathbb{R}$, is a σ-compact topological space, but not a hemicompact topological group group.

(b) Each open subgroup generated by a compact neighbourhood is a hemicompact topological group.

Remark 6. A Hausdorff topological group is called a locally hemicompact topological group if its underlying topological space which is locally hemicompact.

Example 6. (a) Every discrete group is a locally hemicompact topological group.

(b) Every hemicompact topological group is a locally hemicompact topological group.

Proposition 12. [10]. *A locally hemicompact topological group has an open subgroup which is a hemicompact topological group.*

Theorem 8. *Open subgroup of locally hemicompact topological group are locally hemicompact topological group.*

Proof. Follows from Proposition 4.2, [10].

Theorem 9. *A locally compact topological group is locally hemicompact topological group.*

Proof. Let G be a locally compact topological group and $g \in G$, for compact sets $K_1, K_2, ...$ suppose that $g \in K_1$ and $K_n \subseteq K_{n+1}$. Hence $g \in U := \cup_{n\in\mathbb{N}} K_n$ and U is a hemicompact topological group. where $(K_n)_{n\in\mathbb{N}}$ is k_ω-sequence. □

Theorem 10. *The countable disjoint unions of hemicompact topological group are hemicompact topological group.*

Proof. Suppose that $G = \coprod_{j\in\mathbb{N}} G_j$, with each G_j for each $j \in \mathbb{N}$ is a hemicompact topological group and $(K_n^{(j)})_{n\in\mathbb{N}}$ is k_ω-sequence for G_j, then $K_n := \cup_{j\leqslant n} K_n^{(j)}$ is a k_ω-sequence for G. □

Proposition 13. *A first countable hemicompact topological group G is locally compact topological group.*

Proof. Suppose that there is $g \in G$ without compact neighbourhood. Let $U_n \supset U_{n+1} \supset ...$ be a countable basis for the neighbourhood of g, Let $... \subset K_n \subset K_{n+1} \subset ...$ be an admissible sequence of G. and $g_n \in U_n - K_n$ for each $n \in \mathbb{N}$. The compact set $K := \{g_n : n \in \mathbb{N}\} \cup \{g\}$ is not contained in any K_n for each $n \in \mathbb{N}$.

□

Proposition 14. *A σ-compact and locally compact topological group is hemicompact topological group.*

Proof. Let G be a locally compact topological group and $\cup_{i\in I}U_i$ be an open cover with compact closure. G is σ-compact topological group, hence there exists a sequence $(K_n)_{n\in\mathbb{N}}$ of compacts sets such that $X = \cup_{n\in\mathbb{N}}K_n$. For all K_n there exists a finite family of $(U_i)_{i\in I}$ covers K_n. Let $U_n = \cup_{i\in I}U_i$ for all $n \in \mathbb{N}$ and $K'_n := \overline{\cup_{k=1}^{n} U_k}$. Then $(K'_n)_{n\in\mathbb{N}}$ is a sequence of compacts sets. Let the set $K \subset G$ be compact, hence there exists a finite family of $(U_i)_{i\in I}$ covering K. Thus $K \subset K_n$ for some $n \in \mathbb{N}$. □

6. Conclusion

Analogous to the definition hemicompact space we studied the definition of hemicompact group and prove theorems concerning the relations between:

- Locally compact topological group with locally hemicompact topological group.
- σ-compact and locally compact topological group with hemicompact topological group.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] H. Alzaareer, Differential calculus on multiple products, Indag. Math. 30 (2019), 1036–1060. https://doi.org/10.1016/j.indag.2019.07.008.

[2] H. Alzaareer, Lie groups of C^k-maps on non-compact manifolds and the fundamental theorem for Lie group-valued mappings, J. Group Theory. 24 (2021), 1099–1134. https://doi.org/10.1515/jgth-2018-0200.

[3] H. Alzaareer, Lie group structures on groups of maps on non-compact spaces and manifolds, Ph.D. Thesis, Universität Paderborn, 2013.

[4] L. Außenhofer, On the Hausdorff variety generated by all locally k_ω groups and on Schwartz groups, Topol. Appl. 159 (2012), 2248–2257. https://doi.org/10.1016/j.topol.2011.11.058.

[5] P.E. Long, An introduction to general topology, Merrill, Columbus, Ohio, 1971.

[6] R. Engilking, General topology, Helderman Verlag, Berlin, 1989.

[7] J.R. Munkres, Topology, Pearson Education, Noida, 2014.

[8] S.P. Franklin, B.V. Smith Thomas, A survey of k_ω-spaces, Topol. Proc. 2 (1977), 111–124.

[9] H. Glöckner, Completeness of locally k_ω-groups and related infinite-dimensional Lie groups, Topol. Appl. 228 (2017), 277–284. https://doi.org/10.1016/j.topol.2017.05.007.

[10] H. Glöckner, R. Gramlich, T. Hartnick, Final group topologies, Kac-Moody groups and Pontryagin duality, Isr. J. Math. 177 (2010), 49–101. https://doi.org/10.1007/s11856-010-0038-5.

[11] M. Graev, Free topological groups, American Mathematical Society, Providence, 1951.

[12] L. Holá , D. Holý, W. Moors, Usco mappings, in: USCO and Quasicontinuous Mappings, De Gruyter, 2021: pp. 1–58. https://doi.org/10.1515/9783110750188-001.

[13] T. Husain, Topology and maps, Springer, New York, 2012.

[14] M. Manetti, Topology, Springer, Cham, 2015. https://doi.org/10.1007/978-3-319-16958-3.

[15] E. Michael, Bi-quotient maps and cartesian products of quotient maps, Ann. l'inst. Fourier. 18 (1968), 287–302. https://doi.org/10.5802/aif.301.

[16] T. Bray, W. Jaworski, Mathematics and statistics honours project, Carleton University, 2021.

[17] L. Kramer, Locally compact groups and lie groups, Lecture, 2020.

[18] T.B. Singh, Elements of topology, Chapman and Hall/CRC, 2013. https://doi.org/10.1201/b14870.

[19] A. Bialynicki-Birula, B. Bojarski, Z. Ciesielski, et al. Dissertationes mathematicae-(rozprawy matematyczne), Dissertationes Mathematicae-Rozprawy Matematyczne, 2001.

[20] M. Mhailan, M. Abu Hammad, M. Al Horani, et al. On fractional vector analysis, J. Math. Comput. Sci. 10 (2020), 2320-2326. https://doi.org/10.28919/jmcs/4863.

[21] T.B. Singh, Introduction to topology, Springer, Singapore, 2019.

[22] K.B. Iturbe, Grupos topológicos, Thesis, University of the Basque Countryb 2016. http://hdl.handle.net/10810/18039.