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SOME NEW RATIONAL CONTRACTIONS APPROACH TO THE SOLUTION OF INTEGRAL EQUATIONS VIA UNIQUE COMMON FIXED POINT THEOREMS IN COMPLEX VALUED G_b-METRIC SPACES

SIDRA AKBAR[1], SAIF UR REHMAN[1,*], MOHAMMED M.M. JARADAT[2], MUHAMMAD IMRAN HAIDER[1], FARYAL GUL[3]

[1]Institute of Numerical Sciences, Department of Mathematics, Gomal University, Dera Ismail Khan 29220, Pakistan

[2]Mathematics Program, Department of Mathematics and Statistics, College of Arts and Sciences, Qatar University 2017, Doha, Qatar

[3]Gomal Research Institute of Computing, Faculty of Computing, Gomal University, Dera Ismail Khan 29220, Pakistan

*Corresponding author: saif.urrehman27@yahoo.com

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Abstract. This paper aims to prove some generalized common fixed point theorems for three self-mappings in complex valued G_b-metric spaces under the new modified rational contraction conditions. We prove the uniqueness of common fixed point in complex valued G_b-metric spaces without the continuity of self-mappings with supportive trivial and non-trivial illustrative examples. Moreover, we study the approach of Urysohn type integral equations in complex valued G_b-metric spaces to support our main work. By using this concept, one can prove different types of coincidence points and common fixed point results for single-valued contraction conditions in complex valued G_b-metric spaces with the application of different types of integral equations.

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1. Introduction

In 1922, the concept of fixed point (FP) theory was presented by Banach [1] and proved a "Banach contraction principle" which is stated as: "a single-valued contraction map on a complete metric space has a unique FP". Later on, many mathematicians contributed their ideas to the problem of FP by using different types of metric spaces, mappings, and applications.

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In 1989, Bakhtin [2] established the notion of $b-$metric space and proved some of its classical properties. After that, Czerwik [3] used the approach of Bakhtin [2] and proved some FP-results for non-linear set-valued contraction conditions in $b-$metric spaces. While in [4], Akkouchi used an implicit relation approach and presented common fixed point theorems (CFP-theorems) on $b-$metric spaces for single-valued contractive type mappings. Aghajani et al. [5] proved some generalized CFP-results in partially ordered $b-$metric spaces by using the approach of weak-contraction condition. Further, Aydi et al. [6] proved some modified set-valued Quasi-contraction results for FP in $b-$metric spaces. Later on, Roshan et al. [7] proved CFP-theorems in $b-$metric spaces and concluded that the $b-$metric is not necessarily a continuous map. In this direction, some more related FP-results can be found in (e.g, see; [8–13]). In 2014, Mukheimer [14] proved some CFP theorems on complex valued $b-$metric ($CVb-$metric) spaces. Recently, Mehmood et al. [15,16], proved some CFP-results under the rational type-contraction conditions in CV$b-$metric spaces by using the compatibility self-mappings with an application.

Mustafa and Sims [17] introduced the generalized concept of metric space which is known as $G-$metric space. They used Dhage's theory and proved CFP-results in $G-$metric spaces. Further, Mustafa et al. [18] established some modified contraction results for FP in the said space. In [19], Chugh et al. presented the P property in $G-$metric space and proved some results. While Saadati et al. [20] used the concept of $G-$metric spaces to introduce $\Omega-$distance on a generalized partially ordered $G-$metric spaces and proved FP-theorem involving $\Omega-$distance.

In 2014, Aghajani et al. [21] combined the concept of $b-$metric and $G-$metric spaces, and introduced the new concept of generalized $b-$metric space (G_b-metric space) and established a CFP-theorem by using weakly compatible single-valued mappings. Aydi [22] improved and generalized some well-known existing results in the literature and proved some coupled fixed point and tripled coincidence point results in G_b-metric spaces. Gupta in [23] extended and improved some published results and proved FP-results in G_b-metric spaces. In [24] Makran et al. proved a CFP-theorem by using multi-valued maps and established its integral type application. In [25] Mustafa et al. established some tripled coincidence point results in partially ordered G_b-metric spaces and presented an Integral type application.

Ege [26] introduced the concept of a complex valued G_b-metric (CVG_b-metric) space and proved some FP-results in the sense of Banach Kannan contraction principles. Later on, Ege [27] proved a CFP-theorem via $\alpha-$series and obtained new results in CVG_b-metric spaces. Ansari et al. [28] used the concept of C-class functions in CVG_b-metric spaces and proved some FP-theorems in CVG_b-metric spaces. Recently, in 2020, Ege et al. [29] introduced complex $C-$class function in CVG_b-metric spaces to established some FP-theorem by using the complex $C-$class function, $\alpha-$admissible mapping,

$\alpha-(F,\Psi,\Phi)-$contractive type mappings. Recently, Mehmood et al. [30], established some CFP-results in CVG_b-metric spaces with an application.

In this paper, use the approach of Ege [26] and Mehmood et al. [30], and study some new generalized product type rational contraction results in CVG_b-metric spaces based on single-valued mappings. We prove the uniqueness of CFP for three self-mappings under the generalized rational contraction conditions with illustrative examples. Further, to support our results, we establish an application of the UTIEs for the existence of a unique common solution to verify the validity of our findings.

2. Preliminaries

In this section, we present the preliminary concepts related to our main work.
Let the set of complex-numbers is denoted by $\mathbb{C}$ and $v_i, v_{ii} \in \mathbb{C}$. Define $\leq$ as: $v_i \leq v_{ii}$, iff $R(v_i) \leq R(v_{ii})$ and $I(v_i) \leq I(v_{ii})$. Where R and I denotes the real part and imaginary part of $\mathbb{C}$ respectively. Accordingly $v_i \leq v_{ii}$, if any one of the following holds:

i)- $R(v_i) = R(v_{ii})$ and $I(v_i) = I(v_{ii})$,

ii)- $R(v_i) < R(v_{ii})$ and $I(v_i) = I(v_{ii})$,

iii)- $R(v_i) = R(v_{ii})$ and $I(v_i) < I(v_{ii})$,

iv)- $R(v_i) < R(v_{ii})$ and $I(v_i) < I(v_{ii})$.

In special case, we can write $v_i \lesssim v_{ii}$ if $v_i \neq v_{ii}$ and one of (ii), (iii), and (iv) is satisfied.

Remark 2.1. [14] The following presented properties can be hold and verified:

i)- if $\alpha_1, \alpha_2 \in \mathbb{R}$ and $\alpha_1 \leq \alpha_2 \Rightarrow \alpha_1 y \leq \alpha_2 y \quad \forall x \in \mathbb{C}$,

ii)- $0 \leq v_i \lesssim v_{ii} \Rightarrow |v_i| < |v_{ii}|$,

iii)- $v_i \leq v_{ii}$ and $v_{ii} < v_{iii} \Rightarrow v_i < v_{iii}$.

Definition 2.2. [26] Let $V \neq \emptyset$ set and $b > 1$. A mapping $G : V^3 \to \mathbb{C}$ is called a CVG_b-metric if G holds the following axioms:

i)- $G(u, w, x) = 0$ if $u = w = x$,

ii)- $0 < G(u, u, w)$ for all $u, w \in V$ with $u \neq w$,

iii)- $G(u, u, w) \leq G(u, w, x)$ for all $u, w, x \in V$ with $w \neq x$,

iv)- $G(u, w, x) = G(p\{u, w, x\})$, where p is a permutation of u, w, x,

v)- $G(u, w, x) \leq b[G(u, a, a) + G(a, w, x)]$ for all $u, w, x, a \in V$.

Then a pair (V, G) is called a CVG_b-metric space.

Example 2.3. Let $V = [0, \infty)$ and a metric $G : V^3 \to \mathbb{C}$ is defined by:

$$G(u, w, x) = \left(\frac{|3u - 3w|}{4} + \frac{|3w - 3x|}{4} + \frac{|3x - 3u|}{4}\right)(1 + i), \ \forall u, w, x \in V.$$

Then (V, G) is a CVG_b-metric space with constant $b = 2$.

Proposition 2.4. [26] *Let (V, G) be a CVG_b-metric space. Then, $\forall\, u, w, x \in V$,*

i)- $G(u, w, x) \leq b(G(u, u, w) + G(u, u, x))$,

ii)- $G(u, w, w) \leq 2bG(u, u, w)$.

Definition 2.5. [26] Let (V, G) be a CVG_b-metric space, let $u \in V$ and $\{u_m\}$ be a sequence in V. Then, a sequence:

i)- $\{u_m\}$ is CVG_b-convergent to u if for every $a > 0$ in $\mathbb{C}$, $\exists\, m_0 \in \mathbb{N}$ such that $G(u, u_m, u_j) < a,\ \forall\, m, j \geq m_0$.

ii)- $\{u_m\}$ is called CVG_b-Cauchy if for every $a > 0$ in $\mathbb{C}$, $\exists\, m_0 \in \mathbb{N}$ such that $G(u_m, u_j, u_k) < a,\ \forall\, m, j, k \geq m_0$.

iii)- If every CVG_b-Cauchy sequence is CVG_b-convergent in (V, G), then a pair (V, G) is called CVG_b-complete.

Proposition 2.6. [26] *Let (V, G) be a CVG_b-metric space and $\{u_m\}$ be a sequence in V. Then $\{u_m\}$ is CVG_b-convergent to u iff $|G(u, u_m, u_j)| \to 0$ as $m, j \to \infty$.*

Theorem 2.7. [26] *Let (V, G) be a CVG_b-metric space, then for a sequence $\{u_m\}$ in V and a point $u \in V$, the following are equivalent:*

i)- $\{u_m\}$ *is CVG_b-convergent to u,*

ii)- $|G(u_m, u_m, u)| \to 0$ *as* $m \to \infty$,

iii)- $|G(u_m, u, u)| \to 0$ *as* $m \to \infty$.

Proposition 2.8. [26] *Let (V, G) be a CVG_b-metric space and $\{u_m\}$ be a sequence in V. Then $\{u_m\}$ is a CVG_b-convergent to u iff $|G(u, u_m, u_j)| \to 0$ as $m, j \to \infty$.*

Proof: *Suppose that $\{u_m\}$ is CVG_b-convergent to u and let*

$$\beta = \frac{\varepsilon}{\sqrt{2}} + i\frac{\varepsilon}{\sqrt{2}} \quad \forall\, \varepsilon > 0.$$

Then, $0 < \beta \in \mathbb{C}$ and there is $m_0 \in \mathbb{N}$ such that $G(u, u_m, u_j) < \beta$ for $m, j \geq m_0$. Thus, $|G(u, u_m, u_j)| < |\beta| = \varepsilon$ for $m, j \geq m_0$ and so $|G(u, u_m, u_j)| \to 0$ as $m, j \to \infty$. Suppose that $|G(u, u_m, u_j)| \to 0$ as $m, j \to \infty$. For a given $\beta \in \mathbb{C}$ with $\beta > 0$, there exists $\delta > 0$ such that for $u \in \mathbb{C}$,

$$|u| < \delta \Rightarrow u < \beta.$$

Considering $\delta > 0$ and there is $m_0 \in \mathbb{N}$ such that $|G(u, u_m, u_j)| < \delta$ for $m, j \geq m_0$. This implies that $G(u, u_m, u_j) < \beta$ for $m, j \geq n_0$, i.e., $\{u_m\}$ is CVG_b-convergent to u.

3. Main Result

Theorem 3.1. *Let (V, G) be a complete CVG_b-metric space with constant $b > 1$ and $J_1, J_2, J_3 : V \to V$ be mappings satisfying:*

$$G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x) + \eta_2 \left(\frac{G(u, J_1u, J_1u) \cdot G(w, J_2w, J_2w) \cdot G(x, J_3x, J_3x)}{1 + G(w, J_2w, J_3x) \cdot G(J_2w, J_3x, J_3x)} \right), \tag{3.1}$$

for all $u, w, x \in V$ and $\eta_1, \eta_2 \in [0, \frac{1}{2})$ with $(\eta_1 + \eta_2) < \frac{1}{2}$. Then the mappings J_1, J_2 and J_3 have a unique CFP in V.

Proof. Let $u_0 \in V$ br the arbitrary point. Let the iterative sequences $\{u_n\}_{n\geq 0}$ in V be defined by

$$u_{3n+1} = J_1 u_{3n}, \quad u_{3m+2} = J_2 u_{3m+1}, \quad \text{and} \quad u_{3m+3} = J_3 u_{3m+2} \quad \forall n \geq 0. \tag{3.2}$$

Now by the view of (3.1), we have

$$\begin{aligned} &G(u_{3m+1}, u_{3m+2}, u_{3m+3}) = G(J_1u_{3m}, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(u_{3m}, u_{3m+1}, u_{3m+2}) \\ &+ \eta_2 \left(\frac{G(u_{3m}, J_1u_{3m}, J_1u_{3m}) \cdot G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{1 + G(u_{3m+1}, J_2u_{3m+1}, J_3u_{3m+2}) \cdot G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2})} \right) \\ &= \eta_1 G(u_{3m}, u_{3m+1}, u_{3m+2}) \\ &+ \eta_2 \left(\frac{G(u_{3m}, u_{3m+1}, u_{3m+1}) \cdot G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{1 + G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})} \right). \end{aligned}$$

This implies that,

$$\begin{aligned} &|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \leq \eta_1 |G(u_{3m}, u_{3m+1}, u_{3m+2})| \\ &+ \eta_2 \left(\frac{|G(u_{3m}, u_{3m+1}, u_{3m+1})| \cdot |G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|1 + G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})|} \right) \\ &\leq \eta_1 |G(u_{3m}, u_{3m+1}, u_{3m+2})| \\ &+ \eta_2 \left(\frac{|G(u_{3m}, u_{3m+1}, u_{3m+2})| \cdot |G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|} \right). \end{aligned}$$

After simplification, we get that

$$|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \leq \alpha_1 |G(u_{3m}, u_{3m+1}, u_{3m+2})|, \quad \text{where } \alpha_1 = (\eta_1 + \eta_2) < \frac{1}{2}. \tag{3.3}$$

Again by the view of (3.1), we have

$$\begin{aligned} &G(u_{3m+2}, u_{3m+3}, u_{3m+4}) = G(J_1u_{3m+3}, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \\ &+ \eta_2 \left(\frac{G(u_{3m+3}, J_1u_{3m+3}, J_1u_{3m+3}) \cdot G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{1 + G(u_{3m+1}, J_2u_{3m+1}, J_3u_{3m+2}) \cdot G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2})} \right) \\ &= \eta_1 G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \\ &+ \eta_2 \left(\frac{G(u_{3m+3}, u_{3m+4}, u_{3m+4}) \cdot G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{1 + G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})} \right). \end{aligned}$$

This implies that

$$|G(u_{3m+2}, u_{3m+3}, u_{3m+4})| \leq \eta_1 |G(u_{3m+1}, u_{3m+2}, u_{3m+3})| + \eta_2 \left(\frac{|G(u_{3m+3}, u_{3m+4}, u_{3m+4})| \cdot |G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|} \right).$$

After simplification, we get that

$$|G(u_{3m+2}, u_{3m+3}, u_{3m+4})| \leq \alpha_2 |G(u_{3m+1}, u_{3m+2}, u_{3m+3})|, \quad \text{where } \alpha_2 = \frac{\eta_1}{1-\eta_2} < \frac{1}{2}. \tag{3.4}$$

Now, again by the view of (3.1), we have

$$\begin{aligned} G(u_{3m+3}, u_{3m+4}, u_{3m+5}) &= G(J_1 u_{3m+3}, J_2 u_{3m+4}, J_3 u_{3m+2}) \leq \eta_1 G(u_{3m+2}, u_{3m+3}, u_{3m+4}) \\ &+ \eta_2 \left(\frac{G(u_{3m+3}, J_1 u_{3m+3}, J_1 u_{3m+3}) \cdot G(u_{3m+4}, J_2 u_{3m+4}, J_2 u_{3m+4}) \cdot G(u_{3m+2}, J_3 u_{3m+2}, J_3 u_{3m+2})}{1 + G(u_{3m+4}, J_2 u_{3m+4}, J_3 u_{3m+2}) \cdot G(J_2 u_{3m+4}, J_3 u_{3m+2}, J_3 u_{3m+2})} \right) \\ &= \eta_1 G(u_{3m+2}, u_{3m+3}, u_{3m+4}) \\ &+ \eta_2 \left(\frac{G(u_{3m+3}, u_{3m+4}, u_{3m+4}) \cdot G(u_{3m+4}, u_{3m+5}, u_{3m+5}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{1 + G(u_{3m+4}, u_{3m+5}, u_{3m+3}) \cdot G(u_{3m+5}, u_{3m+3}, u_{3m+3})} \right). \end{aligned}$$

This implies that

$$|G(u_{3m+3}, u_{3m+4}, u_{3m+5})| \leq \eta_1 |G(u_{3m+2}, u_{3m+3}, u_{3m+4})| + \eta_2 \left(\frac{|G(u_{3m+3}, u_{3m+4}, u_{3m+4})| \cdot |G(u_{3m+4}, u_{3m+5}, u_{3m+5})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|G(u_{3m+4}, u_{3m+5}, u_{3m+3})| \cdot |G(u_{3m+5}, u_{3m+3}, u_{3m+3})|} \right).$$

After simplification, we get that

$$|G(u_{3m+3}, u_{3m+4}, u_{3m+5})| \leq \alpha_1 |G(u_{3m+2}, u_{3m+3}, u_{3m+4})|, \quad \text{where } \alpha_1 = (\eta_1 + \eta_2) < 1. \tag{3.5}$$

Let us define $\alpha := \max\{\alpha_1, \alpha_2\} < \frac{1}{2}$. Now from (3.4), (3.4), (3.5), and by induction, we have that

$$\begin{aligned} &|G(u_{3m+2}, u_{3m+3}, u_{3m+4})| \leq \alpha |G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \\ &\leq \alpha^2 |G(u_{3m}, u_{3m+1}, u_{3m+2})| \leq \cdots \leq \alpha^{3m+2} |G(u_0, u_1, u_2)| \to 0, \quad \text{as } m \to \infty. \end{aligned}$$

This shows that the sequence $\{u_m\}$ is contractive in CVG_b-metric space (V, G). Let there are natural numbers j and m such that $j > m$, then we have

$$\begin{aligned} &|G(u_m, u_j, u_j)| \leq b\,|G(u_m, u_{m+1}, u_{m+1})| + b\,|G(u_{m+1}, u_j, u_j)| \\ &\leq b\,|G(u_m, u_{m+1}, u_{m+1})| + b^2\,|G(u_{m+1}, u_{m+2}, u_{m+2})| + \cdots + b^{j-m}\,|G(u_{j-1}, u_j, u_j)| \\ &\leq b\alpha^m\,|G(u_0, u, u)| + b^2\alpha^{m+1}\,|G(u_0, u, u)| + \cdots + b^{j-m}\alpha^{j-1}\,|G(u_0, u, u)| \\ &\leq [b\alpha^m + b^2\alpha^{m+1} + \cdots + b^{j-m}\alpha^{j-1}]\,|G(u_0, u, u)| \\ &= [b\alpha^m + b^2\alpha^{m+1} + \cdots + b^{j-m}\alpha^{j-1}]\,|G(u_0, u, u)| \\ &= b\alpha^m[1 + b\alpha + b^2\alpha^2 + \cdots + b^{j-(m+1)}\alpha^{j-(m+1)}]\,|G(u_0, u, u)| \end{aligned}$$

$$= b\alpha^m \sum_{t=0}^{j-(m+1)} b^t \alpha^t \,|G(u_0,u,u)| \leq b\alpha^m \sum_{t=0}^{\infty} b^t \alpha^t \,|G(u_0,u,u)|$$
$$= \frac{b\alpha^m}{1-b\alpha}\,|G(u_0,u,u)| \to 0, \quad \text{as } m \to \infty.$$

By using Proposition 2.4 (i), we have, $|G(u_m,u_j,u_k)| \leq b\,(|G(u_m,u_j,u_j)| + |G(u_m,u_k,u_k)|)$ for $m,j,k \in \mathbb{N}$ with $k > j > m$. If we apply limit $m,j,k \to \infty$, we get that $|G(u_m,u_j,u_k)| \to 0$. This implies that, $\{u_m\}$ is a CVG_b-Cauchy sequence. Since, (V,G) is a complete CVG_b-metric space, $\exists\, \rho \in V$ such that $u_m \to \rho$ as $m \to \infty$, or $\lim_{m\to\infty} u_m = \rho$. Now, we shall prove that $J_1\rho = \rho$, then from (3.1), we have

$$G(J_1\rho, u_{3m+2}, u_{3m+3}) = G(J_1\rho, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(\rho, u_{3m+1}, u_{3m+2})$$
$$+ \eta_2 \left(\frac{G(\rho, J_1\rho, J_1\rho)\cdot G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1})\cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{1 + G(u_{3m+1}, J_2u_{3m+1}, J_3u_{3m+2})\cdot G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2})} \right)$$
$$= \eta_1 G(\rho, u_{3m+1}, u_{3m+2})$$
$$+ \eta_2 \left(\frac{G(\rho, J_1\rho, J_1\rho)\cdot G(u_{3m+1}, u_{3m+2}, u_{3m+2})\cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{1 + G(u_{3m+1}, u_{3m+2}, u_{3m+3})\cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})} \right).$$

This implies that,

$$|G(J_1\rho, u_{3m+2}, u_{3m+3})| \leq \eta_1 |G(\rho, u_{3m+1}, u_{3m+2})|$$
$$+ \eta_2 \left(\frac{|G(\rho, J_1\rho, J_1\rho)|\cdot|G(u_{3m+1}, u_{3m+2}, u_{3m+2})|\cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|1 + G(u_{3m+1}, u_{3m+2}, u_{3m+3})\cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})|} \right).$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain $G(J_1\rho, \rho, \rho) = 0$ this implies $J_1\rho = \rho$. Hence,

$$J_1\rho = \rho. \tag{3.6}$$

Next, we have to show that $J_2\rho = \rho$, then from (3.1), we have

$$G(u_{3m+1}, J_2\rho, u_{3m+3}) = G(J_1u_{3m}, J_2\rho, J_3u_{3m+2}) \leq \eta_1 G(u_{3m}, \rho, u_{3m+2})$$
$$+ \eta_2 \left(\frac{G(u_{3m}, J_1u_{3m}, J_1u_{3m})\cdot G(\rho, J_2\rho, J_2\rho)\cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{1 + G(\rho, J_2\rho, J_3u_{3m+2})\cdot G(J_2\rho, J_3u_{3m+2}, J_3u_{3m+2})} \right)$$
$$= \eta_1 G(u_{3m}, \rho, u_{3m+2})$$
$$+ \eta_2 \left(\frac{G(u_{3m}, u_{3m+1}, u_{3m+1})\cdot G(\rho, J_2\rho, J_2\rho)\cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{1 + G(\rho, J_2\rho, u_{3m+3})\cdot G(J_2\rho, u_{3m+3}, u_{3m+3})} \right).$$

This implies that,

$$|G(u_{3m+1}, J_2\rho, u_{3m+3})| \leq \eta_1 |G(u_{3m}, \rho, u_{3m+2})|$$
$$+ \eta_2 \left(\frac{|G(u_{3m}, u_{3m+1}, u_{3m+1})|\cdot|G(\rho, J_2\rho, J_2\rho)|\cdot|G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{|1 + G(\rho, J_2\rho, u_{3m+3})\cdot G(J_2\rho, u_{3m+3}, u_{3m+3})|} \right).$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain $G(\rho, J_2\rho, \rho) = 0$ implies that $J_2\rho = \rho$. Hence,

$$J_2\rho = \rho. \tag{3.7}$$

Next, we have to prove that $J_3\rho = \rho$, then from (3.1), we have

$$G(u_{3m+1}, u_{3m+2}, J_3\rho) = G(J_1u_{3m}, J_2u_{3m+1}, J_3\rho) \leq \eta_1 G(u_{3m}, u_{3m+1}, \rho)$$
$$+ \eta_2 \left(\frac{G(u_{3m}, J_1u_{3m}, J_1u_{3m}) \cdot G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(\rho, J_3\rho, J_3\rho)}{1 + G(u_{3m+1}, J_2u_{3m+1}, J_3\rho) \cdot G(J_2u_{3m+1}, J_3\rho, J_3\rho)} \right)$$
$$= \eta_1 G(u_{3m}, u_{3m+1}, \rho)$$
$$+ \eta_2 \left(\frac{G(u_{3m}, u_{3m+1}, u_{3m+1}) \cdot G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(\rho, J_3\rho, J_3\rho)}{1 + G(u_{3m+1}, u_{3m+2}, J_3\rho) \cdot G(u_{3m+2}, J_3\rho, J_3\rho)} \right).$$

This implies that,

$$|G(u_{3m+1}, u_{3m+2}, J_3\rho)| \leq \eta_1 |G(u_{3m}, u_{3m+1}, \rho)|$$
$$+ \eta_2 \left(\frac{|G(u_{3m}, u_{3m+1}, u_{3m+1})| \cdot |G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(\rho, J_3\rho, J_3\rho)|}{|1 + G(u_{3m+1}, u_{3m+2}, J_3\rho) \cdot G(u_{3m+2}, J_3\rho, J_3\rho)|} \right).$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain $G(\rho, \rho, J_3\rho) = 0$ implies that $J_3\rho = \rho$. Hence,

$$J_3\rho = \rho. \tag{3.8}$$

Hence, from (3.6), (3.7), and (3.8), it is proved that the mappings $J_1,\ J_2$, and J_3 have a CFP, that is, $J_1\rho = J_2\rho = J_3\rho = \rho$. Now, we have to prove the uniqueness of CFP. Let, there exists $\rho^* \in V$ be the other CFP of the three self-mappings $J_1,\ J_2$, and J_3, such that $J_1\rho^* = J_2\rho^* = J_3\rho^* = \rho^*$. Then, by the view of (3.1), we have

$$G(\rho, \rho^*, \rho^*) = G(J_1\rho, J_2\rho^*, J_3\rho^*) \leq \eta_1 G(\rho, \rho^*, \rho^*)$$
$$+ \eta_2 \left(\frac{G(\rho, J_1\rho, J_1\rho) \cdot G(\rho^*, J_2\rho^*, J_2\rho^*) \cdot G(\rho^*, J_3\rho^*, J_3\rho^*)}{1 + G(\rho^*, J_2\rho^*, J_3\rho^*) \cdot G(J_2\rho^*, J_3\rho^*, J_3\rho^*)} \right).$$

This implies that

$$|G(\rho, \rho^*, \rho^*)| \leq \eta_1 |G(\rho, \rho^*, \rho^*)| + \eta_2 \left(\frac{|G(\rho, \rho, \rho)| \cdot |G(\rho^*, \rho^*, \rho^*)| \cdot |G(\rho^*, \rho^*, \rho^*)|}{|1 + G(\rho^*, \rho^*, \rho^*)| \cdot |G(\rho^*, \rho^*, \rho^*)|} \right) = \eta_1 |G(\rho, \rho^*, \rho^*)|.$$

Hence,

$$|G(\rho, \rho^*, \rho^*)| \leq \eta_1 |G(\rho, \rho^*, \rho^*)| \Rightarrow (1 - \eta_1)|G(\rho, \rho^*, \rho^*)| \leq 0,$$

is a contradiction. Hence, $|G(\rho, \rho^*, \rho^*)| = 0$ implies that $\rho = \rho^*$, proved that the three self-mappings $J_1,\ J_2$, and J_3 have a unique CFP in V.

By using $\eta_2 = 0$ in Theorem 3.2, we get the following corollary.

Corollary 3.2. *Let (V, G) be a complete CVG_b-metric space with coefficient $b > 1$ and $J_1, J_2, J_3 : V \to V$ be mappings satisfying:*

$$G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x), \tag{3.9}$$

for all $u, w, x \in V$, $\eta_1 \in [0, \frac{1}{3})$ such that $b\eta_1 < 1$. Then the three self-mappings $J_1,\ J_2$, and J_3 have a unique CFP in V.

Example 3.3. Let (G, V) be a CVG_b-metric space, let $V = [0, 1]$ and $G : V^3 \to \mathbb{R}$ is defined by:

$$G(u, w, x) = \max\{|u - w|, |w - x|, |x - u|\}(1 + i), \quad \text{for all } u, w, x \in V. \tag{3.10}$$

Next, we define the mappings, $J_1, J_2, J_3 : V \to V$ by $J_1 v = J_2 v = J_3 v = \frac{2v}{15} + \frac{4}{15}$ for all $v \in [0, 1]$. Then, we have to calculate the terms of (3.1), that are,

$$\begin{aligned}
&G(J_1u, J_2w, J_3x) = \frac{2}{15}G(u, w, x), \quad G(u, J_1u, J_1u) = \frac{1}{15}|13u - 4|(1 + i),\\
&G(w, J_2w, J_2w) = \frac{1}{15}|13w - 4|(1 + i), \quad G(x, J_3x, J_3x) = \frac{1}{15}|13x - 4|(1 + i),\\
&G(w, J_2w, J_3x) = \frac{1}{15}\max\{|13w - 4|, 2|w - x|, |2x - 4 + 15w|\}(1 + i),\\
&\text{and } G(J_2w, J_3x, J_3x) = \frac{2}{15}|w - x|(1 + i).
\end{aligned} \tag{3.11}$$

Now, we justify the inequality (3.1) by using (3.10) and (3.11) with $\eta_1 = \frac{2}{15}$ and $\eta_2 = \frac{2}{7}$, we have that

$$\begin{aligned}
&G(J_1u, J_2w, J_3x) = \frac{2}{15}G(u, w, x) \le \frac{2}{15}G(u, w, x)\\
&+ \frac{2}{7}\left(\frac{(|13u - 4|(1 + i)) \cdot (|13w - 4|(1 + i)) \cdot (|13x - 4|(1 + i))}{3375 + 30\left(\max\{|13w - 4|, 2|w - x|, |2x - 4 + 15x|\}(1 + i)\right) \cdot (|w - x|(1 + i))}\right)\\
&= \frac{2}{15}G(u, w, x) + \frac{2}{7}\left(\frac{G(u, J_1u, J_1u) \cdot G(w, J_2w, J_2w) \cdot G(x, J_3x, J_3x)}{1 + G(w, J_2w, J_3x) \cdot G(J_2w, J_3x, J_3x)}\right).
\end{aligned}$$

Hence, all the hypothesis of Theorem 3.1 are satisfied with $\eta_1 = \frac{2}{15}$ and $\eta_2 = \frac{2}{7}$, and $(\eta_1 + \eta_2) = \frac{2}{15} + \frac{2}{7} = \frac{44}{105} < \frac{1}{2}$. The three self-mappings $J_1,\ J_2$, and J_3 have a unique CFP, that is,

$$J_1(4/13) = J_2(4/13) = J_3(4/13) = \frac{2(4/13)}{15} + \frac{4}{15} = \frac{4}{13} \in V = [0, 1].$$

Theorem 3.4. *Let (V, G) be a complete CVG_b-metric space with coefficient $b > 1$ and $J_1, J_2, J_3 : V \to V$ be mappings satisfying:*

$$G(J_1u, J_2w, J_3x) \le \eta_1 G(u, w, x) + \eta_2\left(\begin{array}{c}\left(\frac{G(u, w, x) \cdot G(u, J_1u, J_1u)}{(1 + G(u, w, w))}\right)\\ \cdot\left(\frac{G(w, J_2w, J_2w) \cdot G(x, J_3x, J_3x)}{(1 + G(J_1u, x, x)) \cdot (1 + G(J_2w, J_3x, J_3x))}\right)\end{array}\right), \tag{3.12}$$

for all $u, w, x \in V$, $\eta_1, \eta_2 \in [0, \frac{1}{3})$, such that $\eta_1 + \eta_2 < \frac{1}{3}$, then J_1, J_2 and J_3 have a unique CFP in V.

Proof. Let $u_0 \in V$ br the arbitrary point. Let the iterative sequences $\{u_n\}_{n\ge0}$ in V be defined by

$$u_{3m+1} = J_1u_{3m}, \quad u_{3m+2} = J_2u_{3m+1}, \quad \text{and} \quad u_{3m+3} = J_3u_{3m+2} \quad \forall n \ge 0. \tag{3.13}$$

Now by view of (3.12), we have

$$\begin{aligned}
G(u_{3m+1}, u_{3m+2}, u_{3m+3}) &= G(J_1u_{3m}, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(u_{3m}, u_{3m+1}, u_{3m+2}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m}, u_{3m+1}, u_{3m+2}) \cdot G(u_{3m}, J_1u_{3m}, J_1u_{3m})}{(1 + G(u_{3m}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{(1 + G(J_1u_{3m}, u_{3m+2}, u_{3m+2})) \cdot (1 + G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2}))} \right) \end{array} \right) \\
&= \eta_1 G(u_{3m}, u_{3m+1}, u_{3m+2}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m}, u_{3m+1}, u_{3m+2}) \cdot G(u_{3m}, u_{3m+1}, u_{3m+1})}{(1 + G(u_{3m}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{(1 + G(u_{3m+1}, u_{3m+2}, u_{3m+2})) \cdot (1 + G(u_{3m+2}, u_{3m+3}, u_{3m+3}))} \right) \end{array} \right).
\end{aligned}$$

This implies that,

$$\begin{aligned}
|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| &\leq \eta_1 |G(u_{3m}, u_{3m+1}, u_{3m+2})| \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(u_{3m}, u_{3m+1}, u_{3m+2})| \cdot |G(u_{3m}, u_{3m+1}, u_{3m+1})|}{(|1 + G(u_{3m}, u_{3m+1}, u_{3m+1})|)} \right) \\ \cdot \left(\dfrac{|G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{(|1 + G(u_{3m+1}, u_{3m+2}, u_{3m+2})|) \cdot (|1 + G(u_{3m+2}, u_{3m+3}, u_{3m+3})|)} \right) \end{array} \right).
\end{aligned} \tag{3.14}$$

The rational terms,

$$\frac{|G(u_{3m}, u_{3m+1}, u_{3m+2})| \cdot |G(u_{3m}, u_{3m+1}, u_{3m+1})|}{(|1 + G(u_{3m}, u_{3m+1}, u_{3m+1})|)} \leq |G(u_{3m}, u_{3m+1}, u_{3m+2})|,$$

and

$$\frac{|G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{(|1 + G(u_{3m+1}, u_{3m+2}, u_{3m+2})|) \cdot (|1 + G(u_{3m+2}, u_{3m+3}, u_{3m+3})|)} \leq 1.$$

Then, after simplification (3.14), we get that

$$|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \leq a_1 |G(u_{3m}, u_{3m+1}, u_{3m+2})|, \quad \text{where } a_1 = (\eta_1 + \eta_2) < \frac{1}{3}. \tag{3.15}$$

Again by view of (3.12) and by using the symmetric property of (V, G), we have that

$$\begin{aligned}
G(u_{3m+2}, u_{3m+3}, u_{3m+4}) &= G(J_1u_{3m+3}, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(u_{3m+3}, u_{3m+1}, u_{3m+2}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m+3}, u_{3m+1}, u_{3m+2}) \cdot G(u_{3m+3}, J_1u_{3m+3}, J_1u_{3m+3})}{(1 + G(u_{3m+3}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{(1 + G(J_1u_{3m+3}, u_{3m+2}, u_{3m+2})) \cdot (1 + G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2}))} \right) \end{array} \right) \\
&= \eta_1 G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m+1}, u_{3m+2}, u_{3m+3}) \cdot G(u_{3m+3}, u_{3m+4}, u_{3m+4})}{(1 + G(u_{3m+3}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{(1 + G(u_{3m+4}, u_{3m+2}, u_{3m+2})) \cdot (1 + G(u_{3m+2}, u_{3m+3}, u_{3m+3}))} \right) \end{array} \right).
\end{aligned}$$

This implies that,

$$|G(u_{3m+2},u_{3m+3},u_{3m+4})| \leq \eta_1 |G(u_{3m+1},u_{3m+2},u_{3m+3})| + \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(u_{3m+1},u_{3m+2},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+3},u_{3m+2},u_{3m+1})|} \right) \\ \cdot \left(\dfrac{|G(u_{3m+1},u_{3m+2},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+4},u_{3m+3},u_{3m+2})| \cdot |1+G(u_{3m+2},u_{3m+3},u_{3m+1})|} \right) \end{array} \right). \tag{3.16}$$

The rational terms,

$$\frac{|G(u_{3m+1},u_{3m+2},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+3},u_{3m+2},u_{3m+1})|} \leq |G(u_{3m+2},u_{3m+3},u_{3m+4})|,$$

and

$$\frac{|G(u_{3m+1},u_{3m+2},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+4},u_{3m+3},u_{3m+2})| \cdot |1+G(u_{3m+2},u_{3m+3},u_{3m+1})|} \leq 1.$$

Then, after simplification (3.16), we get that

$$|G(u_{3m+2},u_{3m+3},u_{3m+4})| \leq a_2 |G(u_{3m+1},u_{3m+2},u_{3m+3})|, \quad \text{where } a_2 = \frac{\eta_1}{1-\eta_2} < \frac{1}{3}. \tag{3.17}$$

Again by the view of (3.12) and by using the symmetric property of (V, G), we have that

$$\begin{aligned} G(u_{3m+3},u_{3m+4},u_{3m+5}) &= G(J_1 u_{3m+3}, J_2 u_{3m+4}, J_3 u_{3m+2}) \leq \eta_1 G(u_{3m+3},u_{3m+4},u_{3m+2}) \\ &+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m+3},u_{3m+4},u_{3m+2}) \cdot G(u_{3m+3}, J_1 u_{3m+3}, J_1 u_{3m+3})}{(1+G(u_{3m+3},u_{3m+4},u_{3m+4}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+4}, J_2 u_{3m+4}, J_2 u_{3m+4}) \cdot G(u_{3m+2}, J_3 u_{3m+2}, J_3 u_{3m+2})}{(1+G(J_1 u_{3m+3},u_{3m+2},u_{3m+2})) \cdot (1+G(J_2 u_{3m+4}, J_3 u_{3m+2}, J_3 u_{3m+2}))} \right) \end{array} \right) \\ &= \eta_1 G(u_{3m+3},u_{3m+4},u_{3m+2}) \\ &+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m+3},u_{3m+4},u_{3m+2}) \cdot G(u_{3m+3},u_{3m+4},u_{3m+4})}{(1+G(u_{3m+3},u_{3m+4},u_{3m+4}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+4},u_{3m+5},u_{3m+5}) \cdot G(u_{3m+2},u_{3m+3},u_{3m+3})}{(1+G(u_{3m+4},u_{3m+2},u_{3m+2})) \cdot (1+G(u_{3m+5},u_{3m+3},u_{3m+3}))} \right) \end{array} \right). \end{aligned}$$

This implies that,

$$|G(u_{3m+3},u_{3m+4},u_{3m+5})| \leq \eta_1 |G(u_{3m+2},u_{3m+3},u_{3m+4})| + \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(u_{3m+2},u_{3m+3},u_{3m+4})| \cdot |G(u_{3m+3},u_{3m+4},u_{3m+4})|}{|1+G(u_{3m+3},u_{3m+4},u_{3m+4})|} \right) \\ \cdot \left(\dfrac{|G(u_{3m+4},u_{3m+5},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+4},u_{3m+3},u_{3m+2})| \cdot |1+G(u_{3m+5},u_{3m+4},u_{3m+3})|} \right) \end{array} \right). \tag{3.18}$$

The rational terms,

$$\frac{|G(u_{3m+2},u_{3m+3},u_{3m+4})| \cdot |G(u_{3m+3},u_{3m+4},u_{3m+4})|}{|1+G(u_{3m+3},u_{3m+4},u_{3m+4})|} \leq |G(u_{3m+2},u_{3m+3},u_{3m+4})|,$$

and

$$\frac{|G(u_{3m+4},u_{3m+5},u_{3m+3})| \cdot |G(u_{3m+2},u_{3m+3},u_{3m+4})|}{|1+G(u_{3m+4},u_{3m+3},u_{3m+2})| \cdot |1+G(u_{3m+5},u_{3m+4},u_{3m+3})|} \leq 1.$$

Then, after simplification (3.18), we get that

$$|G(u_{3m+3}, u_{3m+4}, u_{3m+5})| \leq a_1 |G(u_{3m+2}, u_{3m+3}, u_{3m+4})|, \quad \text{where } a_1 = (\eta_1 + \eta_2) < \frac{1}{3}. \tag{3.19}$$

Let us define $a := \max\{a_1, a_2\} < \frac{1}{3}$. Now from (3.15), (3.17), (3.19), and by induction, we have

$$\begin{aligned}
&|G(u_{3m+2}, u_{3m+3}, u_{3m+4})| \leq a|G(u_{3m+1}, u_{3m+2}, u_{3m+3})| \\
&\leq a^2|G(u_{3m}, u_{3m+1}, u_{3m+2})| \leq \cdots \leq a^{3m+2}|G(u_0, u, w)| \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

This shows that the sequence $\{u_m\}$ is contractive in CVG_b-metric space (V, G). Let $m, j \in \mathbb{N}$ and $j > m$, then we have

$$\begin{aligned}
|G(u_m, u_j, u_j)| &\leq b\,|G(u_m, u_{m+1}, u_{m+1})| + b\,|G(u_{m+1}, u_j, u_j)| \\
&\leq b\,|G(u_m, u_{m+1}, u_{m+1})| + b^2\,|G(u_{m+1}, u_{m+2}, u_{m+2})| + \cdots + b^{j-m}\,|G(u_{j-1}, u_j, u_j)| \\
&\leq ba^m\,|G(u_0, u, u)| + b^2 a^{m+1}\,|G(u_0, u, u)| + \cdots + b^{j-m} a^{m-1}\,|G(u_0, u, u)| \\
&\leq [ba^m + b^2 a^{m+1} + \cdots + b^{j-m} a^{j-1}]\,|G(u_0, u, u)| \\
&= [ba^m + b^2 a^{m+1} + \cdots + b^{j-m} a^{j-1}]\,|G(u_0, u, u)| \\
&= ba^m[1 + ba + b^2a^2 + \cdots + b^{j-(m+1)} a^{j-(m+1)}]\,|G(u_0, u, u)| \\
&= ba^m \sum_{t=0}^{j-(m+1)} b^t a^t\,|G(u_0, u, u)| \leq ba^m \sum_{t=0}^{\infty} b^t a^t\,|G(u_0, u, u)| \\
&= \frac{ba^m}{1 - ba}\,|G(u_0, u, u)| \to 0, \quad \text{as } m \to \infty.
\end{aligned}$$

By using Proposition 2.4 (i), we have, $|G(u_m, u_j, u_k)| \leq b\,(|G(u_m, u_j, u_j)| + |G(u_m, u_k, u_k)|)$ for $m, j, k \in \mathbb{N}$ with $m > j > k$. By using limits $m, j, k \to \infty$, we obtain $|G(u_m, u_j, u_k)| \to 0$. This implies that, $\{u_m\}$ is a CVG_b-Cauchy sequence. Since, V is complete CVG_b-metric space, $\exists\, \rho \in V$ such that, $u_m \to \rho$, as $m \to \infty$, or $\lim_{m\to\infty} u_m = \rho$. We have to show that $J_1\rho = \rho$, by contrary case, let $J_1\rho \neq \rho$. Now from (3.12), we have

$$\begin{aligned}
G(J_1\rho, u_{3m+2}, u_{3m+3}) &= G(J_1\rho, J_2u_{3m+1}, J_3u_{3m+2}) \leq \eta_1 G(\rho, u_{3m+1}, u_{3m+2}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(\rho, u_{3m+1}, u_{3m+2}) \cdot G(\rho, J_1\rho, J_1\rho)}{(1 + G(\rho, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{(1 + G(J_1\rho, u_{3m+2}, u_{3m+2})) \cdot (1 + G(J_2u_{3m+1}, J_3u_{3m+2}, J_3u_{3m+2}))} \right) \end{array} \right) \\
&= \eta_1 G(\rho, u_{3m+1}, u_{3m+2}) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(\rho, u_{3m+1}, u_{3m+2}) \cdot G(\rho, J_1\rho, J_1\rho)}{(1 + G(\rho, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{(1 + G(J_1\rho, u_{3m+2}, u_{3m+2})) \cdot (1 + G(u_{3m+2}, u_{3m+3}, u_{3m+3}))} \right) \end{array} \right).
\end{aligned}$$

This implies that,

$$|G(J_1\rho, u_{3m+2}, u_{3m+3})| \leq \eta_1 |G(\rho, u_{3m+1}, u_{3m+2})| + \eta_2 \begin{pmatrix} \left(\dfrac{|G(\rho, u_{3m+1}, u_{3m+2})| \cdot |G(\rho, J_1\rho, J_1\rho)|}{(|1+G(\rho, u_{3m+1}, u_{3m+1})|)}\right) \\ \cdot \left(\dfrac{|G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{(|1+G(J_1\rho, u_{3m+2}, u_{3m+2})|) \cdot (|1+G(u_{3m+2}, u_{3m+3}, u_{3m+3})|)}\right) \end{pmatrix}.$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain

$$|G(J_1\rho, \rho, \rho)| \leq \eta_1 |G(\rho, \rho, \rho)| + \eta_2 \begin{pmatrix} \left(\dfrac{|G(\rho, \rho, \rho)| \cdot |G(\rho, J_1\rho, J_1\rho)|}{(|1+G(\rho, \rho, \rho)|)}\right) \\ \cdot \left(\dfrac{|G(\rho, \rho, \rho)| \cdot |G(\rho, \rho, \rho)|}{(|1+G(J_1\rho, \rho, \rho)|) \cdot (|1+G(\rho, \rho, \rho)|)}\right) \end{pmatrix}.$$

After simplification, we get that $|G(J_1\rho, \rho, \rho)| \leq 0$ is a contradiction. Hence,

$$J_1\rho = \rho. \tag{3.20}$$

Next, we have to show that $J_2\rho = \rho$, by contrary case, let $J_2\rho \neq \rho$. Now from (3.12), we have

$$\begin{aligned} G(u_{3m+1}, J_2\rho, u_{3m+3}) &= G(J_1u_{3m}, J_2\rho, J_3u_{3m+2}) \leq \eta_1 G(u_{3m}, \rho, u_{3m+2}) \\ &+ \eta_2 \begin{pmatrix} \left(\dfrac{G(u_{3m}, \rho, u_{3m+2}) \cdot G(u_{3m}, J_1u_{3m}, J_1u_{3m})}{(1+G(u_{3m}, \rho, \rho))}\right) \\ \cdot \left(\dfrac{G(\rho, J_2\rho, J_2\rho) \cdot G(u_{3m+2}, J_3u_{3m+2}, J_3u_{3m+2})}{(1+G(J_1u_{3m}, u_{3m+2}, u_{3m+2})) \cdot (1+G(J_2\rho, J_3u_{3m+2}, J_3u_{3m+2}))}\right) \end{pmatrix} \\ &= \eta_1 G(u_{3m}, \rho, u_{3m+2}) \\ &+ \eta_2 \begin{pmatrix} \left(\dfrac{G(u_{3m}, \rho, u_{3m+2}) \cdot G(u_{3m}, u_{3m+1}, u_{3m+1})}{(1+G(u_{3m}, \rho, \rho))}\right) \\ \cdot \left(\dfrac{G(\rho, J_2\rho, J_2\rho) \cdot G(u_{3m+2}, u_{3m+3}, u_{3m+3})}{(1+G(u_{3m+1}, u_{3m+2}, u_{3m+2})) \cdot (1+G(J_2\rho, u_{3m+3}, u_{3m+3}))}\right) \end{pmatrix}. \end{aligned}$$

This implies that,

$$|G(u_{3m+1}, J_2\rho, u_{3m+3})| \leq \eta_1 |G(u_{3m}, \rho, u_{3m+2})| + \eta_2 \begin{pmatrix} \left(\dfrac{|G(u_{3m}, \rho, u_{3m+2})| \cdot |G(u_{3m}, u_{3m+1}, u_{3m+1})|}{(|1+G(u_{3m}, \rho, \rho)|)}\right) \\ \cdot \left(\dfrac{|G(\rho, J_2\rho, J_2\rho)| \cdot |G(u_{3m+2}, u_{3m+3}, u_{3m+3})|}{(|1+G(u_{3m+1}, u_{3m+2}, u_{3m+2})|) \cdot (|1+G(J_2\rho, u_{3m+3}, u_{3m+3})|)}\right) \end{pmatrix}.$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain

$$|G(\rho, J_2\rho, \rho)| \leq \eta_1 |G(\rho, \rho, \rho)| + \eta_2 \begin{pmatrix} \left(\dfrac{|G(\rho, \rho, \rho)| \cdot |G(\rho, \rho, \rho)|}{(|1+G(\rho, \rho, \rho)|)}\right) \\ \cdot \left(\dfrac{|G(\rho, J_2\rho, J_2\rho)| \cdot |G(\rho, \rho, \rho)|}{(|1+G(\rho, \rho, \rho)|) \cdot (|1+G(J_2\rho, \rho, \rho)|)}\right) \end{pmatrix}.$$

After simplification, we obtain $|G(\rho, J_2\rho, \rho)| \leq 0$, which is a contradiction. Hence,

$$J_2\rho = \rho. \tag{3.21}$$

Now we shall show that $J_3\rho = \rho$, let by contrary case if, $J_3\rho \neq \rho$. Then from (3.12), we have

$$\begin{aligned}
G(u_{3m+1}, u_{3m+2}, J_3\rho) &= G(J_1u_{3m}, J_2u_{3m+1}, J_3\rho) \leq \eta_1 G(u_{3m}, u_{3m+1}, \rho) \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m}, u_{3m+1}, \rho) \cdot G(u_{3m}, J_1u_{3m}, J_1u_{3m})}{(1 + G(u_{3m}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, J_2u_{3m+1}, J_2u_{3m+1}) \cdot G(\rho, J_3\rho, J_3\rho)}{(1 + G(J_1u_{3m}, \rho, \rho)) \cdot (1 + G(J_2u_{3m+1}, J_3\rho, J_3\rho))} \right) \end{array} \right) \\
&= \eta_1 G(u_{3m}, u_{3m+1}, \rho) + \eta_2 \left(\begin{array}{c} \left(\dfrac{G(u_{3m}, u_{3m+1}, \rho) \cdot G(u_{3m}, u_{3m+1}, u_{3m+1})}{(1 + G(u_{3m}, u_{3m+1}, u_{3m+1}))} \right) \\ \cdot \left(\dfrac{G(u_{3m+1}, u_{3m+2}, u_{3m+2}) \cdot G(\rho, J_3\rho, J_3\rho)}{(1 + G(u_{3m+1}, \rho, \rho)) \cdot (1 + G(u_{3m+2}, J_3\rho, J_3\rho))} \right) \end{array} \right).
\end{aligned}$$

This implies that,

$$\begin{aligned}
|G(u_{3m+1}, u_{3m+2}, J_3\rho)| &\leq \eta_1 |G(u_{3m}, u_{3m+1}, \rho)| \\
&+ \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(u_{3m}, u_{3m+1}, \rho)| \cdot |G(u_{3m}, u_{3m+1}, u_{3m+1})|}{(|1 + G(u_{3m}, u_{3m+1}, u_{3m+1})|)} \right) \\ \cdot \left(\dfrac{|G(u_{3m+1}, u_{3m+2}, u_{3m+2})| \cdot |G(\rho, J_3\rho, J_3\rho)|}{(|1 + G(u_{3m+1}, \rho, \rho)|) \cdot (|1 + G(u_{3m+2}, J_3\rho, J_3\rho)|)} \right) \end{array} \right).
\end{aligned}$$

Now, by applying limit $m \to \infty$ on the above inequality, we obtain

$$|G(\rho, \rho, J_3\rho)| \leq \eta_1 |G(\rho, \rho, \rho)| + \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(\rho, \rho, \rho)| \cdot |G(\rho, \rho, \rho)|}{(|1 + G(\rho, \rho, \rho)|)} \right) \\ \cdot \left(\dfrac{|G(\rho, \rho, \rho)| \cdot |G(\rho, J_3\rho, J_3\rho)|}{(|1 + G(\rho, \rho, \rho)|) \cdot (|1 + G(\rho, J_3\rho, J_3\rho)|)} \right) \end{array} \right).$$

After simplification, we get that $|G(\rho, \rho, \rho)| \leq 0$ is a contradiction. Hence,

$$J_3\rho = \rho. \tag{3.22}$$

From (3.20), (3.21), and (3.22), we get that ρ is a CFP of J_1, J_2 and J_3 i.e.,

$$J_1\rho = J_2\rho = J_3\rho = \rho.$$

Uniqueness: Assume that $\rho^* \in V$ is an other CFP of J_1, J_2, and J_3, so that

$$J_1\rho^* = J_2\rho^* = J_3\rho^* = \rho^* \quad \text{and} \quad J_1\rho = J_2\rho = J_3\rho = \rho.$$

Then, from (3.12), we have that

$$G(\rho,\rho^*,\rho^*) = G(J_1\rho, J_2\rho^*, J_3\rho^*) \leq \eta_1 G(\rho,\rho^*,\rho^*) + \eta_2 \left(\begin{array}{c} \left(\dfrac{G(\rho,\rho^*,\rho^*)\cdot G(\rho,J_1\rho,J_1\rho)}{(1+G(\rho,\rho^*,\rho^*))}\right) \\ \cdot\left(\dfrac{G(\rho^*,J_2\rho^*,J_2\rho^*)\cdot G(\rho^*,J_3\rho^*,J_3\rho^*)}{(1+G(J_1\rho,\rho^*,\rho^*))\cdot(1+G(J_2\rho^*,J_3\rho^*,J_3\rho^*))}\right) \end{array} \right).$$

This implies that,

$$|G(\rho,\rho^*,\rho^*)| \leq \eta_1 |G(\rho,\rho^*,\rho^*)| + \eta_2 \left(\begin{array}{c} \left(\dfrac{|G(\rho,\rho^*,\rho^*)|\cdot|G(\rho,\rho,\rho)|}{(|1+G(\rho,\rho^*,\rho^*)|)}\right) \\ \cdot\left(\dfrac{|G(\rho^*,\rho^*,\rho^*)|\cdot|G(\rho^*,\rho^*,\rho^*)|}{(|1+G(\rho,\rho^*,\rho^*)|)\cdot(|1+G(\rho^*,\rho^*,\rho^*)|)}\right) \end{array} \right).$$

After simplification, we get $|G(\rho,\rho^*,\rho^*)| = 0$, implies that $\rho = \rho^*$. Hence proved that J_1, J_2, and J_3 have a unique CFP in V.

By reducing the rational term in Theorem 3.4, we can get the following two corollaries.

Corollary 3.5. *Let (V,G) be a complete CVG_b-metric space with constant $b > 1$ and $J_1, J_2, J_3 : V \to V$ be mappings satisfying:*

$$G(J_1u, J_2w, J_3x) \leq \eta_1 G(u,w,x) + \eta_2 \left(\frac{G(u,w,x)\cdot G(u,J_1u,J_1u)\cdot G(w,J_2w,J_2w)}{(1+G(u,w,w))\cdot(1+G(J_1u,x,x))}\right), \quad (3.23)$$

for all $u,w,x \in V$, $\eta_1,\eta_2 \in [0,\frac{1}{3})$, such that $\eta_1+\eta_2 < \frac{1}{3}$, then J_1, J_2 and J_3 have a unique CFP in V.

Corollary 3.6. *Let (V,G) be a complete CVG_b-metric space with constant $b > 1$ and $J_1, J_2, J_3 : V \to V$ be mappings satisfying:*

$$G(J_1u, J_2w, J_3x) \leq \eta_1 G(u,w,x) + \eta_2 \left(\frac{G(u,w,x)\cdot G(u,J_1u,J_1u)\cdot G(x,J_3x,J_3x)}{(1+G(u,w,w))\cdot(1+G(J_2w,J_3x,J_3x))}\right), \quad (3.24)$$

for all $u,w,x \in V$, $\eta_1,\eta_2 \in [0,\frac{1}{3})$, such that $\eta_1+\eta_2 < \frac{1}{3}$, then J_1, J_2 and J_3 have a unique CFP in V.

Example 3.7. Let (V,G) be a CVG_b-metric space, where $V = [0,1]$ and $G : V^3 \to \mathbb{C}$ with $G(u,w,x) = \left(\frac{4}{9}(|u-w|+|w-x|+|x-u|)\right)^2(1+i)$, for all $u,w,x \in V$. Now we define $J_1, J_2, J_3 : V \to V$ as

$$J_1v = J_2v = J_3v = \frac{v}{7}.$$

Notice that,

$$\left\{ \begin{array}{c} |G(u,w,x)|, \left(\dfrac{|G(u,w,x)|\cdot|G(u,J_1u,J_1u)|}{(|1+G(u,w,w)|)}\right) \\ \cdot\left(\dfrac{|G(w,J_2w,J_2w)|\cdot|G(x,J_3x,J_3x)|}{(|1+G(J_1u,x,x)|)\cdot(|1+G(J_2w,J_3x,J_3x)|)}\right) \end{array} \right\} \geq 0.$$

In all regards, it is enough to show that $G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x)$, for all $u, w, x \in [0, 1]$ and $\eta_1, \eta_2 \in [0, \frac{1}{3})$, we have

$$
\begin{aligned}
G(J_1u, J_2w, J_3x) &= \left(\frac{4}{9}(|J_1u - J_2w| + |J_2w - J_3x| + |J_3x - J_1u|\right)^2 (1 + i) \\
&= \frac{1}{49}\left(\frac{4|u - w|}{9} + \frac{4|w - x|}{9} + \frac{4|x - u|}{9}\right)^2 (1 + i).
\end{aligned} \tag{3.25}
$$

And,

$$
G(u, w, x) = \left(\frac{4|u - w|}{9} + \frac{4|w - x|}{9} + \frac{4|x - u|}{9}\right)^2 (1 + i). \tag{3.26}
$$

Now, we present different cases with $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$ and $b = 2$, implies that $(\eta_1 + \eta_2) = \frac{1}{10} + \frac{1}{20} = \frac{3}{20} < \frac{1}{3}$.

Case-1. Let $u = 0, w = 0, x = 0$, then from (3.25) and (3.26), directly we get that $G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x)$. Hence, (3.12) is satisfied with $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$, and $b = 2$.

Case-2. Let $u = 0, w = 0, x = \frac{1}{2}$, then from (3.25) and (3.26), we find $G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x)$, is satisfy with $\eta_1 = \frac{1}{10}$, i.e,

$$
\begin{aligned}
&\frac{1}{49}\left(\frac{4|0 - 0|}{9} + \frac{4|0 - \frac{1}{2}|}{9} + \frac{4|\frac{1}{2} - 0|}{9}\right)^2 (1 + i) \\
&\qquad \leq \eta_1 \left(\frac{4|0 - 0|}{9} + \frac{4|0 - \frac{1}{2}|}{9} + \frac{4|\frac{1}{2} - 0|}{9}\right)^2 (1 + i) \\
&\Rightarrow \frac{16}{3969}(1 + i) \leq \frac{16}{810}(1 + i).
\end{aligned}
$$

Hence, (3.12) is satisfied with $\eta_1 = \frac{1}{10}$, $\eta_2 = \frac{1}{20}$ and $b = 2$.

Case-3. Let Let $u = \frac{1}{2}, w = \frac{1}{3}, x = \frac{1}{5}$, then from (3.25) and (3.26), we find $G(J_1u, J_2w, J_3x) \leq \eta_1 G(u, w, x)$, is satisfy with $\eta_1 = \frac{1}{10}$, i.e,

$$
\begin{aligned}
&\frac{1}{49}\left(\frac{4\left|\frac{1}{2} - \frac{1}{3}\right|}{9} + \frac{4\left|\frac{1}{3} - \frac{1}{5}\right|}{9} + \frac{4\left|\frac{1}{5} - \frac{1}{2}\right|}{9}\right)^2 (1 + i) \\
&\qquad \leq \eta_1 \left(\frac{4\left|\frac{1}{2} - \frac{1}{3}\right|}{9} + \frac{4\left|\frac{1}{3} - \frac{1}{5}\right|}{9} + \frac{4\left|\frac{1}{5} - \frac{1}{2}\right|}{9}\right)^2 (1 + i) \\
&\Rightarrow \frac{16}{11025}(1 + i) \leq \frac{16}{2250}(1 + i).
\end{aligned}
$$

Hence, (3.12) is satisfied with $\eta_1 = \frac{1}{10}, \eta_2 = \frac{1}{20}$ and $b = 2$.

Case-4. Let $u = \frac{1}{2}, w = 1, x = 1$, then from (3.25) and (3.26), we find $G(J_1 u, J_2 w, J_3 x) \leq \eta_1 G(u, w, x)$, is satisfy with $\eta_1 = \frac{1}{10}$, i.e,

$$\frac{1}{49}\left(\frac{4\left|\frac{1}{2}-1\right|}{9}+\frac{4\left|1-1\right|}{9}+\frac{4\left|1-\frac{1}{2}\right|}{9}\right)^2(1+i)$$
$$\leq \eta_1\left(\frac{4\left|\frac{1}{2}-1\right|}{9}+\frac{4\left|1-1\right|}{9}+\frac{4\left|1-\frac{1}{2}\right|}{9}\right)^2(1+i)$$
$$\Rightarrow \frac{16}{3969}(1+i) \leq \frac{16}{810}(1+i).$$

Hence, inequality (3.12) is satisfied with $\eta_1 = \frac{1}{10}, \eta_2 = \frac{1}{20}$ and $b = 2$. Thus all the conditions of Theorem 3.4 are satisfied with noticing that the point $0 \in V$, which remains fixed under mappings J_1, J_2 and J_3, is indeed unique.

4. Applications

In this section, we establish an application of the NLIEs to support our main work. Let $V = C([k_1, k_2], \mathbb{R})$ be the set of all real-valued continuous functions defined on $[k_1, k_2]$. Now we state and prove a result based on the three NLIEs to get the existing result of a common solution to uplift our work. Consider the UTIEs are;

$$\begin{aligned} u(q) &= \int_{k_1}^{k_2} Q_1(q, r, u(r))dr + k_1(q), \\ w(q) &= \int_{k_1}^{k_2} Q_2(q, r, w(r))dr + k_2(q), \\ x(q) &= \int_{k_1}^{k_2} Q_3(q, r, x(r))dr + k_3(q), \end{aligned} \tag{4.1}$$

where $q, r \in [k_1, k_2],\ u, w, x, \hbar_i \in V$, where $i = 1, 2, 3,$ and $V = C([k_1, k_2], \mathbb{R})$ be set of all $\mathbb{R}-$valued continuous functions based on $[k_1, k_2]$ and $Q_i : [k_1, k_2]^2 \times \mathbb{R} \to \mathbb{R}$ where $i = 1, 2, 3$.

Theorem 4.1. *Let $V = C([k_1, k_2], \mathbb{R})$, where $[k_1, k_2] \subseteq \mathbb{R}$ and a CVG_b-metric $G : V^3 \to \mathbb{C}$ is defined as,*

$$G(u, w, x) = \Big(\ ||u(q) - w(q)|| + ||w(q) - x(q)|| + ||x(q) - u(q)||\ \Big)\sqrt{1 + k_1^2}e^{i\cot k_1}, \tag{4.2}$$

for all $u, w, x \in V$ and $q \in [k_1, k_2]$. Let $Q_1, Q_2, Q_3 : [k_1, k_2] \times [k_1, k_2] \times \mathbb{R} \to \mathbb{R}$ be the $\mathbb{R}-$valued functions. We note that $F_u, F_w, F_x \in V$ such that

$$\begin{aligned} F_u(q) &= \int_{k_1}^{k_2} Q_1(q, r, u(r))dr, \\ F_w(q) &= \int_{k_1}^{k_2} Q_2(q, r, w(r))dr, \\ F_x(q) &= \int_{k_1}^{k_2} Q_3(q, r, x(r))dr. \end{aligned} \tag{4.3}$$

If there exists $\mu \in (0,1)$ *such that, for all* $u, w, x \in V$,

$$\left(\begin{array}{c} ||F_u(q) + k_1(q) - F_w(q) - k_2(q)|| \\ +||F_w(q) + k_2(q) - F_x(q) - k_3(q)|| \\ +||F_x(q) + k_3(q) - F_u(q) - k_1(q)|| \end{array} \right) \sqrt{1+k_1^2} e^{i \cot k_1} \leq \mu \mathbf{M} \left(\mathbb{F}_{(u,w,x)}, u, w, x \right), \tag{4.4}$$

where $\mathbb{F}_{(u,w,x)} = F_u, F_w, F_x$ *and*

$$\mathbf{M} \left(\mathbb{F}_{(u,w,x)}, u, w, x \right) = \max \left\{ A_1 \left(\mathbb{F}_{(u,w,x)}, u, w, x \right), A_2 \left(\mathbb{F}_{(u,w,x)}, u, w, x \right) \right\}, \tag{4.5}$$

with

$$A_1 \left(\mathbb{F}_{(u,w,x)}, u, w, x \right) = \Big(\ ||u-w|| + ||w-x|| + ||x-u|| \ \Big) \sqrt{1+k_1^2} e^{i \cot k_1},$$

and

$$\begin{aligned} & A_2 \left(\mathbb{F}_{(u,w,x)}, u, w, x \right) \\ &= \frac{8 \Big(\ \|F_u + k_1 - u\| \cdot \|F_w + k_2 - w\| \cdot \|F_x + k_3 - x\| \ \Big) \left(\sqrt{1+k_1^2} e^{i \cot k_1} \right)}{\left(\sqrt{1+k_1^2} e^{i \cot k_1} \right)^{-2} + 2 \left(\begin{array}{c} \|F_w + k_2 - w\| \\ +\|F_w + k_2 - F_x - k_3\| \\ +\|F_x + k_3 - w\| \end{array} \right) \cdot (\|F_w + k_2 - F_x - k_3\|)}. \end{aligned}$$

Then the three UTIEs i.e., (4.1) have a unique common solution.

Proof. Define $J_1, J_2, J_3 : V \to V$ as

$$\begin{aligned} J_1 u &= J_1 u(q) = F_u(q) + k_1(q) = F_u + k_1, \quad u(q) = u, \\ J_2 w &= J_2 w(q) = F_w(q) + k_2(q) = F_w + k_2, \quad w(q) = w, \\ J_3 x &= J_3 x(q) = F_x(q) + k_3(q) = F_x + k_3, \quad x(q) = x. \end{aligned} \tag{4.6}$$

Then, the following two cases occur;

i)- If $A_1 \left(\mathbb{F}_{(u,w,x)}, u, w, x \right)$ is the maximum term in (4.5), then from (4.2), (4.4), and (4.6), we have that

$$G(J_1 u, J_2 w, J_3 x) \leq \mu \Big(\ ||u-w|| + ||w-x|| + ||x-u|| \ \Big) \sqrt{1+k_1^2} e^{i \cot k_1} = \mu G(u, w, x),$$

for all $u, w, x \in V$. Thus, the maps J_1, J_2 and J_3 satisfy all the hypothesis of Theorem 3.2 with $\mu = \eta_1$ and $\eta_2 = 0$ in (3.1). Then, the three UTIEs (4.1) have a unique common solution in V.

ii)- If $A_2\left(\mathbb{F}_{(u,w,x)},u,w,x\right)$ is the maximum term in (4.5), then from (4.2), (4.4), and (4.6), we have that

$$\begin{aligned}
&G(J_1u,J_2w,J_3x)\\
&\leq \mu\frac{8\left(\ \|F_u+k_1-u\|\cdot\|F_w+k_2-w\|\cdot\|F_x+k_3-x\|\ \right)\left(\sqrt{1+k_1^2}e^{i\cot k_1}\right)}{\left(\sqrt{1+k_1^2}e^{i\cot k_1}\right)^{-2}+2\begin{pmatrix}\|F_w+k_2-w\|\\+\|F_w+k_2-F_x-k_3\|\\+\|F_x+k_3-w\|\end{pmatrix}\cdot\left(\|F_w+k_2-F_x-k_3\|\right)}\\
&=\mu\left(\frac{G(u,J_1u,J_1u)\cdot G(w,J_2w,J_2w)\cdot G(x,J_3x,J_3x)}{1+G(w,J_2w,J_3x)\cdot G(J_2w,J_3x,J_3x)}\right),
\end{aligned}$$

for all $u,w,x\in V$. Thus, the maps $J_1,\ J_2$, and J_3 satisfy all the hypothesis of Theorem 3.2 with $\mu=\eta_1$ and $\eta_2=0$ in (3.1). Then, the three UTIEs (4.1) have a unique common solution in V.

5. Conclusion

We presented some new generalized product type rational contraction results in CVG_b-metric spaces for three self-mappings. We proved the uniqueness of a CFP by using the new generalized product type rational contraction conditions for single-valued mappings in CVG_b-metric spaces without continuity of self-mappings. In support of the results, we presented two illustrative examples in CVG_b-metric spaces for three self-mappings. Furthermore, we presented an application of the three UTIEs to get the existing result of a unique common solution to support our main work. In this direction, authors can contribute their ideas to the problems of FP, coincidence points, and CFP on CVG_b-metric spaces by using different types of contraction conditions without the continuity of single-valued mappings with the applications of different types of integral equations.

Authors' Contributions

All the authors contributed equally to this work. The authors have read and approved the final version of the manuscript.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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