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A COPULA GOVERNING SKEWED PROCESSES AND APPLICATION

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ABSTRACT. We study the evolution copula which governs a pair of processes one of which is conditioned by the exceeding of a peak. We start by clarifying the copula that links different values of the process along its evolution and after we make clear how to treat the conditioning to any event that involves extreme values of some regular processes. As a prototype and using classical measures of asymmetry, we focus on the Wiener process and emphasize skewness produced by the conditioning operation.

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1. Introduction

Dependence between walks of a given process was recently explored by many researchers, among others, we cite for instance the recent papers [10] and [21]. The most popular process in stochastic literature is Wiener's one since it has large applications in finance and market modeling. The idea of this work is to study the dependence of a price evolution, once when at a previous time, the price has exceeded some fixed prior level. Precisely, if $(W_t)_{t\geq 0}$ denotes a given process, we expect to understand the behavior of the process $W_t/M_t \geq a$, where $a \in \mathbb{R}$ and $M_t = \sup_{0\leq s\leq t} W_s$. By a simple change of variable and taking in account the symmetry of W_t law, one may without further efforts, determine the behavior and the evolution of W_t/m_t where $m_t = \inf_{0\leq s\leq t} W_s$. The definition of Wiener process is rewritten, as a preliminary at the first subsection (2.1).

The aim of the current paper is to describe the aspect of dependence, on one hand, between any pair $(W_t, W_s),\ t, s \in \mathbb{R}$, of the Brownian motion and on the other, but more important one, how the conditioned processes (W_t, M_t) and (W_t, m_t) evolve for large values of t. The main tool used here is the copula which summarizes the dependence property between random variables and allows, in many

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important statistical cases, to rebuild the distribution of a given random vector with a well known copula and known margins. Let us give an overview on this latter notion.

A copula, is introduced as a tool for modeling the dependence structure between random variables. As usual, the unavoidable book of Nelson [12] will be our sure reference. Copulas make a link between a multivariate joint distribution and its uni-variate marginal ones via the famous Sklar's theorem [1] or [2]. This latter result gives a bridge between the joint distribution and its margins using a copula. It was indeed a turning point in the statistical analysis where problems of retrieval are evoked. We cite among others [22] or more recently [4].

The current paper is organized as follows: First we recall some definitions and results about Wiener processes and Brownian copulas that will be treated later in the article. We focus on the copula joining the Brownian motion at two arbitrary times hence a copula governing diffusion process. We estimate some concordance parameters mainly β of Blomqvist because of its simplicity. We recall some important results on asymmetry [23] and on classical parameters of concordance mainly Kendal's τ and ρ of Spearman. To take in account the asymmetry of data, some new asymmetric copulas were suggested to describe asymmetric phenomena like it was done in [5]. For more details on asymmetry and orders on copulas, we refer to [6]. In the third section, we clarify the construction of conditioned Brownian motion and give some of its properties. Finally at the fourth section, we study the obtained skewed process after conditioning. One should understand by *study* the determination of the copula that governs the dynamical system. As a prototype of motivation, one may think to fluctuation price behavior when it has exceeded some critical level. In this context, we give an application to theses results in section 6.

2. Preliminaries

2.1. **Wiener's process.** Consider a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A Wiener process $(W_t)_{t\geq 0}$ with natural filtration $\mathcal{F}_t = \sigma\left\{W_s^{-1}(A) \mid s \leq t, , A \in \mathcal{B}(\mathbb{R})\right\}$ where $\mathcal{B}(\mathbb{R})$ is the Borel σ-algebra on $\mathbb{R}$ is defined as follows:

Definition 1. *A one-dimensional Brownian motion is a real-valued process $W_t, t \geq 0$ that has the following properties:*

(a) If $t_0 < t_1 < \cdots < t_n$, then $W_{t_0}, W_{t_1} - W_{t_0}, \ldots, W_{t_n} - W_{t_{n-1}}$ are independent.

(b) If $s, t \geq 0$, then

$$P(W_{s+t} - W_s \in A) = \int_A (2\pi t)^{-1/2} \exp\left(-x^2/2t\right) dx$$

(c) With probability $1, t \to W_t$ is continuous.

We denote by

$$M_t = \sup_{0\leq s\leq t} W_s,$$

and

$$m_t = \inf_{0\leq s\leq t} W_s$$

respectively the associated running maxima and minima processes. Owing to the continuity of trajectories, theses quantities exist. The following results are well known

Proposition 1.

$$\mathbb{P}\left(M_t > a\right) = 2\mathbb{P}\left(W_t > a\right)$$

Proposition 2. *We have the following reflection principle:*

$$\mathbb{P}\left(W_t \leq a/M_t > b\right) = 2\mathbb{P}\left(W_t > 2b - a/M_t > a\right)$$

2.2. **Bivariate copula.**

Definition 2. *A copula C is a bifunction on $\mathbb{I}^2$ into $\mathbb{I}$ which satisfies the following conditions for all u, v, u_1, v_1, u_2, v_2 in $\mathbb{I}$*

(1) *Border conditions: $C(0, v) = C(u, 0) = 0$.*
(2) *Uniform margins: $C(1, v) = v$ and $C(u, 1) = u$.*
(3) *The $C-$volume property:*

$$V_C(R) = C(u_2, v_2) - C(u_2, v_1) - C(u_1, v_2) + C(u_1, v_1) \geq 0$$

for all rectangle $R = [u_1, u_2] \times [v_1, v_2] \subset \mathbb{I}^2$ with $u_1 < u_2$ and $v_1 < v_2$.

For convenience, we write $\mathcal{C}$ to denote the set of all bivariate copulas. Each element of $\mathcal{C}$ is framed between Fréchet-Hoeffding bounds W and M given by $W(u, v) = \max(u + v - 1, 0)$ and $M(u, v) = \min(u, v)$. Precisely, we have

$$\forall (x, y) \in \mathbb{I}^2 : \quad W(x, y) \leq C(x, y) \leq M(x, y).$$

These bounds (M and W) are also copulas but in higher dimensions, say for multivariate copulas, W is not a copula.
Statistically speaking, Fréchet-Hoeffding bounds M and W model respectively the co-monotonicity and counter monotonicity of empirical variables X and Y. The copula $\Pi : \ (x, y) \in \mathbb{I}^2 \mapsto xy$ characterizes the total independence between the two variables.

Theorem 1 (Sklar's theorem)**.** *Let H be two-dimensional distribution function on a probability space (Ω, p) with marginal distribution functions F and G. Then there exists a copula C such that*

$$\forall (x, y) \in \mathbb{R}^2 : \quad H(x, y) = C(F(x), G(y)). \tag{1}$$

If F and G are continuous then the copula C is unique.

The equality (1) Means simply

$$\forall (x,y) \in \mathbb{R}^2 : \quad \mathbb{P}\left(X \leq x, Y \leq y\right) = C(\mathbb{P}(X \leq x), \mathbb{P}(Y \leq y)).$$

Definition 3. *The survival copula of a given copula C is the function $\widehat{C}$ defined by the formula*

$$\forall u, v \in \mathbb{I} : \quad \widehat{C}(u,v) = u + v - 1 + C(1-u, 1-v)$$

The copula $\widehat{C}$ satisfies

$$\forall (x,y) \in \mathbb{R}^2 : \quad \mathbb{P}\left(X > x, Y > y\right) = \widehat{C}(\mathbb{P}(X > x), \mathbb{P}(Y > y)).$$

Proposition 3. *Let X and Y be continuous random variables. If f denotes a strictly increasing function. Then*

$$C_{f(X),f(Y)} = C_{X,Y}$$

For a pair of continuous random vectors (X_1, X_2) and (Y_1, Y_2), we denote by $\mathcal{Q}$ ([12, pages,159-182]) the difference of two probabilities

$$\mathcal{Q} = P((X_1 - X_2)(Y_1 - Y_2) > 0) - P((X_1 - X_2)(Y_1 - Y_2) < 0)$$

If the corresponding copulas are C_1 and C_2, then

$$\mathcal{Q}(C_1, C_2) = 4 \int_{\mathbb{I}^2} C_1(u,v)\, dC_2(u,v) - 1$$

The usual first three measures of concordance may be defined in terms of the concordance function $\mathcal{Q}$.

Kendall's tau of C is defined by

$$\tau(C) = \mathcal{Q}(C, C)$$

Spearman's rho by

$$\rho(C) = 3\mathcal{Q}(C, \Pi)$$

Gini's gamma by

$$\gamma(C) = \mathcal{Q}(C, M) + \mathcal{Q}(C, W).$$

Blomqvist's Beta by

$$\beta(C) = 4C\left(\frac{1}{2}, \frac{1}{2}\right) - 1.$$

2.3. **Tail dependence.** The following results are extracted from [12, page, 214]

Definition 4. *Let X and Y be continuous random variables with distribution functions F and G, respectively. The upper tail dependence parameter λ_U is the limit (if it exists)*

$$\lambda_U = \lim_{t\to 1^-} P\left(Y > G^{(-1)}(t)/X > F^{(-1)}(t)\right) \tag{2}$$

The lower tail dependence parameter λ_L is the limit (if it exists)

$$\lambda_L = \lim_{t\to 0^+} P\left(Y \leq G^{(-1)}(t)/X \leq F^{(-1)}(t)\right) \tag{3}$$

Proposition 4. *Let X and Y be continuous random variables with distribution functions F and G, respectively, and let C be the copula of X and Y. If the limits exist, then the upper and lower tail dependence are given by:*

$$\lambda_U = 2 - \lim_{t\to 1^-} \frac{1 - C(t,t)}{1-t}$$

and

$$\lambda_L = \lim_{t\to 0^+} \frac{C(t,t)}{t}.$$

3. Brownian Copula

We start by describing a Brownian copula $C_{s,t}$ which links the two processes W_s and W_t for two strictly positive reals s and t with $s < t$. Φ_t denotes the cumulative distribution function of W_t and ϕ_t its derivative and shortly $\Phi = \Phi_1$ and $\phi = \phi_1$.

$$\mathbb{P}(W_t \leq y/W_s = x) = \mathbb{P}(W_t - W_s \leq y - x/W_s = x) = \mathbb{P}(W_{t-s} \leq y - x/W_s = x).$$

Hence

$$\mathbb{P}(W_s \leq x, W_t \leq y) = \int_{-\infty}^{x} \mathbb{P}(W_{t-s} \leq y - z) d\Phi_s(z).$$

So

$$\mathbf{B}_{s,t}\left(\Phi_s(x), \Phi_t(y)\right) = \int_{-\infty}^{x} \Phi\left(\frac{y-z}{\sqrt{t-s}}\right) \mathrm{d}\Phi_s(z).$$

With the change of variables $u = \Phi_s(x), v = \Phi_t(y)$ and $w = \Phi_s(z)$, we obtain

$$\mathrm{B}_{s,t}\left(u,v\right) = \int_{0}^{u} \Phi\left(\frac{\sqrt{t}\Phi^{-1}\left(v\right) - \sqrt{s}\Phi^{-1}(w)}{\sqrt{t-s}}\right) \mathrm{d}w$$

It is exactly the Gaussian copula with correlation coeficient $\sqrt{\frac{s}{t}}$. When t is tending to infinity, or s is tending to 0, the Brownian copula is tending to the independence one.
The density of the Brownian copula is

$$b_{s,t}\left(u,v\right) = \sqrt{\frac{t}{t-s}}\phi\left(\frac{\sqrt{t}\Phi^{-1}\left(v\right) - \sqrt{s}\Phi^{-1}(u)}{\sqrt{t-s}}\right) \frac{1}{\phi\left(\Phi^{-1}\left(v\right)\right)}$$

4. Diffusion Process Copula

Proposition 5. *Let X_t be a diffusion process satisfying the following stochastic differential equation:*

$$dX_t = \mu X_t dt + \sigma X_t dW_t \text{ and } X_0 > 0.$$

The classical solution of such SPDE is given explicitly, using the well known Itô's formula by

$$X_t = X_0 exp\left(\sigma W_t + (\mu - \frac{\sigma^2}{2})t\right).$$

Proof. Since the mapping $w \mapsto X_0 exp\left(\sigma w + (\mu - \frac{\sigma^2}{2})t\right)$ is strictly increasing, the copula linking the two processes X_s and X_t is the Brownian copula $B_{s,t}$. □

5. A Parametrized Copula Depending on Time

5.1. **A skewed Brownian motion.** It is well known that conditioning a variable on a measurable subset from another random variable causes a skewness parameter [21]. Here we consider a strictly positive real $a > 0$ and denote shortly $\alpha = \alpha_{a,t}$ the probability $\mathbb{P}(M_t > a)$ which equals $2(1 - \Phi_t(a))$ since $\alpha = 2\mathbb{P}(W_t > a)$ and $\alpha_x = \alpha_{x,t}$ for any real $x > 0$.
We denote X_t the process $W_t/(M_t > a)$ obtained by conditioning the standard Brownian motion W_t by the measurable set $M_t > a$ and we aim to determine the distribution function $F_{a,t}$ of such process $X_{a,t}$. For any real number x satisfying $x \geq a$, we have

$$\begin{aligned}\mathbb{P}(X_{a,t} \leq x) &= \tfrac{1}{\alpha}\mathbb{P}\left(W_t \leq x, M_t > a\right)\\ &= \tfrac{1}{\alpha}\left[\mathbb{P}(W_t \leq x) - \mathbb{P}(W_t \leq x, M_t \leq a)\right]\\ &= \tfrac{1}{\alpha}\left[\mathbb{P}(W_t \leq x) - \mathbb{P}(M_t \leq a)\right]\\ &= \tfrac{1}{\alpha}\left[\Phi_t(x) - 1 + \alpha)\right]\end{aligned}$$

On the other hand if $x \leq a$

$$\begin{aligned}\mathbb{P}(X_{a,t} \leq x) &= \mathbb{P}\left(W_t \leq x/M_t > a\right)\\ &= \mathbb{P}(W_t > 2a - x/M_t > a)\\ &= \tfrac{1}{\alpha}\mathbb{P}(W_t > 2a - x, M_t > a)\\ &= \tfrac{1}{\alpha}\left[\mathbb{P}(W_t > 2a - x) - \mathbb{P}(W_t > 2a - x, M_t \leq a)\right]\\ &= \tfrac{1}{\alpha}\mathbb{P}(W_t > 2a - x)\\ &= \tfrac{1}{\alpha}\Phi_t(x - 2a)\end{aligned}$$

Finally the distribution function of the process $X_{a,t}$ is given by

$$F_{a,t}(x) = \begin{cases} \dfrac{\Phi_t(x - 2a)}{\alpha} \text{ if } x \leq a \\ \\ 1 - \dfrac{\Phi_t(-x)}{\alpha} \text{ elsewhere.} \end{cases}$$

Its density function $f_{a,t}$ is resulting as

$$f_{a,t}(x) = \begin{cases} \dfrac{\phi_t(x-2a)}{\alpha} \text{ if } x \leq a \\ \\ \dfrac{\phi_t(x)}{\alpha} \text{ elsewhere} \end{cases}$$

The representative curve of $f_{a,t}$ is symmetrical with respect to the vertical axis $x = a$. So graphically, one has immediately $\mathbb{E}(X_{a,t}) = a$. Let us confirm this result by a direct calculation:

$$\begin{array}{rcl} \alpha\mathbb{E}(X_{a,t}) & = & \int_{-\infty}^{a} x\phi_t(2a-x)\,dx + \int_{a}^{+\infty} x\phi_t(x)\,dx \\ & = & 2a\int_{-\infty}^{a}\phi_t(u)\,du \\ & = & 2a(1-\Phi_t(a)) \\ & = & \alpha a \end{array}$$

which confirms the intuitive geometrical remark above ensuring that $\mathbb{E}(X_t) = a$.

To calculate the variance, we start by the second moment $\mathbb{E}(X_{a,t}^2)$.

$$\begin{array}{rcl} \alpha\mathbb{E}(X_{a,t}^2) & = & \int_{-\infty}^{a} x^2\phi_t(2a-x)\,dx + \int_{a}^{+\infty} x^2\phi_t(x)\,dx \\ & = & \int_{a}^{+\infty}\left((2a-x)^2 + x^2\right)\phi_t(x)\,dx \\ & = & 4a^2\int_{a}^{+\infty}\phi_t(x)\,dx - 4a\int_{a}^{+\infty} x\phi_t(x)\,dx + 2\int_{a}^{+\infty} x^2\phi_t(x)\,dx \\ & = & (2a^2+t)\alpha - 2at\phi_t(a) \end{array}$$

Finally

$$V(X_{a,t}) = a^2 + t - \frac{2at}{\alpha}\phi_t(a)$$

5.2. **A time parametrized copula.** We are interested in this paragraph to search the copula $C_{a,t}$ linking the two processes M_t and $X_{a,t}$ which will be denoted shortly X_t in the sequel.

Let be x and y two real numbers. The joint distribution function H of M_t and X_t is given by:

$$\begin{array}{rcl} H(x,y) & = & \mathbb{P}(M_t \leq x, X_t \leq y) \\ & = & \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x, X_t \leq y) \\ & = & \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(X_t \leq y / M_t > x) \end{array}$$

- ▶ First case: If $x \leq a$.

$$\begin{array}{rcl} H(x,y) & = & \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(X_t \leq y / M_t > x) \\ & = & \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(X_t \leq y) \\ & = & \mathbb{P}(X_t \leq y) - (1-\mathbb{P}(M_t \leq x))\mathbb{P}(X_t \leq y) \end{array}$$

We put $u = \mathbb{P}(M_t \leq x)$ and $v = \mathbb{P}(X_t \leq y)$. Then

$$\begin{array}{rcl} u = \mathbb{P}(M_t \leq x) & \iff & 1-u = 2(1-\Phi_t(x)) \\ & \iff & x = \Phi_t^{-1}\left(\frac{1+u}{2}\right) \end{array}$$

Thus the condition $x \leq a$ is equivalent to $u \leq 1-\alpha$ and the value of the copula $C_{a,t}$ in this case is $C_{a,t}(u,v) = uv$

- Second case: If $a < x \leq y$

$$\begin{aligned} v = \mathbb{P}(X_t \leq y) &\iff \quad v = 1 - \frac{\Phi_t(-y)}{\alpha} \\ &\iff \quad y = \Phi_t^{-1}(1-\alpha+\alpha v) \end{aligned}$$

The condition $a \leq x \leq y$ becomes $1-\alpha \leq u \leq 1-2\alpha+2\alpha v$ The joint distribution function in this case is

$$\begin{aligned} H(x,y) &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(X_t \leq y/M_t > x) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(W_t \leq y/M_t > x) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(W_t > 2x - y/M_t > x) \\ &= \mathbb{P}(X_t \leq y) - (1-\Phi_t(-y)) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\left(1 - \frac{\Phi_t(-y)}{\alpha_x}\right) \end{aligned}$$

Then

$$C_{a,t}(u,v) = u + (1-\alpha)v - 1 + \alpha$$

- Third case: If $a \leq y \leq x$ This condition is equivalent to $1-\alpha \leq 1-2\alpha+2\alpha v \leq u$ since $a \leq y$. The joint distribution function in this case become

$$\begin{aligned} H(x,y) &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(W_t \leq y/M_t > x) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)F_{x,t}(y) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\left(\frac{\Phi_t(y-2x)}{\alpha_x}\right) \\ &= \mathbb{P}(X_t \leq y) - \Phi_t(y-2x) \end{aligned}$$

Then

$$C_{a,t}(u,v) = v - \Phi_t\left(\Phi_t^{-1}(1-\alpha+\alpha v) - 2\Phi_t^{-1}(\frac{1+u}{2})\right)$$

and using the relationship between Φ_t and Φ we can write

$$C_{a,t}(u,v) = v - \Phi\left(\Phi^{-1}(1-\alpha+\alpha v) - 2\Phi^{-1}\left(\frac{1+u}{2}\right)\right)$$

- Fourth case: If $y \leq a \leq x$

We have

$$\begin{aligned} v = \mathbb{P}(X_t \leq y) &\iff \quad v = \frac{\Phi_t(y-2a)}{\alpha} \\ &\iff \quad y = \Phi_t^{-1}(\alpha v) + 2a \end{aligned}$$

Then the condition $y \leq a \leq x$ is equivalent to $2a + \Phi_t^{-1}(\alpha v) \leq a \leq \Phi_t^{-1}(\frac{1+u}{2})$ and finally to $v \leq \frac{1}{2}$ and $u \geq 1-\alpha$.

For this condition

$$\begin{aligned} H(x,y) &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)\mathbb{P}(W_t \leq y / M_t > x) \\ &= \mathbb{P}(X_t \leq y) - \mathbb{P}(M_t > x)F_{x,t}(y) \\ &= \mathbb{P}(X_t \leq y) - \Phi_t(y - 2x) \end{aligned}$$

Then $C_{a,t}(u,v) = v - \Phi_t\left(2a + \Phi_t^{-1}(\alpha v) - 2\Phi_t^{-1}(\frac{1+u}{2})\right)$

Finally the copula $C_{a,t}$ may be written as

$$C_{a,t}(u,v) = \begin{cases} uv \text{ if } u \leq 1-\alpha \\ u + (1-\alpha)v - 1 + \alpha \text{ if } 1-\alpha \leq u \leq 1 - 2\alpha + 2\alpha v \\ v - \Phi\left(\Phi^{-1}(1-\alpha+\alpha v) - 2\Phi^{-1}(\frac{1+u}{2})\right) \text{ if } 1-2\alpha+2\alpha v \leq u \text{ and } v \geq \frac{1}{2} \\ v - \Phi\left(\frac{2a}{\sqrt{t}} + \Phi^{-1}(\alpha v) - 2\Phi^{-1}(\frac{1+u}{2})\right) \text{ elsewhere.} \end{cases} \tag{4}$$

5.3. **Some parameters of concordance.** As mentioned at introduction, we give some characteristics of the family

5.3.1. *Tail dependence parameters.*

$$\begin{aligned} \lambda_L(C_{a,t}) &= \lim_{s \to 0^+} \frac{C_{a,t}(s,s)}{s}. \\ &= 0. \end{aligned}$$

$$\begin{aligned} \lambda_U(C_{a,t}) &= \lim_{s \to 0^+} 2 - \frac{1 - C_{a,t}(s,s)}{1-s}. \\ &= \alpha. \\ &= 2\Phi\left(-\frac{a}{\sqrt{t}}\right). \\ &= \alpha_{a,t}. \end{aligned}$$

5.3.2. *β of Blomqvist.* To estimate β, it is enough to remark that

$$\begin{aligned} \beta(C_{a,t}) &= 4C_{a,t}\left(\tfrac{1}{2}, \tfrac{1}{2}\right) - 1. \\ &= \begin{cases} 0 \text{ if } \alpha < \frac{1}{2} \\ 1 - 4\Phi\left(\frac{a}{\sqrt{t}} - 2\Phi^{-1}\left(\frac{3}{4}\right)\right) \text{ if } \alpha \geq \frac{1}{2} \end{cases}. \end{aligned}$$

5.4. **Limit cases.** We have theses limit cases when one of the two parameters a or t tends to 0 or to $+\infty$.

$$\lim_{t \to 0^+} C_{a,t} = \lim_{a \to +\infty} C_{a,t} = \Pi.$$

And for other cases,

$$\lim_{a \to 0^+} C_{a,t} = \lim_{t \to +\infty} C_{a,t}(u,v) = \begin{cases} u \text{ if } 1+u \leq 2v \\ v - \Phi\left(\Phi^{-1}(v) - 2\Phi^{-1}(\frac{1+u}{2})\right) \text{ elsewhere.} \end{cases}$$

We give some remarks about theses limit cases:

- First, when a tends to 0, the event $(M_t > a)$ tends to a certain event and the process $X_{a,t}$ is tending to W_t then it seems natural that the limit copula is C_{M_t,W_t} which coincides with C_{M_1,W_1}
- Second, even the two processes M_t and W_t are depending on time, the copula joining them is not.

5.5. **The survival copula.** As already done for the running maxima-process of $(W_t)_t$, we consider the process $Y_t = (W_t/m_t < -a)_t$ obtained using conditioning W_t by the event $(m_t < -a)$ and denote, for more simplicity, $Z_t = -W_t$, $n_t = \min\limits_{0\le s\le t} Z_s$ and $N_t = \max\limits_{0\le s\le t} Z_s$.
One remarks that, using reflection principle of the Brownian motion, the process Z_t is also a standard Brownian motion since it is the reflected of (W_t) from 0, we can use easily the fact that we have both the two relations: $n_t = -M_t$ and $N_t = -m_t$. The results established above lead to:

$$\begin{aligned} Y_{a,t} &= W_t/_{m_t<-a} \\ &= -\big(-W_t/-N_t < -a\big) \\ &= -\left(Z_t/_{N_t>a}\right) \\ &\overset{d}{=} -X_{a,t}. \end{aligned}$$

We conclude that $\mathbb{E}\left(Y_{a,t}\right) = -\mathbb{E}\left(X_{a,t}\right)$ and the two processes have same variance.
Now we search the copula linking the two processes m_t and $Y_{a,t}$.

$$\begin{aligned} \mathbb{P}(m_t \le x, Y_{a,t} \le y) &= \mathbb{P}\left(-N_t \le x, -Z_t/_{N_t>a} \le y\right) \\ &= \mathbb{P}\left(N_t > -x, Z_t/_{N_t>a} > -y\right) \\ &= 1 - \mathbb{P}\left(N_t \le -x\right) - \mathbb{P}\left(Z_t/_{N_t>a} \le -y\right) + \mathbb{P}\left(N_t \le -x, Z_t/_{N_t>a} \le -y\right) \\ &= 1 - (1-u) - (1-v) + C_{a,t}(1-u, 1-v) \\ &= u + v - 1 + C_{a,t}(1-u, 1-v) \\ &= \widehat{C_{a,t}}(u,v) \end{aligned}$$

With $u = \mathbb{P}(m_t \le x) = \mathbb{P}(N_t > -x)$ and $v = \mathbb{P}(Y_{a,t} \le y) = \mathbb{P}(Z_t/_{N_t>a} > -y)$
The copula linking M_t and $X_{a,t}$ coincides with the survival one of the copula joining m_t and $Y_{a,t}$.
By recalling limit cases above, when a is tending to 0, or t is tending to $+\infty$, the following equality holds

$$C_{m_t,W_t} = \widehat{C_{M_t,W_t}}.$$

Since the copula C_{M_t,W_t} is independent on time as seen above, we conclude that it is also the same for the copula C_{m_t,W_t}.
When a tends to $+\infty$ or t tends to 0, the copula $C_{m_t,Y_{a,t}}$ converges to the independence one.

5.6. **One parameter family.** One can easily see from expression (4) that the double parameter family $(C_{a,t})_{a>0,t>0}$ can be expressed only using one parameter taking into account that the underlying double parameter family of copulas depends only on $\frac{a}{\sqrt{t}}$. Therefore one has $\alpha = 2\left(1-\Phi\left(\frac{a}{\sqrt{t}}\right)\right) \in]0,1]$. Hence the copula governing these processes becomes

$$C_\alpha(u,v) = \begin{cases} uv \text{ if } u \leq 1-\alpha \\ u+(1-\alpha)v-1+\alpha \text{ if } 1-\alpha \leq u \leq 1-2\alpha+2\alpha v \\ v-\Phi\left(\Phi^{-1}(1-\alpha+\alpha v)-2\Phi^{-1}(\frac{1+u}{2})\right) \text{ if } 1-2\alpha+2\alpha v < u \text{ and } v \geq \frac{1}{2} \\ v-\Phi\left(2\Phi^{-1}(1-\frac{\alpha}{2})+\Phi^{-1}(\alpha v)-2\Phi^{-1}(\frac{1+u}{2})\right) \text{ elsewhere.} \end{cases} \tag{5}$$

The copula $C_{a,t}$ is constant along each parabolic curve given by its cartesian equation is $a = c\sqrt{t}$, where $c = \Phi^{-1}\left(1-\frac{\alpha}{2}\right) > 0$.

And for showing well this dependence on α, we construct 500 data scatterplots for each copula C_α for some values 0, $\frac{1}{2}$, $\frac{3}{4}$, and 1 of the unified parameter α. And a simple calculation allows by discretizing the domain I^2, we obtained for example for the case $\alpha = \frac{1}{2}$ the asymmetry $\frac{1}{4}$.

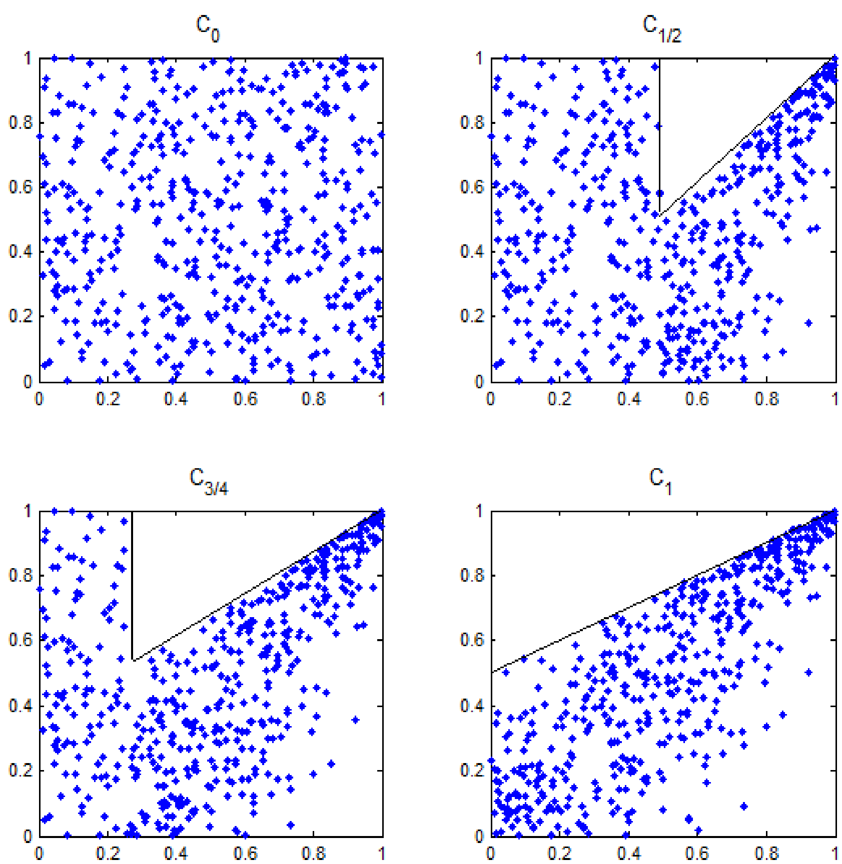

Figure 1. 500 data scatterplots for copulas C_0, $C_{\frac{1}{2}}$, $C_{\frac{3}{4}}$ and C_1

6. Application

The diffusion process X_t defined in section 4 modelizes a price of an option or of a product at time t. Our aim is to describe the dependence between its maximum $U_t = \underset{0\leq s\leq t}{X_s}$ and the price itself when the maximum U_t has surely exceeded some known level $a > 0$.
If we denote f the function $f : x \mapsto X_0 exp\left(\sigma x + (\mu - \frac{\sigma^2}{2})t\right)$ which is strictly increasing. Since $X_t = f(W_t)$, we have $U_t = f(M_t)$.
Always from the increasiness of the function f, we conclude that the mesurable set $\{M_t > a\}$ is equal to $\{U_t > f(a)\}$.
We come back to the process $f\,(W_t/M_t > a)$ which equals:

$$\begin{aligned} f\,(W_t/M_t > a) &= f\,(W_t \mathbb{1}_{M_t>a}) \\ &= \begin{cases} f(W_t)(w) \text{ if } w \in M_t > a \\ f(0) \text{ elsewhere} \end{cases} \\ &= \begin{cases} X_t(w) \text{ if } w \in U_t > f(a) \\ f(0) \text{ elsewhere} \end{cases} \\ &= X_t/U_t > f(a) \end{aligned}$$

One can easily conclude that the copula joining the processes U_t and $X_t/U_t > f(a)$ is the copula $C_{a,t}$ seen above since theses processes U_t and $X_t/U_t > f(a)$ can be written as $U_t = f(M_t)$ and $X_t/U_t > f(a) = f\,(W_t/M_t > a)$.
The price X_t knowing that it has exceeded a fixed level $a >$ at a time before t and its maxima U_t have the dependence $C_{f(a),t}$.

7. Conclusion

Even if the algebraic expression of the copula C_α seems complicated, mainly on the white area in figure 1, its implementation is not difficult and applications, as expected, in many fields is surely promising. We are content with this one exposed above.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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