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MODELLING WITH FRACTIONAL ORDER CONTINUOUS AND DISCRETE (DELTA) INTEGRATION

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ABSTRACT. In this paper, we use the discrete case methodology to the continuous situation in order to identify closed-form of fractional order continuous and discrete integration. We have derived several theorems and formula using Riemann Liouville fractional integral with gamma function. Also, we develop the discrete analog of the continuous version for ν^{th} fractional order integration. The novelty of this article is the introduction of fractional order exponential functions and obtaining its related theorems.

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1. INTRODUCTION

The continuous fractional calculus has made significant progress (see the publications by Miller and Ross [12], Oldham and Spanier [13], and Podlubny [14]). However, discrete fractional calculus has only recently attracted a lot of attention (see the articles by Atici, Eloe and Goodrich, and references in [1]- [7], Miller and Ross [12], and M. Holm [8]- [10]). Inspite of many mathematicians like Euler, Fourier, Laplace, Lebiniz who brings up about derivatives of arbitrary order, the tautochrone problem's integral equation occurs in the formulation, and Niels Henrik Abel was the first to solve it using fractional operations in 1823. Abel's integral equation is if the time of slide is a known constant K, then

$$K = \int_{0}^{\kappa} (\kappa - s)^{-1/2} u(s) ds. \tag{1}$$

Abel hypothesized about more basic integral equations using the following kernels: $(\kappa - s)^{\alpha}$. A particular example of a definite fractional integration for $\alpha = \frac{1}{2}$ is the integral mentioned in equation (1),

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apart from the multiplicative factor $1/\Gamma(\frac{1}{2})$. The function u in the integrands of integral equation (1) is unknown and have to be determined. Abel noted the right side of equation (1) as $\sqrt{\pi}[d^{-1/2}/d\kappa^{-1/2}]u(\kappa)$ and he operated both sides of the equation with $d^{1/2}/d\kappa^{1/2}$ to obtain

$$\frac{d^{1/2}}{d\kappa^{1/2}}K = \sqrt{\pi}u(\kappa). \tag{2}$$

According to the fractional operators (with appropriate conditions on u), we have $D^{1/2}D^{-1/2}u = D^0u = u$. In order to determined $u(\kappa)$, the fractional derivative is computed for order $\frac{1}{2}$ of the constant K in equation (2) which is an astonishing achievement of Abel in the fractional caluclus. Note that the fractional order derivative of a constant need not be zero. In the later part, we employ the same concept to discrete case by delta operator. Mathematicians have described Abel's solution as "elegant". Later that it was Fourier's integral formula and Abel's solution that caught Liouville's interest when he attempted to solve differential equations involving fractional operators and produced the first significant study of fractional calculus. This theoretical development began with the well-known outcome for integral order derivatives: $D^m e^{a\kappa} = a^m e^{a\kappa}$, which he naturally expanded to derivatives of any order

$$D^\nu e^{a\kappa} = a^\nu e^{a\kappa}, \;\; a > 0. \tag{3}$$

According to his presumption, any derivative of a function $u(\kappa)$ that may be expanded into a series of terms of the form

$$u(\kappa) = \sum_{n=0}^{\infty} c_n e^{a_n\kappa}, \quad Re\; a_n > 0, \tag{4}$$

Applying equation (3) on equation (4) may yield

$$D^\nu u(\kappa) = \sum_{n=0}^{\infty} c_n a_n^\nu e^{a_n\kappa}, \; Re\; a_n > 0. \tag{5}$$

As Liouville's original formula for a fractional derivative formula (5) is well-known, it naturally generalizes a derivative of any order ν, where ν can be any number. The drawback of this restriction is that it only applies to functions of the form (4). Liouville may have been aware of these limitations and thus he created a definite integral connected to the gamma function, that is

$$I = \int_0^\infty x^{a-1}e^{-\kappa x}dx, \;\; a > 0, \;\; \kappa > 0. \tag{6}$$

The change of variable $\kappa x = s$ yields $I = \kappa^{-a}\int_0^\infty s^{a-1}e^{-s}ds = \kappa^{-a}\Gamma(a)$ or $\kappa^{-a} = \frac{1}{\Gamma(a)}I$. Then, Liouville operates with D^ν on both sides of the equation above, to obtain, according to Liouville's basic assumption,

$$D^\nu\kappa^{-a} = \frac{1}{\Gamma(a)}D^\nu\int_0^\infty x^{a-1}e^{-\kappa x}dx = \frac{1}{\Gamma(a)}\int_0^\infty x^{a-1}(D^\nu e^{-\kappa x})dx = \frac{(-1)^\nu}{\Gamma(a)}\int_0^\infty x^{a+\nu-1}e^{-\kappa x}dx$$

Thus Liouville obtains a fractional derivative as:

$$D^{\nu}\kappa^{-a} = \frac{(-1)^{\nu}\Gamma(a+\nu)}{\Gamma(a)}\kappa^{-a-\nu}, \ \ a > 0. \tag{7}$$

However, Liouville's definitions were too restricted to hold up over time. The equation (6) is only applicable to functions of the type κ^{-a} (with $a > 0$), while the equation (5) is only applicable to functions of the class (4). Both cannot be used for a wide range of functions.

G. F. Bernhard Riemann formulated his theory of fractional integration which was anonymously published in his Gesammmelte Werke [1892]. In search of a Taylor series generalisation, he derived the fractional derivatives as

$$D^{-\nu}u(\kappa) = \frac{1}{\Gamma(\nu)}\int\limits_{c}^{\kappa}(\kappa - s)^{\nu-1}u(s)ds \ + \ \psi(\kappa). \tag{8}$$

Later that Riemann felt it necessary to include a complimentary function $\psi(\kappa)$ in his definition due to the ambiguity in the bottom limit of integration c. In essence, this supplementary function aims to quantify the deviance from the principle of exponents. The definition was created using the contour integration technique:

$${}_{c}D_{\kappa}^{-\nu}u(\kappa) = \frac{1}{\Gamma(\nu)}\int\limits_{c}^{\kappa}(\kappa - s)^{\nu-1}f(s)ds, \ \ Re\,\nu \ > \ 0 \tag{9}$$

for integration to an arbitrary order.

When $\kappa > c$ in equation (9), we get the equation (8) but without a complementary function. When $c = 0$, the most prevalent form appears.

$${}_{0}D_{\kappa}^{-\nu}u(\kappa) = \frac{1}{\Gamma(\nu)}\int\limits_{0}^{\kappa}(\kappa - s)^{\nu-1}f(s)ds, \ \ Re\,\nu \ > \ 0. \tag{10}$$

This fractional integral is also known as the Riemann-Liouville fractional integral.

Also, we will develop the extended analog of the exponential function for arbitrary order integral and differential operators. Then we will develop the discrete analog of the above-mentioned extended operator with help of Newton's forward difference operator as

$$\Delta_{0}^{-n}u(\kappa) = \frac{1}{(n-1)!}\Delta^{-1}\left\{(\kappa - s - 1)^{(n-1)}u(s)\right\}\Big|_{0}^{\kappa}, \tag{11}$$

where $\kappa^{(n)} = \kappa(\kappa-1)(\kappa-2)(\kappa-3)...(\kappa-(n-1))$ represents falling factorial.

2. Fractional Order Continuous Integration

In this section, we provide proof to the equation (10) when ν is any positive integer. [16]The Euler Gamma function is defined as,

$$\Gamma(m) = \int_{0}^{\infty} x^{m-1}e^{-x}dx \ , \ Re(m) \ > \ 0.$$

The most significant properties of Gamma function are

(i) $\Gamma(m+1) = m\,\Gamma(m)$, (ii) $\Gamma(m+1) = m!$, and (iii) $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$.

Lemma 2.1. *For positive integer n and $\nu > 0$, we have $D_0^{-\nu}\kappa^n = \dfrac{\Gamma(n+1)\kappa^{\nu+n}}{\Gamma(\nu+n+1)}$, and hence $D_0^{-\nu}\dfrac{\kappa^n}{n!} = \dfrac{\kappa^{\nu+n}}{(\nu+n)!}$, where $(\nu+n)! = \Gamma(\nu+n+1)$.*

Proof. Applying $u(\kappa) = \kappa$ in equation (10) and integration by parts, we get,

$$\Gamma(\nu)D_0^{-\nu}\kappa = \int_0^\kappa (\kappa - s)^{\nu-1} s ds = \frac{\Gamma(2)\kappa^{\nu+1}}{\Gamma(\nu+2)}, \nu > 0.$$

Again, applying equation (10) to $u(\kappa) = \kappa^2$ and usual integration by parts, we get $D_0^{-\nu}\kappa^2 = \dfrac{1}{\Gamma\nu}\int_0^\kappa (\kappa - s)^{\nu-1} s^2 ds = \dfrac{\Gamma(3)\kappa^{\nu+2}}{\Gamma(\nu+3)}, \nu > 0$.

For $u(\kappa) = \kappa^3$, it is easy to obtain, $D_0^{-\nu}\kappa^3 = \dfrac{\Gamma(4)\kappa^{\nu+3}}{\Gamma(\nu+4)}$, $\nu > 0$.

Proceeding like this and by mathematical induction on $'n'$, we get Lemma 2.1. □

An alternate proof for Riemann Liouville fractional integral defined as (10) is given below.

Theorem 2.2. *Let u be a real valued continuous function on $[0, \infty)$ and $n \in \mathbb{N}$. Let $D_0^{-1}u(\kappa) = \int_0^\kappa u(s)ds$, $D_0^{-n}u(\kappa) = D_0^{-(n-1)}(D_0^{-1}u(\kappa))$ and $\dfrac{d}{ds}D_0^{-1}u(s) = u(s)$. Then*

$$(n-1)!D_0^{-n}u(\kappa) = \int_0^\kappa (\kappa - s)^{n-1} u(s) ds. \tag{12}$$

Proof. The proof is proved by inductive method. From the given condition $D_0^{-1}u(\kappa) = \int_0^\kappa u(s)ds = \int_0^\kappa (\kappa - s)^0 u(s)ds$, equation (12) is trivial when $n = 1$.
Assume that equation (12) is true for $n > 1$. From the given conditions, we have

$$D_0^{-(n+1)}u(\kappa) = D_0^{-n}(D_0^{-1}u(\kappa)). \tag{13}$$

Replacing $u(\kappa)$ by $D_0^{-1}u(\kappa)$ in equation (12), we find

$$(n-1)!D_0^{-n}(D_0^{-1}u(\kappa)) = \int_0^\kappa (\kappa - s)^{n-1} D_0^{-1}u(s)ds,$$

which is the same as

$$(n-1)!D_0^{-(n+1)}u(\kappa) = \int_0^\kappa D_0^{-1}u(s)\frac{d}{ds}\Big[\frac{(\kappa - s)^n}{n}(-1)\Big]ds. \tag{14}$$

By applying $\int udv = uv - \int vdu$ on equation (14), we obtain

$$n!D_0^{-(n+1)}u(\kappa) = \Big[D_0^{-1}u(s)(\kappa-s)^n(-1)\Big]_0^\kappa + \int\limits_0^\kappa (\kappa-s)^n \frac{d}{ds}D_0^{-1}u(s)ds.$$

Since, $D_0^{-1}(u(0)) = \int\limits_0^0 u(s)ds = 0$, we find

$$n!D_0^{(n+1)}u(\kappa) = \int\limits_0^\kappa (\kappa-s)^n u(s)ds.$$

Hence, we obtain equation (12) for $n = n-1$. □

Lemma 2.3. *For positive integer n, $D_0^{-n}e^\kappa = e^\kappa - \sum\limits_{r=0}^{n-1}\frac{\kappa^r}{r!}$, and hence $\lim\limits_{n\to\infty} D_0^{-n}e^\kappa = 0$.*

Proof. By Bernoulli's integration formula $(n-1)!D_0^{-n}e^\kappa = \int\limits_0^\kappa (k-s)^{n-1}e^s ds$, we derive

$$(n-1)!D_0^{-n}e^\kappa = \Big[(\kappa-s)^{n-1}e^s + (n-1)(\kappa-s)^{n-2}e^s + \cdots + (n-1)!e^s\Big]_0^\kappa$$

$$= 0 + 0 + \cdots + (n-1)!e^\kappa - \kappa^{n-1} - (n-1)\kappa^{n-2} - (n-1)(n-2)\kappa^{n-3} - \cdots - (n-1)!$$

$$D_0^{-n}e^\kappa = e^\kappa - \frac{\kappa^{n-1}}{(n-1)!} - \frac{\kappa^{n-2}}{(n-2)!} - \frac{\kappa^{n-3}}{(n-3)!} - \cdots - 1 \quad = e^\kappa - \sum_{r=0}^{n-1}\frac{\kappa^r}{r!}.$$

Now the proof follows by taking limit on both sides. □

Theorem 2.4. *If $u(\kappa)$ has maclaurin series expansion as $u(\kappa) = \sum\limits_{r=0}^{\infty}\frac{u^{(r)}(0)\kappa^r}{\Gamma(r+1)}$, then $D_0^{-\nu}u(\kappa) = \sum\limits_{r=0}^{\infty}\frac{u^r(0)\kappa^{\nu+r}}{\Gamma(\nu+r+1)}$, $\nu+r+1 \notin \{0,-1,-2,\cdots\}$ for $r \in \mathbb{N}(0)$.*

Proof. The proof follows by linearity of $D_0^{-\nu}$ and the Lemma 2.1. □

From Theorem 2.4, we can easily obtain the following corollary.

Corollary 2.5. *Let $\nu+r+1 \notin \{0,-1,-2,\cdots\}$ for $r \in \mathbb{N}(0)$. Then for $\nu > 0$, (i) $D_0^{-\nu}e^\kappa = \sum\limits_{r=0}^{\infty}\frac{\kappa^{r+\nu}}{\Gamma(\nu+r+1)}$,*
(ii) $D_0^{-\nu}\sin\kappa = \sum\limits_{r=1}^{\infty}\frac{(-1)^{r-1}\,\kappa^{\nu+2r-1}}{\Gamma(\nu+2r)}$,
(iii) $D_0^{-\nu}\cos\kappa = \sum\limits_{r=0}^{\infty}\frac{(-1)^r\,\kappa^{\nu+2r}}{\Gamma(\nu+2r+1)}$ (iv) $D_0^{-\nu}(1-\kappa)^{-1} = \sum\limits_{r=0}^{\infty}\frac{r!\kappa^{\nu+r}}{\Gamma(\nu+r+1)}$, $|\kappa| < 1$.

Definition 2.6. For $\nu > 0$, the ν^{th} exponential function with shift $t > 0$ is defined by

$$e(\kappa^\nu, t) = \frac{\kappa^\nu}{\Gamma(\nu+1)} + \frac{\kappa^{\nu+t}}{\Gamma(\nu+t+1)} + \frac{\kappa^{\nu+2t}}{\Gamma(\nu+2t+1)} + \cdots \tag{15}$$

Note that $e(\kappa^0, 1) = e^\kappa$ and $e(\kappa^\nu, 1) = \frac{k^\nu}{\Gamma(\nu+1)} + \frac{k^{\nu+t}}{\Gamma(\nu+t+1)} + \frac{k^{\nu+2t}}{\Gamma(\nu+2t+1)} + \cdots$

Theorem 2.7. *The fractional differential equation $D^\nu u(\kappa) = e^\kappa$ has a solution $u(\kappa) = e(\kappa^\nu)$ and hence $D_0^{-\nu}e^\kappa = e(\kappa^\nu, 1)$.*

Proof. From the linearity of $D_0^{-\nu}$ and applying equation (10) for $u(\kappa) = e^{\kappa}$, we get
$D_0^{-\nu} e^{\kappa} = \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} e^s ds = \sum\limits_{n=0}^{\infty} \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \frac{s^n}{n!} ds = \sum\limits_{n=0}^{\infty} D_0^{-\nu} \left(\frac{\kappa^n}{n!}\right)$
Now the proof follows from Lemma 2.1 and by definition of $e(\kappa^{\nu}, 1)$. ☐

Definition 2.8. The ν^{th} sine function is defined by

$$\sin(\kappa^{\nu}, t) = \frac{\kappa^{\nu+1}}{\Gamma(\nu+2)} - \frac{\kappa^{\nu+2t+1}}{\Gamma(\nu+2t+2)} + \frac{\kappa^{\nu+4t+1}}{\Gamma(\nu+4t+2)} - \cdots, \qquad \nu > 0. \tag{16}$$

Theorem 2.9. *The fractional differential equation $D^{\nu} u(\kappa) = \sin \kappa$ has a solution $u(\kappa) = \sin(\kappa^{\nu}, t)$ and hence $D_0^{-\nu} \sin \kappa = \sin(\kappa^{\nu}, 1)$.*

Proof. From the linearity of $D_0^{-\nu}$ and applying equation (10) for $u(\kappa) = \sin \kappa$, we get
$D_0^{-\nu} \sin \kappa = \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \sin s \, ds = \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \left\{ \sum\limits_{n=0}^{\infty} \frac{(-1)^n s^{2n+1}}{(2n+1)!} \right\} ds$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \left\{ \frac{s^{2n+1}}{(2n+1)!} \right\} ds = \sum_{n=0}^{\infty} (-1)^n D_0^{-\nu} \left\{ \frac{\kappa^{2n+1}}{(2n+1)!} \right\}.$$

Now the proof follows from Lemma 2.1 and by definition of $\sin(\kappa^{\nu}, 1)$. ☐

Definition 2.10. The ν^{th} cosine function is defined by

$$\cos(\kappa^{\nu}, t) = \frac{\kappa^{\nu}}{\Gamma(\nu+1)} - \frac{\kappa^{\nu+2t}}{\Gamma(\nu+2t+1)} + \frac{\kappa^{\nu+4t}}{\Gamma(\nu+4t+1)} - \cdots, \qquad \nu > 0 \tag{17}$$

Theorem 2.11. *The fractional differential equation $D^{\nu} u(\kappa) = \cos \kappa$ has a solution $u(\kappa) = \cos(\kappa^{\nu}, t)$ and hence $D_0^{-\nu} \cos \kappa = \cos(\kappa^{\nu}, 1) = \sum\limits_{r=0}^{\infty} (-1)^r e(\kappa^{\nu}, 2)$.*

Proof. From the linearity of $D_0^{-\nu}$ and applying equation (10) for $u(\kappa) = \cos \kappa$, we get
$D_0^{-\nu} \cos \kappa = \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \cos s ds = \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \left\{ \sum\limits_{n=0}^{\infty} \frac{(-1)^n s^{2n}}{(2n)!} \right\} ds$

$$= \sum_{n=0}^{\infty} (-1)^n \frac{1}{\Gamma(\nu)} \int\limits_0^{\kappa} (\kappa - s)^{\nu-1} \frac{s^{2n}}{(2n)!} ds = \sum_{n=0}^{\infty} (-1)^n D_0^{-\nu} \left(\frac{\kappa^{2n}}{(2n)!} \right)$$

Now the proof follows from Lemma 2.1 and by definition of $\cos(\kappa^{\nu}, 1)$. ☐

3. Fractional Discrete (Delta) Integration

In this section, we develop the discrete analog of the continuous version given in equation (10).

Lemma 3.1. [11] *Let u, v be real valued functions. Then*

$$\Delta^{-1}\{u(\kappa)v(\kappa)\} = u(\kappa)\Delta^{-1}v(\kappa) - \Delta^{-1}\{\Delta^{-1}v(\kappa+1)\Delta u(\kappa)\}.$$

Lemma 3.2. [11] *Let u, v be real valued functions. Then*
$\Delta^{-1}(\kappa - s)^{(n)} = -\frac{(\kappa - s)^{(n+1)}}{(n+1)}$*, keeping κ as constant and s as variable.*

Theorem 3.3. *For $n \in \mathbb{N}$, if $\Delta_0^{-1}u(\kappa) = \Delta^{-1}u(\kappa)|_0^\kappa$, $\Delta_0^{-n}u(\kappa) = \Delta_0^{-(n-1)}\left(\Delta_0^{-1}u(\kappa)\right)$, then*

$$(n-1)!\Delta_0^{-n}u(\kappa) = \Delta^{-1}\left\{(\kappa-s-1)^{(n-1)}u(s)\right\}\Big|_0^\kappa, \tag{18}$$

where $\kappa^{(n)} = (\kappa-1)(\kappa-2)(\kappa-3)\cdots(\kappa-(n-1))$ is the falling factorial.

Proof. We prove this theorem by induction on $'n'$. From the given condition $\Delta_0^{-1}u(\kappa) = \Delta^{-1}u(s)\Big|_0^t = \Delta^{-1}\left\{(\kappa-s-1)^{(0)}u(s)\right\}|_0^\kappa$, so equation (18) is trivial when $n=1$. We assume that equation (18) is true for $n>1$.

From the given condition, we have

$$\Delta_0^{-(n+1)}u(\kappa) = \Delta_0^{-n}\left(\Delta_0^{-1}u(\kappa)\right). \tag{19}$$

Replacing $u(\kappa)$ by $\Delta_0^{-1}u(\kappa)$ in equation (18), we have

$$(n-1)!\Delta_0^{-n}\left(\Delta_0^{-1}u(\kappa)\right) = \Delta^{-1}\left\{(\kappa-s-1)^{(n-1)}\Delta_0^{-1}u(s)\right\}\Big|_0^\kappa$$

$$(n-1)!\Delta_0^{-(n+1)}u(\kappa) \quad = \Delta^{-1}\left\{\Delta_0^{-1}u(s)\left(\kappa-s-1\right)^{(n-1)}\right\}\Big|_0^\kappa$$

Applying Lemma 3.1, we arrive

$$(n-1)!\Delta_0^{-(n+1)}u(\kappa) = \left[\Delta_0^{-1}u(s)\Delta^{-1}(\kappa-s-1)^{(n-1)} - \Delta^{-1}\left\{\Delta^{-1}(\kappa-s)^{(n-1)}u(s)\right\}\right]\Big|_0^\kappa.$$

By Lemma 3.2, it is easy to obtain

$$(n-1)!\Delta_0^{-(n+1)}u(\kappa) = \left[\Delta_0^{-1}u(s)\frac{(-1)(\kappa-s)^n}{n} - \Delta^{-1}\left\{\frac{(-1)(\kappa-s-1)^{(n)}}{n}u(s)\right\}\right]\Big|_0^\kappa.$$

Again applying Lemma 3.2,

$$n!\Delta_0^{-(n+1)}u(\kappa) = \left[0 - 0 + \Delta^{-1}\left\{(\kappa-s-1)^{(n)}u(s)\right\}\right]\Big|_0^\kappa$$

$$= \left[\Delta^{-1}\left\{(\kappa-s-1)^{(n)}u(s)\right\}\right]\Big|_0^\kappa.$$

By induction, the proof is complete. □

Definition 3.4. For $\nu \in \mathbb{R}^+$, the fractional delta integration is defined as

$$\Gamma(\nu)\Delta_0^{-\nu}u(\kappa) = \Delta^{-1}\left\{(\kappa-s-1)^{(\nu-1)}u(s)\right\}\Big|_0^\kappa = \sum_{s=0}^{\kappa-\nu}(\kappa-s-1)^{(\nu-1)}u(s), \tag{20}$$

where $\kappa^{(\nu)} = \dfrac{\Gamma(\kappa+1)}{\Gamma(\kappa+1-\nu)}$.

Theorem 3.5. *For $\nu > 0$, $\Gamma(\nu+n+1)\Delta_0^{-\nu}\kappa^{(n)} = \Gamma(n+1)\kappa^{(\nu+n)}$.*

Proof. Applying $u(\kappa) = \kappa^{(n)}$ in the first part of equation (20),

$$\Gamma(\nu)\Delta_0^{-\nu}\kappa^{(n)} = \Delta^{-1}\{s^{(n)}(\kappa-s-1)^{(\nu-1)}\}|_0^\kappa$$

$$= \left[\kappa^{(n)}\Delta^{-1}\{(\kappa-s-1)^{(\nu-1)}\} - \Delta^{-1}\{\Delta^{-1}(\kappa-s-2)^{(\nu-1)}\Delta s^{(n)}\}\right]\Big|_0^\kappa$$

$$= \left[s^{(n)}\frac{(-1)(\kappa-s)^{(\nu)}}{\nu} - \Delta^{-1}\{\frac{(-1)(\kappa-s-1)^{(\nu)}}{\nu} \times ns^{(n-1)}\}\right]\Big|_0^\kappa$$

$$= \frac{1}{\nu}[0-0] + \frac{n}{\nu}\Delta^{-1}\{s^{(n-1)}(\kappa - s - 1)^{(\nu)}\}|_0^\kappa$$
$$= \frac{n}{\nu}\Big[s^{(n-1)}\Delta^{-1}(\kappa - s - 1)^{(\nu)} - \Delta^{-1}\{\Delta^{-1}(\kappa - s - 2)^{(\nu)}\Delta s^{(n-1)}\}\Big]\Big|_0^\kappa$$
$$\Gamma(\nu+1)\Delta_0^{-\nu}\kappa^{(n)} = n\Big[s^{(n-1)}\frac{(-1)(\kappa - s)^{(\nu+1)}}{(\nu+1)} - \Delta^{-1}\{\frac{(-1)(\kappa - s - 1)^{(\nu+1)}}{(\nu+1)} \times (n-1)s^{(n-2)}\}\Big]\Big|_0^\kappa$$
$$\Gamma(\nu+1)\Delta_0^{-\nu}\kappa^{(n)} = \frac{n}{(\nu+1)}\Big[0-0\Big] + \frac{n(n-1)}{(\nu+1)}\Delta^{-1}\{s^{(n-2)}(\kappa - s - 1)^{(\nu+1)}\}|_0^\kappa.$$

On continuing like this, we arrive on

$$\Gamma(\nu+n-1)\Delta_0^{-\nu}\kappa^{(n)} = n(n-1)...2\Big[s^{(1)}\Delta^{-1}\{(\kappa - s - 1)^{(\nu+n-2)}\} - \Delta^{-1}\{\Delta^{-1}(\kappa - s - 2)^{(\nu+n-2)}\Delta s\}\Big]\Big|_0^\kappa$$
$$= n(n-1)\cdots 2\Big[s^{(1)}\frac{(-1)(\kappa - s)^{(\nu+n-1)}}{(\nu+n-1)} - \Delta^{-1}\{\frac{(-1)(\kappa - s - 1)^{(\nu+n-1)}}{(\nu+n-1)} \times 1\}\Big]\Big|_0^\kappa$$
$$= \frac{n(n-1)\cdots 2}{(\nu+n-1)}\Big[0-0\Big] + \frac{\Gamma(n+1)}{(\nu+n-1)}\Big[\Delta^{-1}\{(\kappa - s - 1)^{(\nu+n-1)}\}\Big]\Big|_0^\kappa$$
$$\Gamma(\nu+n)\Delta_0^{-\nu}\kappa^{(n)} = \Gamma(n+1)\Big[\frac{(-1)(\kappa - s)^{(\nu+n)}}{(\nu+n)}\Big]\Big|_0^\kappa$$
$$\Gamma(\nu+n+1)\Delta_0^{-\nu}\kappa^{(n)} = \Gamma(n+1)\kappa^{(\nu+n)}.$$
□

Lemma 3.6. *For $\nu > 0$ and $n \in \mathbb{N}$, $\Delta_0^{-\nu}\kappa^n = \sum_{r=1}^{n} S_r^n \frac{r!\kappa^{(\nu+r)}}{\Gamma(\nu+r+1)}$, where S_r^n is the stirling number of second kind.*

Proof. From the Stirling numbers of the second kind S_r^n, we have $\kappa^n = \sum_{r=1}^{n} S_r^n \kappa^{(r)}$. Applying $\Delta_0^{-\nu}$ on both sides, $\Delta_0^{-\nu}\kappa^n = \sum_{r=1}^{n} S_r^n \Delta_0^{-\nu}\kappa^{(r)}$. Using the similar procedure of Theorem 3.5, we get Lemma 3.6. □

Corollary 3.7. *For $\nu > 0$ and $n \in \mathbb{N}$,*

(i) $\Delta_0^{-\nu}\kappa = S_1^1 \frac{\kappa^{(\nu+1)}}{\Gamma(\nu+2)}$,

(ii) $\Delta_0^{-\nu}\kappa^2 = S_1^2 \frac{1!\kappa^{(\nu+1)}}{\Gamma(\nu+2)} + S_2^2 \frac{2!\kappa^{(\nu+2)}}{\Gamma(\nu+3)}$,

(iii) $\Delta_0^{-\nu}\kappa^3 = S_1^3 \frac{1!t^{(\nu+1)}}{\Gamma(\nu+2)} + S_2^3 \frac{2!\kappa^{(\nu+2)}}{\Gamma(\nu+3)} + S_3^3 \frac{3!\kappa^{(\nu+3)}}{\Gamma(\nu+4)}$,

(iv) $\Delta_0^{-\nu}\kappa^4 = S_1^4 \frac{1!\kappa^{(\nu+1)}}{\Gamma(\nu+2)} + S_2^4 \frac{2!\kappa^{(\nu+2)}}{\Gamma(\nu+3)} + S_3^4 \frac{3!\kappa^{(\nu+3)}}{\Gamma(\nu+4)} + S_4^4 \frac{4!\kappa^{(\nu+4)}}{\Gamma(\nu+5)}$.

Theorem 3.8. *Let $u(\kappa) = \sum_{r=0}^{\infty} \frac{\Delta^r u(0)}{r!}\kappa^{(r)}$, then*

$$\Delta_0^{-\nu}u(\kappa) = \sum_{r=0}^{\infty} \Delta^r u(0) \frac{t^{(\nu+r)}}{\Gamma(\nu+r+1)}. \tag{21}$$

Proof. Since $u(\kappa) = \sum_{r=0}^{\infty} \frac{\Delta^r u(0)}{r!}\kappa^{(r)}$, taking $\Delta_0^{-\nu}$ on both side, we get

$$\Delta_0^{-\nu} u(\kappa) = \sum_{r=0}^{\infty} \frac{\Delta^r u(0)}{r!}\Delta_0^{-\nu}\kappa^{(r)}.$$

By Theorem 3.5, we get equation (21). □

From Theorem 3.8, we can easily obtain the following corollary,

Corollary 3.9. *For $\nu > 0$ and $(k+1-\nu) \in \mathbb{Z}$,*

(i) $\Delta_0^{-\nu} e^{\kappa} = \sum_{r=0}^{\infty} (e-1)^r \frac{\kappa^{(\nu+r)}}{\Gamma(\nu+r+1)}$

(ii) $\Delta_0^{-\nu} 2^{\kappa} = \sum_{r=0}^{\infty} \frac{\kappa^{(\nu+r)}}{\Gamma(\nu+r+1)}$

(iii) $\Delta_0^{-\nu} \sin\kappa = \sum_{r=0}^{\infty} \Delta^r \sin(0) \frac{\kappa^{(\nu+r)}}{\Gamma(\nu+r+1)}$

(iv) $\Delta_0^{-\nu} \cos\kappa = \sum_{r=0}^{\infty} \Delta^r \cos(0) \frac{\kappa^{(\nu+r)}}{\Gamma(\nu+r+1)}$

Example 3.10. *Applying $\nu = 2,\ \kappa = 5$ in Corollary 3.9 (i), we get $\Delta_0^{-2} e^k = 47.018494$.*

Applying $\nu = 0.5,\ \kappa = 2.5$ in Corollary 3.9 (i), we get $\Delta_0^{-0.5} e^k = 9.122$.

Applying $\nu = 2.5,\ \kappa = 2.5$ in Corollary 3.9 (i), we get $\Delta_0^{-2.5} e^k = 1$.

Applying $\nu = 2.5,\ \kappa = 5.5$ in Corollary 3.9 (i), we get $\Delta_0^{-2.5} e^k = 57.014928$.

Example 3.11. *Applying $\nu = 3,\ \kappa = 6$ in Corollary 3.9 (ii), we get $\Delta_0^{-3} 2^k = 42$.*

Applying $\nu = 3.5,\ \kappa = 6.5$ in Corollary 3.9 (ii), we get $\Delta_0^{-3.5} 2^k = 52.187828$.

Example 3.12. *Applying $\nu = 2,\ \kappa = 4$ in Corollary 3.9 (iii), we get $\Delta_0^{-2} \sin k = 0.0698043$.*

Applying $\nu = 2.4,\ \kappa = 4.4$ in Corollary 3.9 (iii), we get $\Delta_0^{-2.4} \sin k = 0.076785$.

Example 3.13. *Applying $\nu = 3,\ \kappa = 5$ in Corollary 3.9 (iv), we get $\Delta_0^{-3} \cos k = 9.9893391$.*

Applying $\nu = 3.2,\ \kappa = 5.2$ in Corollary 3.9 (iv), we get $\Delta_0^{-3.2} \cos k = 10.919023$.

4. Conclusion

In this study, the basic idea of fractional order integration is defined and its closed form are discussed. Also, we develop the discrete analog of fractional order continuous integration. Several theorem have been obtained based on fractional order continuous and discrete delta integration. Theorem 3.8 is used to obtain formula for certain types of infinite series involving gamma function.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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