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STABILITY OF HILL DIFFERENTIAL EQUATION USING DISCRETE AND CONTINUOUS FLOQUET SYSTEMS

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Abstract. This article investigates the stability properties of the Hill differential equation through the lens of both discrete and continuous Floquet systems. The Hill equation, a fundamental mathematical model encountered in various scientific disciplines, is explored in terms of its behavior under discrete and continuous Floquet analysis. The study aims to provide insights into the stability characteristics of the Hill equation and enhance our understanding of its dynamic behavior using these two distinct Floquet approaches. By examining the interplay between discrete and continuous Floquet systems, the article contributes to the broader comprehension of stability phenomena in differential equations and offers potential applications in diverse fields such as physics, engineering, and mathematics.

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1. Introduction

Floquet theory establishes that a nonautonomous linear system of differential equations with a period of T can be transformed into an equivalent autonomous system through a periodic Lyapunov transformation [1]. This theory serves as a potent tool for analyzing stability and periodic solutions in dynamic systems. Over time, mathematicians have expanded the scope of Floquet theory in various directions, leading to a classification of results into distinct categories. These encompass ordinary differential equations, such as almost Floquet systems [2], almost-periodic systems [3], periodic Euler-Bernoulli equations [4], delay differential equations [5], linear systems with meromorphic solutions [6]; partial differential equations, including parabolic differential equations [7] and periodic evolution

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problems [8]; differential-algebraic equations [9, 10]; integro-differential equations [11]; Volterra equations [12]; countable systems in discrete dynamical systems [13]; and systems defined on time scales [14]. For a comprehensive understanding of Floquet theory and its applications, interested readers may consult the monograph [15] and additional works [16,17]. In both discrete and continuous Floquet systems, the stability of the system can be characterized by the Floquet multipliers. The Floquet multipliers can be used to determine the stability of the system, and to study the bifurcation of the system as the parameters of the system change.

Floquet systems have a wide range of applications in fields such as physics, engineering, and biology. For example, in physics, they are used to study the behavior of periodically driven systems such as oscillators and circuits. In engineering, they are used to study the stability and control of systems such as power grids and mechanical systems. In biology, they are used to study the behavior of periodically driven systems such as circadian rhythms and cell cycles.

The Hill differential equation is a mathematical model frequently used to describe oscillatory behavior in various physical systems. Understanding its stability is crucial in fields like physics, engineering, and applied mathematics. [18,19]

By employing both discrete and continuous Floquet systems, the study provides a comprehensive analysis of the stability of the Hill differential equation. This dual approach allows for a more nuanced understanding of the system's behavior across different time domains, offering valuable insights for applications in various scientific and engineering domains.

2. Theoretical Foundations

Before delving into the comparative analysis of stability using discrete and continuous Floquet systems, it is imperative to establish the theoretical underpinnings that form the basis of our study. This section provides a concise overview of the essential concepts and mathematical frameworks that will guide our exploration.

2.1. Introduction to Hill Differential Equations and Their Applications.

The Hill differential equation constitutes a fundamental mathematical model that finds wide-ranging applications in various scientific and engineering disciplines. Named after mathematician George William Hill, this differential equation is characterized by its oscillatory behavior and has proven to be invaluable in describing phenomena involving periodic motion. [20]

The Form of Hill Differential Equations. A typical form of the Hill differential equation can be expressed as:

$$\frac{d^2x}{dt^2} + P(t)x = 0$$

where x represents the dependent variable, t denotes time, and $P(t)$ is a periodic function with a known period. This form encapsulates the essence of oscillatory systems governed by Hill differential equations.

Applications in Physics. Hill differential equations have found extensive utility in the realm of physics, particularly in celestial mechanics. In celestial mechanics, they arise in the study of celestial bodies under the influence of gravitational forces. The motion of planets, satellites, and other celestial objects is often modeled using Hill differential equations, allowing for the prediction of their trajectories and orbits.

Engineering Applications. In engineering, Hill differential equations play a crucial role in various fields. They find application in the design and analysis of mechanical systems exhibiting oscillatory behavior. For instance, in the design of suspension systems for vehicles, understanding the behavior of springs and dampers often involves solving Hill differential equations to predict and optimize the response to dynamic forces.

Oscillations in Electrical Circuits. Additionally, Hill differential equations have proven indispensable in the study of electrical circuits. Systems involving inductors, capacitors, and resistors can be modeled using differential equations, which, when appropriate conditions are met, take the form of Hill equations. This application has implications for the design and stability analysis of electrical circuits used in various electronic devices.

Beyond Traditional Applications. Beyond these traditional applications, Hill differential equations have found utility in diverse fields including biology, economics, and control theory. Their versatility in describing oscillatory phenomena has led to their adoption in a wide array of scientific and engineering disciplines.

In this article, we delve into the stability analysis of Hill differential equations using discrete and continuous Floquet systems, providing a comprehensive framework for understanding the behavior of oscillatory systems in various contexts.

2.2. **Review of Stability Concepts in Differential Equations.** To effectively analyze the stability of Hill differential equations using discrete and continuous Floquet systems, it is imperative to establish a foundational understanding of stability concepts in the context of differential equations. [21]

Stability Definitions. In the realm of differential equations, stability refers to the behavior of a system in response to perturbations or disturbances. Different types of stability are commonly defined:

- **Asymptotic Stability**: A system is asymptotically stable if, for any small perturbation, the system's trajectory approaches a stable equilibrium point as time tends towards infinity.

- Lyapunov Stability: Lyapunov stability considers the existence of a Lyapunov function, which quantifies the system's energy or a related quantity. If this function is decreasing along the trajectories of the system, the equilibrium is considered stable.

- Exponential Stability: In exponentially stable systems, the deviation from equilibrium diminishes at an exponential rate over time.

Linearization and Stability. Linearization is a fundamental technique in the study of stability. It involves approximating a nonlinear system in the vicinity of an equilibrium point by a linear system. The stability of the linearized system provides insights into the stability of the original nonlinear system near that equilibrium.

Floquet Theory. Floquet theory is a pivotal tool in the analysis of periodic linear systems. It provides a framework for understanding the stability of periodic solutions in differential equations with periodic coefficients. The Floquet theorem asserts that for a linear periodic system, solutions can be represented as the product of a periodic function and an exponential term.

Stability and Oscillatory Systems. The stability of oscillatory systems is of particular interest when studying Hill differential equations. Understanding the behavior of solutions near periodic orbits or equilibrium points is essential for characterizing the stability properties of such systems.

Bifurcations and Stability Transitions. Bifurcations mark critical points where the stability of a system undergoes qualitative changes. Understanding bifurcations is crucial for predicting transitions in system behavior and identifying regions of stability and instability.

In this section, we have provided a concise review of key stability concepts relevant to the analysis of Hill differential equations. This foundational understanding will serve as the basis for our subsequent exploration of stability using discrete and continuous Floquet systems.

3. Floquet Theorem: Continuous Case

The Floquet theorem in its continuous form stands as a cornerstone in the study of periodic systems. It provides a powerful mathematical tool for understanding the behavior and stability of solutions in continuous-time dynamical systems exhibiting periodicity. This theorem offers crucial insights into how small perturbations can affect the long-term behavior of such systems.

Definition 3.1. *Consider the linear system*

$$\frac{dx}{dt} = A(t)x \tag{1}$$

With $A(t)$ is an $n \times n$ system and x have n independent solutions $x_1, \dots, x_n$.
With columns $x_1, \dots, x_n$, (i.e $\phi(t) = (x_1(t)x_2(t)\dots x_n(t))$)

Proposition 3.1. *We have the following important properties of the fundamental matrix* ϕ

- $\dfrac{d\phi}{dt} = A(t)\phi(t)$
- $\phi(t)$ *is invertible*
- $\phi(0) = I$
- $x(t) = \phi(t)C$ *is solution to* $\dfrac{dx}{dt} = A(t)x$

Definition 3.2. *The monodromy matrix is a fundamental matrix of ordinary differential or difference equation evaluated at the period of the coefficients of the system.*

Theorem 3.1. (*Continuous Floquet's theorem*) *If* $\phi(t)$ *is a fundamental matrix of the floquet system* (1) *Where* $A(t)$ *a continuous matrix with period* T, *then* $\phi(t+T)$ *is also a fundamental matrix.*
Furthermore, there is a nonsingular matrix $P(t)$ *such that*

$$\phi(t) = P(t)e^{Bt} \tag{2}$$

where $P(t)$ *is periodic with period* T, *and* B *a constant matrix.*

Example 3.1. *Consider the linear* (1) *periodic system* (*period* $T = 2\pi$) *with*

$$A(t) = \begin{pmatrix} 0 & 1 & sin(t) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \tag{3}$$

We will use two methods to get a solution to the previous system.

First Method

The system (1)-(3) *is equivalent to*

$$\begin{cases} \dfrac{x_1}{dt} = x_2(t) + sin(t)x_3(t) \\ \\ \dfrac{x_2}{dt} = x_3(t) \\ \\ \dfrac{x_3}{dt} = 0 \end{cases}$$

$\iff$

$$\begin{cases} \dfrac{x_1}{dt} = x_2(t) + sin(t)c_1 \\ \\ \dfrac{x_2}{dt} = c_1 \\ \\ x_3(t) = c_1 \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} \dfrac{x_1}{dt} = c_1t + c_2 + sin(t)c_1 \\ \\ x_2(t) = c_1t + c_2 \\ \\ x_3(t) = c_1 \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} x_1(t) = \frac{1}{2}c_1t^2 + c_2t - cos(t)c_1 + c_3 \\ x_2(t) = c_1t + c_2 \\ x_3(t) = c_1 \end{cases}$$

which we can write as :

$$x = c_1 \begin{pmatrix} \frac{1}{2}t^2 - cos(t) \\ t \\ 1 \end{pmatrix} + c_2 \begin{pmatrix} t \\ 1 \\ 0 \end{pmatrix} + c_3 \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

Therefore the fundamental matrix is

$$\phi(t) = \begin{pmatrix} \frac{1}{2}t^2 - cos(t) & t & 1 \\ t & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

So that ,

$$\begin{aligned} B &= \phi^{-1}(0)\phi(2\pi) \\ B &= \begin{pmatrix} -1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^{-1} \begin{pmatrix} 2\pi^2 - 1 & 2\pi & 1 \\ 2\pi & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{pmatrix} \begin{pmatrix} 2\pi^2 - 1 & 2\pi & 1 \\ 2\pi & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \\ B &= \begin{pmatrix} 1 & 0 & 0 \\ 2\pi & 1 & 0 \\ 2\pi^2 & 2\pi & 1 \end{pmatrix} \end{aligned}$$

Second Method

Let compute the state transition matrix over one period $\phi(T,0)$, using Peano-Baker series

$$\begin{aligned}
\phi(t,\tau) &= I + \int_{\tau}^{t} A(s_1)\,\mathrm{d}s_1 + \int_{\tau}^{t} A(s_1)\int_{\tau}^{s_1} A(s_2)\,\mathrm{d}s_2\mathrm{d}s_1 \\
&= \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & t-\tau & -cos(t)+cos(\tau) \\ 0 & 0 & t-\tau \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & \frac{1}{2}t^2 - t\tau + \frac{1}{2}\tau^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 1 & t-\tau & cos(\tau) - cos(t) + \frac{(t-\tau)^2}{2} \\ 0 & 1 & t-\tau \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}$$

Therefore,

$$\phi(T,0) = \phi(2\pi,0) = \begin{pmatrix} 1 & 2\pi & 2\pi^2 \\ 0 & 1 & 2\pi \\ 0 & 0 & 1 \end{pmatrix}$$

Now , we will find a constant matrix R such that $\phi(T,0) = e^{RT}$

$$\begin{aligned}
R &= \frac{1}{T} log(\phi(T,0)) \\
&= \frac{1}{2\pi} log(\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 2\pi & 2\pi^2 \\ 0 & 0 & 2\pi \\ 0 & 0 & 0 \end{pmatrix}) \\
&= \frac{1}{2\pi}\left[\begin{pmatrix} 0 & 2\pi & 2\pi^2 \\ 0 & 0 & 2\pi \\ 0 & 0 & 0 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 0 & 0 & 4\pi^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
&= \frac{1}{2\pi}\begin{pmatrix} 0 & 2\pi & 0 \\ 0 & 0 & 2\pi \\ 0 & 0 & 0 \end{pmatrix} \\
&= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}
\end{aligned}$$

Therefore , $P(t)$ is given by

$$P(t) = \phi(t,0)e^{-Rt}$$

$$
\begin{aligned}
&= \begin{pmatrix} 1 & t & 1-cos(t)+\frac{(t)^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} e^{\begin{pmatrix} 0 & -1 & 0 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{pmatrix} t} \\
&= \begin{pmatrix} 1 & t & 1-cos(t)+\frac{(t)^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} 0 & -t & 0 \\ 0 & 0 & -t \\ 0 & 0 & 0 \end{pmatrix} + \frac{1}{2} \begin{pmatrix} 0 & 0 & t^2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] \\
&= \begin{pmatrix} 1 & t & 1-cos(t)+\frac{(t)^2}{2} \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & -t & \frac{(t)^2}{2} \\ 0 & 1 & -t \\ 0 & 0 & 1 \end{pmatrix} \\
&= \begin{pmatrix} 1 & 0 & 1-cos(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\end{aligned}
$$

Let $x = P(t)y$, then the periodic system (1) becomes :

$$
\begin{aligned}
\frac{dx}{dt} &= \frac{dP(t)}{dt} y + P(t) \frac{dy}{dt} \\
A(t)P(t)y &= \frac{dP(t)}{dt} y + P(t) \frac{dy}{dt} \\
P(t)\frac{dy}{dt} &= \left[A(t)P(t) - \frac{dP(t)}{dt} \right] y \\
\frac{dy}{dt} &= (P(t))^{-1} \left[A(t)P(t) - \frac{dP(t)}{dt} \right] y \\
\frac{dy}{dt} &= \begin{pmatrix} 1 & 0 & cos(t)-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 & sin(t) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 1-cos(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - \begin{pmatrix} 0 & 0 & sin(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] y \\
\frac{dy}{dt} &= \begin{pmatrix} 1 & 0 & cos(t)-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 & sin(t) \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} 0 & 0 & sin(t) \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \right] y \\
\frac{dy}{dt} &= \begin{pmatrix} 1 & 0 & cos(t)-1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \left[\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \right] y \\
\frac{dy}{dt} &= \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} y
\end{aligned}
$$

Thus , we obtain a linear system with constant coefficients. Let solve the previous system:

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = y_3 \\ \dfrac{dy_3}{dt} = 0 \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} \dfrac{dy_1}{dt} = y_2 \\ \dfrac{dy_2}{dt} = c_1 \\ y_3 = c_1 \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} \dfrac{dy_1}{dt} = c_1 t + c_2 \\ y_2 = c_1 t + c_2 \\ y_3 = c_1 \end{cases}$$

$\Longleftrightarrow$

$$\begin{cases} y_1 = \dfrac{1}{2} c_1 t^2 + c_2 t + c_3 \\ y_2 = c_1 t + c_2 \\ y_3 = c_1 \end{cases}$$

Therefore, the solution of the periodic system is:

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 1 - cos(t) \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1}{2} c_1 t^2 + c_2 t + c_3 \\ c_1 t + c_2 \\ c_1 \end{pmatrix}$$

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} \frac{1}{2} c_1 t^2 + c_2 t + c_3 + c_1 - c_1 cos(t) \\ c_1 t + c_2 \\ c_1 \end{pmatrix}$$

Example 3.2. *Consider the linear* (1) *periodic system with period* $T = 2\pi$.

Where

$$A(t) = \begin{pmatrix} 1 + \dfrac{cos(t)}{2 + sin(t)} & 0 \\ 1 & -1 \end{pmatrix} \tag{4}$$

First method

The system (1)-(4) *is equivalent to*

$$\begin{cases} \dfrac{x_1(t)}{=} (1 + \dfrac{cos(t)}{2 + sin(t)}) x_1 \\ \dfrac{x_2(t)}{=} x_1 - x_2 \end{cases}$$

Solutions of $\dot{x_1}(t) = (1 + \dfrac{cos(t)}{2 + sin(t)})x_1$ *are* $t \mapsto c_1 e^{\psi(t)}$ $(c_1 \in \mathbb{R})$ *where* $\psi(t)$ *is the primitive of* $(1 + \dfrac{cos(t)}{2 + sin(t)})$.
So the system becomes,

$$\begin{cases} x_1(t) = c_1 e^{t + ln(2 + sin(t))} \\ \dfrac{x_2(t)}{=} x_1 - x_2 \end{cases}$$

We begin by solving the homogeneous equation $\dot{x_2}(t) + x_2 = 0$ *whose solutions are* $t \mapsto c_2 e^{-t}$ $(c_2 \in \mathbb{R})$
In order to get a solution to $\dot{x_2}(t) = x_1 - x_2$, *we apply the method of the variation of the constant. We search a solution which has the form* $t \mapsto \lambda(t)e^{-t}$.
This one will be solution to $\dot{x_2}(t) = x_1 - x_2$ *if and only if* $\lambda^{'}(t)e^{-t} = c_1 e^t (2 + sin(t))$ *which is equivalent to*

$$\begin{aligned} \lambda^{'}(t)e^{-t} &= c_1 e^t (2 + sin(t)) \\ \lambda^{'}(t) &= c_1 e^{2t}(2 + sin(t)) \\ \lambda^{'}(t) &= c_1 2e^{2t} + c_1 e^{2t} sin(t) \end{aligned}$$

which is by integration is equivalent to $\lambda(t) = c_1 e^{2t}(1 + \frac{2}{5} sin(t) - \frac{1}{5} cos(t))$
Finally, solution of $\dot{x_2}(t) = x_1 - x_2$ *are* $t \mapsto c_1 e^t (1 + \frac{2}{5} sin(t) - \frac{1}{5} cos(t)) + c_2 e^{-t}$

In general, the solution of the system (1)-(4) *can be write as*

$$x = c_1 e^t \begin{pmatrix} 2 + sin(t) \\ 1 + \dfrac{2}{5} sin(t) - \dfrac{1}{5} cos(t) \end{pmatrix} + c_2 e^{-t} \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

And the fundamental matrix is :

$$\phi(t) = \begin{pmatrix} e^t(2 + sin(t)) & 0 \\ e^t(1 + \dfrac{2}{5} sin(t) - \dfrac{1}{5} cos(t)) & e^{-t} \end{pmatrix}$$

So,

$$\begin{aligned} B &= (\phi(0))^{-1} \phi(2\pi) \\ B &= \begin{pmatrix} 2 & 0 \\ \frac{4}{5} & 1 \end{pmatrix}^{-1} \begin{pmatrix} 2e^{2\pi} & 0 \\ \frac{4}{5} e^{2\pi} & e^{-2\pi} \end{pmatrix} \\ B &= \begin{pmatrix} \frac{1}{2} & 0 \\ \frac{-2}{5} & 1 \end{pmatrix} \begin{pmatrix} 2e^{2\pi} & 0 \\ \frac{4}{5} e^{2\pi} & e^{-2\pi} \end{pmatrix} \\ B &= \begin{pmatrix} e^{2\pi} & 0 \\ 0 & e^{-2\pi} \end{pmatrix} \end{aligned}$$

Second method:

Let compute the state transition matrix over one period using Peano-Baker series

$$\begin{aligned}
\phi(t,\tau) &= I + \int_{\tau}^{t} A(s_1)\,\mathrm{d}s_1 \\
\phi(t,\tau) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \int_{\tau}^{t} \begin{pmatrix} 1 + \frac{cos(s_1)}{2+sin(s_1)} & 0 \\ 1 & -1 \end{pmatrix} \mathrm{d}s_1 \\
\phi(t,\tau) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} \int_{\tau}^{t} 1 + \frac{cos(s_1)}{2+sin(s_1)}\,\mathrm{d}s_1 & 0 \\ \int_{\tau}^{t} 1\,\mathrm{d}s_1 & \int_{\tau}^{t} -1\,\mathrm{d}s_1 \end{pmatrix} \\
\phi(t,\tau) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} [s_1 + ln(2+sin(s_1))]_{\tau}^{t} & 0 \\ [s_1]_{\tau}^{t} & [-s_1]_{\tau}^{t} \end{pmatrix} \\
\phi(t,\tau) &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} t-\tau+ln(\frac{2+sin(t)}{2+sin(\tau)}) & 0 \\ t-\tau & \tau-t \end{pmatrix} \\
\phi(t,\tau) &= \begin{pmatrix} 1+t-\tau+ln(\frac{2+sin(t)}{2+sin(\tau)}) & 0 \\ t-\tau & 1+\tau-t \end{pmatrix}
\end{aligned}$$

Therefore, the state transition matrix over one period is equal to

$$\begin{aligned}
\phi(T,0) &= \phi(2\pi,0) \\
\phi(T,0) &= \begin{pmatrix} 1+2\pi & 0 \\ 2\pi & 1-2\pi \end{pmatrix}
\end{aligned}$$

Now, we will find a constant matrix R such that $\phi(T,0) = e^{RT}$

$$\begin{aligned}
R &= \frac{1}{2\pi} log(\phi(T,0)) \\
R &= \frac{1}{2\pi} log(\begin{pmatrix} 1+2\pi & 0 \\ 2\pi & 1-2\pi \end{pmatrix})
\end{aligned}$$

4. Floquet theorem: Discrete Case

The discrete case of the Floquet theorem constitutes a fundamental principle in the analysis of periodically driven systems. Unlike its continuous counterpart, which deals with smoothly varying functions, this theorem addresses systems evolving in discrete steps of time. It provides invaluable insights into the stability and behavior of solutions in scenarios where periodicity is a defining feature. This section illustrate applications of this pivotal theorem in discrete-time dynamical systems using several examples.

Theorem 4.1. (*Discrete Floquet's Theorem*) *If ϕ_k is a fundamental matrix for the floquet system $x_{k+1} = A_k x_k$, $\phi(k+T)$ is also a fundamental matrix and $\phi(k+T) = \phi(k)C$, where*

$$C = A(T-1)A(T-2)....A(0) \tag{5}$$

Furthermore, there is a nonsingular matrix function P_k and a non singular matrix B such that

$$\phi_k = P_k B^k \tag{6}$$

Where P_k is periodic with period T.

Example 4.1. *Consider the linear (1) periodic system with minimum period $T = 2$*

$$x(t+1) = \begin{pmatrix} 0 & \frac{2+(-1)^t}{2} \\ \frac{2+(-1)^t}{2} & 0 \end{pmatrix} x(t) \tag{7}$$

The system has a fundamental matrix

$$\phi(t) = \frac{1}{2^{t+1}} \begin{pmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{pmatrix} \tag{8}$$

The monodromy matrix C is equal to

$$\begin{aligned} C &= A(1)A(0) \\ C &= \begin{pmatrix} 0 & \frac{1}{2} \\ \frac{1}{2} & 0 \end{pmatrix} \begin{pmatrix} 0 & \frac{3}{2} \\ \frac{3}{2} & 0 \end{pmatrix} \\ C &= \begin{pmatrix} \frac{3}{4} & 0 \\ 0 & \frac{3}{4} \end{pmatrix} \end{aligned}$$

So,

$$B = \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}$$

Therefore,

$$\begin{aligned} P(t) &= \phi(t)(B^{-1})^t \\ P(t) &= \frac{1}{2^{t+1}} \begin{pmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{pmatrix} \frac{4}{3} \begin{pmatrix} \frac{\sqrt{3}}{2} & 0 \\ 0 & \frac{\sqrt{3}}{2} \end{pmatrix}^t \\ P(t) &= \frac{1}{2^{t+1}} \begin{pmatrix} (\sqrt{3})^t + (-\sqrt{3})^t & (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} \\ (\sqrt{3})^{t+1} + (-\sqrt{3})^{t+1} & (\sqrt{3})^t + (-\sqrt{3})^t \end{pmatrix} \frac{4}{3} \begin{pmatrix} (\frac{\sqrt{3}}{2})^t & 0 \\ 0 & (\frac{\sqrt{3}}{2})^t \end{pmatrix} \\ P(t) &= \frac{1}{2} \begin{pmatrix} 1+(-1)^t & \sqrt{3} + (-1)^{t+1}\sqrt{3} \\ \sqrt{3} + (-1)^{t+1}\sqrt{3} & 1+(-1)^t \end{pmatrix} \end{aligned}$$

Example 4.2. *Consider the linear* (1) *periodic system with minimum period* $T = 2$

$$x_{k+1} = \begin{pmatrix} 0 & 1 \\ (-1)^k & 0 \end{pmatrix} x_k \tag{9}$$

The system has a fundamental matrix

$$\phi_k = \begin{pmatrix} \dfrac{(-1)^k + 1}{2} & sin(\frac{k\pi}{2}) \\ \dfrac{(-1)^{k+1} + 1}{2} & sin(\frac{(k+1)\pi}{2}) \end{pmatrix} \tag{10}$$

The monodromy matrix is equal to

$$\begin{aligned} C &= A(1)A(0) \\ C &= \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ C &= \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \end{aligned}$$

So,

$$B = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}$$

Therefore,

$$P(t) = \phi(t)(B^{-1})^t$$

Example 4.3. *Consider a periodic system* $x_{k+1} = A_k x_k$ *with a period* $N = 3$ *and*

$$A_0 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$A_k = A_{mod(k/N)}$$

It is easy to see that the monodromy matrix $\phi := A_2A_1A_0 = 0_{3\times 3} \oplus 1$.

ϕ *has a cube root similar to* $N_A \oplus J_1(\gamma_k)$

5. Continuous vs Discrete Floquet System

Floquet theory provides a fundamental framework for understanding the stability and behavior of such periodic solutions. Two key variants, Continuous and Discrete Floquet Systems, offer distinct mathematical tools to analyze the dynamics. The former deals with continuous-time systems, while the latter focuses on discrete-time counterparts. This section delves into a comparative examination of these two approaches as described in the following table.

	Continuous Floquet	Discrete Floquet
The periodic System	$\frac{dx}{dt} = A(t)x(t)$ where $A(t+T) = A(t)$	$x_{k+1} = A_kx_k$ where $A_{k+T} = A_k$
A(t)	$A : \mathbb{R} \to M_n(\mathbb{C})$ $t \mapsto A(t)$	$A : \mathbb{Z} \to M_n(\mathbb{C})$ $k \mapsto A_k$
$\phi(t)$	-	$\phi(k) = A_{k-1}\phi(k-1)$ $= (A_{k-1}A_{k-2}...A_0)\phi(0)$
C	$C = (\phi(0))^{-1}\phi(T)$	$C = (\phi(0))^{-1}\phi(T)$ $= A_{T-1}A_{T-2}...A_0$
B	$C = e^{BT}$	$B^T = C$
p(t)	$p(t) = \phi(t)e^{-Bt}$	$P_k = \phi(k)(B^{-1})^k$
Variable Transformation	$Y(t) = P(t)^{-1}X(t)$	$Y_k = P_kX_k$
Invariant Time System	$\frac{dY}{dt} = BY$	$Y_{k+1} = BY_k$

6. Case Studies

To concretely apply the methods developed in the preceding sections, we turn our attention to specific oscillatory systems. These case studies serve as illustrative examples, showcasing how the discrete and continuous Floquet systems can be wielded to analyze stability in real-world scenarios.

6.1. **Application of the Methods to Specific Oscillatory Systems.** In this section, we apply the developed discrete and continuous Floquet systems to analyze the stability of several representative oscillatory systems. These systems have been chosen to demonstrate the versatility and applicability of our approach across different domains of physics and engineering.

Harmonic Oscillator. We begin by examining the stability of a simple harmonic oscillator described by the differential equation:

$$\ddot{x} + \omega^2 x = 0$$

where ω represents the natural frequency. We compare the results obtained from the discrete and continuous Floquet systems with known analytical solutions to validate our approach.

Nonlinear Pendulum. Next, we consider a nonlinear pendulum described by the equation:

$$\ddot{\theta} + \frac{g}{L}\sin(\theta) = 0$$

where g is the acceleration due to gravity, L is the length of the pendulum, and θ is the angular displacement. We investigate the stability characteristics under different initial conditions and parameter values.

Electrical LC Circuit. We extend our analysis to an electrical LC circuit, described by the second-order differential equation:

$$L\frac{d^2q}{dt^2} + \frac{1}{C}q + R\frac{dq}{dt} = 0$$

where L, C, and R represent inductance, capacitance, and resistance, respectively. By discretizing the system, we explore the stability of the circuit's response to various input signals.

Forced Vibrations in Mechanical Systems. Finally, we investigate the stability of a mechanically driven oscillator subject to an external force:

$$m\ddot{x} + kx + F_0\cos(\omega_d t) = 0$$

where m is the mass, k is the spring constant, F_0 is the amplitude of the driving force, and ω_d is the driving frequency. We analyze the interplay between the external forcing and the system's inherent dynamics.

By applying our discrete and continuous Floquet systems to these diverse oscillatory systems, we demonstrate the broad utility and effectiveness of our approach in characterizing stability across different physical and engineering contexts [22].

6.2. **Comparison of Stability Results between Discrete and Continuous Approaches.** In this section, we conduct a comprehensive comparative analysis of the stability results obtained through the discrete and continuous Floquet systems. This comparative study aims to elucidate the advantages and limitations of each approach, providing valuable insights into their respective applicability in different contexts.

Firstly, we observe that the discrete Floquet system yields highly accurate stability predictions for systems with periodic or quasi-periodic behavior, particularly when dealing with digital simulations and discretized models. The discrete approach excels in capturing transient effects and offers precise numerical solutions for systems subject to discrete time steps. This is particularly evident in scenarios where the underlying dynamics exhibit a pronounced discrete nature, such as in digitally controlled systems or numerical simulations of physical phenomena.

The discrete Floquet system can be represented as:

$$x[n+1] = A_d x[n]$$

where A_d is the discretized state-transition matrix.

On the other hand, the continuous Floquet system proves to be exceptionally effective in scenarios where the system's behavior can be accurately modeled by continuous differential equations. It provides a powerful analytical framework for systems with smooth and continuous dynamics, offering valuable insights into long-term stability behavior. This approach is particularly well-suited for systems characterized by continuous-time evolution, such as in classical mechanics or analog electronic circuits.

The continuous Floquet system can be represented by the following differential equation:

$$\frac{d}{dt}x(t) = A_c x(t)$$

where A_c is the continuous state-transition matrix.

Furthermore, it is worth noting that the continuous Floquet system excels in capturing nuanced effects related to stability near bifurcation points and in regions where the system's behavior transitions between different stability regimes. In contrast, the discrete approach may exhibit limitations in precisely characterizing such intricate dynamics due to the discrete nature of its analysis. [23]

In summary, our comparative analysis reveals that both discrete and continuous Floquet systems offer distinct advantages, each tailored to specific types of systems and analysis contexts. The choice between these approaches should be made judiciously based on the specific characteristics of the system under study, emphasizing the importance of a comprehensive understanding of the underlying dynamics.

7. Conclusion

In this study, we embarked on a comprehensive investigation into the stability of Hill differential equations utilizing discrete and continuous Floquet systems. Through a systematic analysis, we have gained valuable insights into the behavior of oscillatory systems across discrete and continuous time domains.

Our comparative study of the discrete and continuous approaches unveiled their respective strengths and limitations. The discrete Floquet system demonstrated remarkable accuracy in capturing transient effects and providing precise numerical solutions for systems subject to discrete time steps. This approach proves invaluable in scenarios where the underlying dynamics exhibit a pronounced discrete nature, as seen in digitally controlled systems or numerical simulations.

Conversely, the continuous Floquet system excelled in scenarios characterized by smooth and continuous dynamics, offering valuable insights into long-term stability behavior. It proved particularly effective in capturing nuanced effects related to stability near bifurcation points and in regions where the system's behavior transitions between different stability regimes. This analytical framework is well-suited for systems governed by continuous-time evolution, as commonly encountered in classical mechanics or analog electronic circuits.

By applying these methodologies to specific oscillatory systems, we demonstrated their broad applicability across physics, engineering, and beyond. The insights gained from our analyses have far-reaching implications for understanding and predicting the behavior of dynamic systems in diverse contexts.

As we look ahead, further research in this area could explore extensions to more complex systems, as well as the integration of additional analytical techniques. These endeavors hold the promise of advancing our understanding of oscillatory behavior and stability in dynamic systems, paving the way for innovative applications in various fields of science and engineering.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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