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SPECTRAL POWER GRAPH OF GENERALIZED QUATERNION GROUPS

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Abstract. This paper examines the spectral properties of the power graph of the generalized quaternion group. We focus on the characteristic polynomial of the graph associated with adjacency, Laplacian, and signless Laplacian matrices.

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Introduction

There are various types of spectral graph theory, including adjacency, Laplacian, and signless Laplacian matrices. An analysis of the spectrum of these matrices can provide information regarding the graph. In this paper, the spectra properties of the power graph are examined, one of the finite groups that may be represented by graphs.

A power graph of the group G is denoted by Γ_G and defined as a graph whose vertex set is all the elements of G and two distinct vertices v_p and v_q are adjacent if and only if $v_p^x = v_q$ or $v_q^y = v_p$ for positive integers x and y [4]. Further discussion on this topic can be found in Ali, et al. [2] which discussed some topological indices of the power graph of dihedral and generalized quaternion groups. Meanwhile, the degree of a vertex of this graph is presented by Sehgal and Singh [7].

Several authors have presented the spectral problem of the graph, including the commuting and non-commuting graphs for the dihedral group, which can be seen in [8–11]. Accordingly, Romdhini et al. [12] investigated the spectra theory of the power graph for dihedral groups. Therefore, this research

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aims to formulate the characteristic polynomial of the power graph for the generalized quaternion group associated with adjacency, Laplacian, and signless Laplacian matrices.

This paper is structured as follows: Section 1 introduces the aim of our research. In Section 2, we provide an overview of important background information that will be utilized throughout this work. Moving on to Section 3, we investigate the characteristic polynomial of the power graph for the generalized quaternion group. Lastly, in Section 4, we conclude the main results of this research.

1. Preliminaries

In this section, we present some fundamental definitions and preliminary concepts that will be consistently applied throughout the entirety of our work.

Definition 1.1. [5] Generalized quaternion group (Q_{4n}) with $n \geq 2$ is defined by

$$\left\langle a, b | a^{2n} = e, a^n = b^2, b^{-1}ab = a^{-1} \right\rangle.$$

Throughout the discussion, we denote $\Gamma_{Q_{4n}}$ as the power graph of Q_{4n}. Now, we divide Q_{4n} into three sets for further discussion. Let $V_1 = \{e, a^n\}$, $V_2 = \{a^i b : 0 \leq i \leq 2n-1\}$, and $V_3 = \{a^i : 1 \leq i \leq 2n\} \backslash V_1$. It is clearly that $|V_1| = 2$, $|V_2| = 2n$ and $|V_3| = 2n-2$.

Furthermore, Ali et al [1] have described the degree of every vertex in $\Gamma_{Q_{4n}}$ which are beneficial to construct the Laplacian and signless Laplacian matrices of $\Gamma_{Q_{4n}}$. It is presented below:

Theorem 1.2. [1] *Let $\Gamma_{Q_{4n}}$ be a power graph of Q_{4n} and $deg(v)$ be the degree of vertex v, then*

(1) $deg(v) = 4n-1, \forall v \in V_1$,
(2) $deg(v) = 3, \forall v \in V_2$,
(3) $deg(v) = 2n-1, \forall v \in V_3$.

The foundational theories of the adjacency, Laplacian, and signless Laplacian matrices of $\Gamma_{Q_{4n}}$ are presented below.

Definition 1.3. [3] The adjacency (A) matrix of $\Gamma_{Q_{4n}}$ denoted by $A(\Gamma_{Q_{4n}}) = [a_{ij}]$, where

$$a_{ij} = \begin{cases} 1, & \text{if } v_i \neq v_j \text{ and they are adjacent} \\ 0, & \text{otherwise.} \end{cases}$$

To construct matrices in Definition 1.4 and 1.5, we need the definition of the degree matrix of $\Gamma_{Q_{4n}}$. It is defined as the diagonal matrix of vertex degree of $\Gamma_{Q_{4n}}$ and denoted by $D(\Gamma_{Q_{4n}})$. Now we are moving to the definition of Laplacian and signless Laplacian matrices.

Definition 1.4. [3] The Laplacian (L) matrix of order $n \times n$ associated with $\Gamma_{Q_{4n}}$ is given by $L(\Gamma_{Q_{4n}}) = D(\Gamma_{Q_{4n}}) - A(\Gamma_{Q_{4n}})$.

Definition 1.5. [3] The signless Laplacian (SL) matrix of order $n \times n$ associated with $\Gamma_{Q_{4n}}$ is given by $SL(\Gamma_{Q_{4n}}) = D(\Gamma_{Q_{4n}}) + A(\Gamma_{Q_{4n}})$.

The characteristic polynomial of $A(\Gamma_{Q_{4n}})$ is the determinant of the matrix $\lambda I_n - A(\Gamma_{Q_{4n}})$ and is denoted by $P_{A(\Gamma_{Q_{4n}})}(\lambda)$. The eigenvalues of $A(\Gamma_{Q_{4n}})$ are the roots of $P_{A(\Gamma_{Q_{4n}})}(\lambda) = 0$. Similar definition for matrices $L(\Gamma_{Q_{4n}})$ and $SL(\Gamma_{Q_{4n}})$. In order to formulate $P_{A(\Gamma_{Q_{4n}})}(\lambda)$, $P_{L(\Gamma_{Q_{4n}})}(\lambda)$, and $P_{SL(\Gamma_{Q_{4n}})}(\lambda)$, we need the following theorem to simplify the process.

Theorem 1.6. [6] *If a square matrix* $M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$ *can be partitioned into four blocks where* $|A| \neq 0$, *then the determinant of* M *is*

$$|M| = \begin{vmatrix} A & B \\ O & D - CA^{-1}B \end{vmatrix} = |A| \left| D - CA^{-1}B \right|.$$

Next, the row and column operations are used to prove the main results. Therefore, it is sufficient to write the notation. R_i is the i-th row, $R_i^{'}$ is the new i-th row provided from row operation. Meanwhile, C_i is the i-th column and $C_i^{'}$ is its new column from column operation.

2. Main Results

Within this section, we present the formulation of the characteristic polynomial of $\Gamma_{Q_{4n}}$ throughout the entirety of our work. We focus on the adjacency, Laplacian, and signless Laplacian matrices.

Theorem 2.1. *Let* $\Gamma_{Q_{4n}}$ *be the power graph for* Q_{4n}, *then the characteristic polynomial of* $A(\Gamma_{Q_{4n}})$ *is*

$$\begin{aligned} P_{A(\Gamma_{Q_{4n}})}(\lambda) =& (\lambda^4 - 2(n-1)\lambda^3 - 6n\lambda^2 + 2(4n^2 - 7n - 1)\lambda + 8n^2 - 10n - 1) \\ & (\lambda + 1)^{3n-3}(\lambda - 1)^n. \end{aligned}$$

Proof. From Theorem 1.2 we know that the vertex of V_1 is adjacent to all other vertices in $\Gamma_{Q_{4n}}$. Vertices in V_2 are adjacent to vertices in V_1 and two vertices a^ib and $a^{n+i}b$ are always adjacent, for $0 \leq i \leq n-1$. Meanwhile, every vertex in V_3 is adjacent to all other vertices in V_3. Following Definition 1.3, we can construct $A(\Gamma_{Q_{4n}})$ of the size $2n \times 2n$ as given below:

$$
\begin{array}{c|ccccccccccccccccc}
 & e & a^n & a & \ldots & a^{n-1} & a^{n+1} & \ldots & a^{2n-1} & b & ab & \ldots & a^{n-1}b & a^n b & a^{n+1}b & \ldots & a^{2n-1}b \\
\hline
e & 0 & 1 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
a^n & 1 & 0 & 1 & \ldots & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 & 1 & 1 & \ldots & 1 \\
a & 1 & 1 & 0 & \ldots & 1 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1} & 1 & 1 & 1 & \ldots & 0 & 1 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
a^{n+1} & 1 & 1 & 1 & \ldots & 1 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1} & 1 & 1 & 1 & \ldots & 1 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
b & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 \\
ab & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{n-1}b & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 \\
a^n b & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
a^{n+1}b & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\
a^{2n-1}b & 1 & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0
\end{array} \tag{2.1}
$$

Matrix $A(\Gamma_{Q_{4n}})$ can be partitioned into 16 block matrices as follows:

$$
A(\Gamma_{Q_{4n}}) = \begin{bmatrix} (J-I)_2 & J_{2\times(2n-2)} & J_{2\times n} & J_{2\times n} \\ J_{(2n-2)\times 2} & (J-I)_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\ J_{n\times 2} & 0_{n\times(2n-2)} & 0_n & I_n \\ J_{n\times 2} & 0_{n\times(2n-2)} & I_n & 0_n \end{bmatrix}.
$$

The characteristic polynomial of $A(\Gamma_{Q_{4n}})$ is $|\lambda I_n - A(\Gamma_{Q_{4n}})|$ as follows:

$$
P_{A(\Gamma_{Q_{4n}})}(\lambda) = \begin{vmatrix} (\lambda+1)I_2 - J_2 & -J_{2\times(2n-2)} & -J_{2\times n} & -J_{2\times n} \\ -J_{(2n-2)\times 2} & (\lambda+1)_{2n-2} - J_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\ -J_{n\times 2} & 0_{n\times(2n-2)} & \lambda I_n & I_n \\ -J_{n\times 2} & 0_{n\times(2n-2)} & -I_n & \lambda I_n \end{vmatrix}. \tag{2.2}
$$

We apply the following steps to simplify the determinant in Equation 2.2 as given below:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{2n+i}$, for $i = 1, 2, \ldots, n$.

(2) $C_{2n+i} \longrightarrow C_{2n+i} + C_{3n+i}$, for $i = 1, 2, \ldots, n$.

(3) $R_{2n+1+i} \longrightarrow R_{2n+1+i} - R_{2n+1}$, for $i = 1, 2, \ldots, n-1$.

(4) $C_{2n+1} \longrightarrow C_{2n+1} + C_{2n+2} + \ldots + C_{3n}$.

(5) $C_1 \longrightarrow C_1 + \frac{1}{\lambda-1} C_{2n+1}$.

(6) $C_2 \longrightarrow C_2 + \frac{1}{\lambda-1} C_{2n+1}$.

(7) $R_{3+i} \longrightarrow R_{3+i} - R_3$, for $i = 1, 2, \ldots, 2n-3$.

(8) $C_3 \longrightarrow C_3 + C_4 + \ldots + C_{2n}$.

Then $P_{A(\Gamma_{Q_{4n}})}(\lambda)$ can be expressed as

$$\begin{vmatrix} \lambda - \frac{2n}{\lambda-1} & -1-\frac{2n}{\lambda-1} & 2-2n & -J_{1\times(2n-3)} & -2n & -2J_{1\times(n-1)} & -1 & -J_{1\times(n-1)} \\ -1-\frac{2n}{\lambda-1} & \lambda-\frac{2n}{\lambda-1} & 2-2n & -J_{1\times(2n-3)} & -2n & -2J_{1\times(n-1)} & -1 & -J_{1\times(n-1)} \\ -1 & -1 & \lambda-(2n-3) & -J_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} \\ 0 & 0 & 0_{(2n-3)\times 1} & (\lambda+1)I_{2n-3} & 0_{(2n-3)\times 1} & 0_{(2n-3)\times(n-1)} & 0 & 0_{(2n-3)\times(n-1)} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & \lambda-1 & 0_{1\times(n-1)} & -1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & (\lambda-1)I_{n-1} & J_{(n-1)\times 1} & -I_{n-1} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & \lambda+1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda+1)I_{n-1} \end{vmatrix}.$$

Consequently, by Theorem 1.6, we can obtain the characteristic polynomial of $A(\Gamma_{Q_{4n}})$ as follows:

$$P_{A(\Gamma_{Q_{4n}})}(\lambda) = \begin{vmatrix} \lambda-\frac{2n}{\lambda-1} & -1-\frac{2n}{\lambda-1} & 2-2n \\ -1-\frac{2n}{\lambda-1} & \lambda-\frac{2n}{\lambda-1} & 2-2n \\ -1 & -1 & \lambda-(2n-3) \end{vmatrix} (\lambda+1)^{3n-3}(\lambda-1)^n.$$

It can be seen that the first determinant is a 3×3 matrix, therefore, we can get

$$\begin{aligned} P_{A(\Gamma_{Q_{4n}})}(\lambda) =& (\lambda^4 - 2(n-1)\lambda^3 - 6n\lambda^2 + 2(4n^2 - 7n - 1)\lambda + 8n^2 - 10n - 1) \\ & (\lambda+1)^{3n-3}(\lambda-1)^n. \end{aligned}$$

□

Theorem 2.2. *Let $\Gamma_{Q_{4n}}$ be the power graph for Q_{4n}, then the characteristic polynomial of $L(\Gamma_{Q_{4n}})$ is*

$$P_{L(\Gamma_{Q_{4n}})}(\lambda) = \lambda(\lambda-4n)^2(\lambda-2n)^{2n-3}(\lambda-2)^n(\lambda-4)^n.$$

Proof. The Laplacian matrix of $\Gamma_{Q_{4n}}$ construction depends on the degree and adjacency matrices of $\Gamma_{Q_{4n}}$. The $4n \times 4n$ degree matrix of $\Gamma_{Q_{4n}}$, $D(\Gamma_{Q_{4n}})$, is as follows

$$\begin{array}{c} \\ e \\ a^n \\ a \\ \vdots \\ a^{n-1} \\ a^{n+1} \\ \vdots \\ a^{2n-1} \\ b \\ \vdots \\ a^{n-1}b \\ a^nb \\ \vdots \\ a^{2n-1}b \end{array} \begin{array}{c} \begin{array}{ccccccccccccccc} e & a^n & a & \ldots & a^{n-1} & a^{n+1} & \ldots & a^{2n-1} & b & \ldots & a^{n-1}b & a^nb & \ldots & a^{2n-1}b \end{array} \\ \begin{pmatrix} 4n-1 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 4n-1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 2n-1 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 2n-1 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 2n-1 & \ldots & 1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 2n-1 & 0 & \ldots & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 3 & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 3 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 3 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 0 & 0 & \ldots & 3 \end{pmatrix} \end{array}. \quad (2.3)$$

Based on Definition 1.4, the Laplacian matrix of $\Gamma_{Q_{4n}}$ is $L(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) - A(\Gamma_{D_{2n}})$ as follows:

$$
\begin{array}{c}
e \\ a^n \\ a \\ \vdots \\ a^{n-1} \\ a^{n+1} \\ \vdots \\ a^{2n-1} \\ b \\ \vdots \\ a^{n-1}b \\ a^nb \\ \vdots \\ a^{2n-1}b
\end{array}
\begin{array}{c}
\begin{array}{cccccccccccccccc}
e & a^n & a & \dots & a^{n-1} & a^{n+1} & \dots & a^{2n-1} & b & \dots & a^{n-1}b & a^nb & \dots & a^{2n-1}b
\end{array} \\
\begin{pmatrix}
4n-1 & -1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & 4n-1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 & -1 & \dots & -1 \\
-1 & -1 & 2n-1 & \dots & -1 & -1 & \dots & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \dots & 2n-1 & -1 & \dots & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\
-1 & -1 & -1 & \dots & -1 & 2n-1 & \dots & -1 & 0 & \dots & 0 & 0 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & -1 & \dots & -1 & -1 & \dots & 2n-1 & 0 & \dots & 0 & 0 & \dots & 0 \\
-1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 3 & \dots & 0 & -1 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 3 & 0 & \dots & -1 \\
-1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & -1 & \dots & 0 & 3 & \dots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
-1 & -1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & -1 & 0 & \dots & 3
\end{pmatrix}
\end{array}.
$$

Moreover, $L(\Gamma_{Q_{4n}})$ can be partitioned into 16 block matrices as given below:

$$
L(\Gamma_{Q_{4n}}) = \begin{bmatrix}
4nI_2 - J_2 & -J_{2\times(2n-2)} & -J_{2\times n} & -J_{2\times n} \\
-J_{(2n-2)\times 2} & 2nI_{2n-2} - J_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\
-J_{n\times 2} & 0_{n\times(2n-2)} & 3I_n & -I_n \\
-J_{n\times 2} & 0_{n\times(2n-2)} & -I_n & 3I_n
\end{bmatrix}.
$$

The characteristic polynomial of $L(\Gamma_{Q_{4n}})$ can be obtained from the following determinant:

$$
P_{L(\Gamma_{Q_{4n}})}(\lambda) = \begin{vmatrix}
(\lambda - 4n)I_2 + J_2 & J_{2\times(2n-2)} & J_{2\times n} & J_{2\times n} \\
J_{(2n-2)\times 2} & (\lambda - 2n)I_{2n-2} + J_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\
J_{n\times 2} & 0_{n\times(2n-2)} & (\lambda - 3)I_n & I_n \\
J_{n\times 2} & 0_{n\times(2n-2)} & I_n & (\lambda - 3)I_n
\end{vmatrix}. \tag{2.4}
$$

The row and column operations apply to Equation 2.4 as follows:

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{2n+i}$, for $i = 1, 2, \dots, n$.

(2) $C_{2n+i} \longrightarrow C_{2n+i} + C_{3n+i}$, for $i = 1, 2, \dots, n$.

(3) $R_{2n+1+i} \longrightarrow R_{2n+1+i} - R_{2n+1}$, for $i = 1, 2, \dots, n-1$.

(4) $C_{2n+1} \longrightarrow C_{2n+1} + C_{2n+2} + \dots + C_{3n}$.

(5) $C_1 \longrightarrow C_1 - \frac{1}{\lambda-2} C_{2n+1}$.

(6) $C_2 \longrightarrow C_2 - \frac{1}{\lambda-2} C_{2n+1}$.

(7) $R_{3+i} \longrightarrow R_{3+i} - R_3$, for $i = 1, 2, \dots, 2n-3$.

(8) $C_3 \longrightarrow C_3 + C_4 + \dots + C_{2n}$.

(9) $C_1 \longrightarrow C_1 - \frac{1}{\lambda-2} C_3$.

(10) $C_2 \longrightarrow C_2 - \frac{1}{\lambda-2} C_3$.

Then after performing row and column operations, we get $P_{L(\Gamma_{Q_{4n}})}(\lambda)$ as

$$\begin{vmatrix} \frac{(\lambda-4n)(\lambda-1)}{\lambda-2} & \frac{\lambda-4n}{\lambda-2} & 2n-2 & J_{1\times(2n-3)} & 2n & 2J_{1\times(n-1)} & 1 & J_{1\times(n-1)} \\ \frac{\lambda-4n}{\lambda-2} & \frac{(\lambda-4n)(\lambda-1)}{\lambda-2} & 2n-2 & J_{1\times(2n-3)} & 2n & 2J_{1\times(n-1)} & 1 & J_{1\times(n-1)} \\ 0 & 0 & \lambda-2 & J_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} \\ 0 & 0 & 0_{(2n-3)\times 1} & (\lambda-2n)I_{2n-3} & 0_{(2n-3)\times 1} & 0_{(2n-3)\times(n-1)} & 0 & 0_{(2n-3)\times(n-1)} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & \lambda-2 & 0_{1\times(n-1)} & 1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & (\lambda-2)I_{n-1} & -J_{(n-1)\times 1} & I_{n-1} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & \lambda-4 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-4)I_{n-1} \end{vmatrix}.$$

Following Theorem 1.6, we then can obtain

$$\begin{aligned} P_{L(\Gamma_{Q_{4n}})}(\lambda) &= \begin{vmatrix} \frac{(\lambda-4n)(\lambda-1)}{\lambda-2} & \frac{\lambda-4n}{\lambda-2} \\ \frac{\lambda-4n}{\lambda-2} & \frac{(\lambda-4n)(\lambda-1)}{\lambda-2} \end{vmatrix} (\lambda-2n)^{2n-3}(\lambda-2)^{n+1}(\lambda-4)^{n} \\ &= \lambda(\lambda-4n)^2(\lambda-2n)^{2n-3}(\lambda-2)^{n}(\lambda-4)^{n}. \end{aligned}$$

□

Theorem 2.3. *Let $\Gamma_{Q_{4n}}$ be the power graph for Q_{4n}, then the characteristic polynomial of $SL(\Gamma_{Q_{4n}})$ is*

$$\begin{aligned} P_{SL(\Gamma_{Q_{4n}})}(\lambda) = \big(&\lambda^4 - 2(6n-1)\lambda^3 + 4(12n^2-2n-3)\lambda^2 - 8(8n^3+6n^2-16n+5)\lambda \\ &+32(6n^3-11n^2+6n-1)\big)\,(\lambda-2n+2)^{2n-3}(\lambda-2)^{n}(\lambda-4)^{n-1} \end{aligned}$$

Proof. Based on Equations 2.3 and 2.1 and Definition 1.5, the signless Laplacian matrix of $\Gamma_{Q_{4n}}$ is $SL(\Gamma_{D_{2n}}) = D(\Gamma_{D_{2n}}) + A(\Gamma_{D_{2n}})$ as given below:

$$\begin{array}{c} \\ e \\ a^n \\ a \\ \vdots \\ a^{n-1} \\ a^{n+1} \\ \vdots \\ a^{2n-1} \\ b \\ \vdots \\ a^{n-1}b \\ a^n b \\ \vdots \\ a^{2n-1}b \end{array} \begin{array}{c} \begin{array}{cccccccccccccccc} e & a^n & a & \dots & a^{n-1} & a^{n+1} & \dots & a^{2n-1} & b & \dots & a^{n-1}b & a^n b & \dots & a^{2n-1}b \end{array} \\ \begin{pmatrix} 4n-1 & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & 4n-1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 & 1 & \dots & 1 \\ 1 & 1 & 2n-1 & \dots & 1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 2n-1 & 1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 1 & 1 & \dots & 1 & 2n-1 & \dots & 1 & 0 & \dots & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & \dots & 1 & 1 & \dots & 2n-1 & 0 & \dots & 0 & 0 & \dots & 0 \\ 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 3 & \dots & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 3 & 0 & \dots & 1 \\ 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 1 & \dots & 0 & 3 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 1 & 0 & \dots & 3 \end{pmatrix} \end{array}.$$

$SL(\Gamma_{Q_{4n}})$ can be partitioned into 16 block matrices as given below:

$$SL(\Gamma_{Q_{4n}}) = \begin{bmatrix} (4n-2)I_2 + J_2 & J_{2\times(2n-2)} & J_{2\times n} & J_{2\times n} \\ J_{(2n-2)\times 2} & (2n-2)I_{2n-2} + J_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\ J_{n\times 2} & 0_{n\times(2n-2)} & 3I_n & I_n \\ J_{n\times 2} & 0_{n\times(2n-2)} & I_n & 3I_n \end{bmatrix}.$$

The characteristic polynomial of $SL(\Gamma_{Q_{4n}})$ can be obtained from the following determinant:

$$P_{SL(\Gamma_{Q_{4n}})}(\lambda) = \begin{vmatrix} (\lambda-4n+2)I_2 - J_2 & -J_{2\times(2n-2)} & -J_{2\times n} & -J_{2\times n} \\ -J_{(2n-2)\times 2} & (\lambda-2n+2)I_{2n-2} - J_{2n-2} & 0_{(2n-2)\times n} & 0_{(2n-2)\times n} \\ -J_{n\times 2} & 0_{n\times(2n-2)} & (\lambda-3)I_n & -I_n \\ -J_{n\times 2} & 0_{n\times(2n-2)} & -I_n & (\lambda-3)I_n \end{vmatrix}. \quad (2.5)$$

We apply the following row and column operations to Equation 2.5, then

(1) $R_{3n+i} \longrightarrow R_{3n+i} - R_{2n+i}$, for $i = 1, 2, \ldots, n$.
(2) $C_{2n+i} \longrightarrow C_{2n+i} + C_{3n+i}$, for $i = 1, 2, \ldots, n$.
(3) $R_{2n+1+i} \longrightarrow R_{2n+1+i} - R_{2n+1}$, for $i = 1, 2, \ldots, n-1$.
(4) $C_{2n+1} \longrightarrow C_{2n+1} + C_{2n+2} + \ldots + C_{3n}$.
(5) $C_1 \longrightarrow C_1 + \frac{1}{\lambda-4}C_{2n+1}$.
(6) $C_2 \longrightarrow C_2 + \frac{1}{\lambda-4}C_{2n+1}$.
(7) $R_{3+i} \longrightarrow R_{3+i} - R_3$, for $i = 1, 2, \ldots, 2n-3$.
(8) $C_3 \longrightarrow C_3 + C_4 + \ldots + C_{2n}$.
(9) $C_1 \longrightarrow C_1 + \frac{1}{\lambda-4n+4}C_3$.
(10) $C_2 \longrightarrow C_2 + \frac{1}{\lambda-4n+4}C_3$.

Consequently, we derive $P_{SL(\Gamma_{Q_{4n}})}(\lambda)$ as the following determinant

$$\begin{vmatrix} a & b & 2-2n & -J_{1\times(2n-3)} & -2n & -2J_{1\times(n-1)} & -1 & -J_{1\times(n-1)} \\ b & a & 2-2n & -J_{1\times(2n-3)} & -2n & -2J_{1\times(n-1)} & -1 & -J_{1\times(n-1)} \\ 0 & 0 & \lambda-4n+4 & -J_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & 0 & 0_{1\times(n-1)} \\ 0 & 0 & 0_{(2n-3)\times 1} & (\lambda-2n+2)I_{2n-3} & 0_{(2n-3)\times 1} & 0_{(2n-3)\times(n-1)} & 0 & 0_{(2n-3)\times(n-1)} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & \lambda-4 & 0_{1\times(n-1)} & -1 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & (\lambda-4)I_{n-1} & J_{(n-1)\times 1} & -I_{n-1} \\ 0 & 0 & 0 & 0_{1\times(2n-3)} & 0 & 0_{1\times(n-1)} & \lambda-2 & 0_{1\times(n-1)} \\ 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times 1} & 0_{(n-1)\times(2n-3)} & 0_{(n-1)\times 1} & 0_{n-1} & 0_{(n-1)\times 1} & (\lambda-2)I_{n-1} \end{vmatrix},$$

where $a = \lambda - 4n + 1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4}$ and $b = -1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4}$. Based on Theorem 1.6, we then can obtain

$$\begin{aligned} P_{SL(\Gamma_{Q_{4n}})}(\lambda) = &\begin{vmatrix} \lambda - 4n + 1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4} & -1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4} \\ -1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4} & \lambda - 4n + 1 - \frac{2n}{\lambda-4} - \frac{(2n-2)}{\lambda-4n+4} \end{vmatrix} \\ &(\lambda-4n+4)(\lambda-2n+2)^{2n-3}(\lambda-2)^n(\lambda-4)^n \\ = &\big(\lambda^4 - 2(6n-1)\lambda^3 + 4(12n^2-2n-3)\lambda^2 - 8(8n^3+6n^2-16n+5)\lambda \\ &+32(6n^3-11n^2+6n-1)\big)(\lambda-2n+2)^{2n-3}(\lambda-2)^n(\lambda-4)^{n-1}. \end{aligned}$$

□

3. Conclusion

An analysis of the spectral properties of the power graph of the generalized quaternion group is presented in this paper. Specifically, we presented the characteristic polynomial of the graph whose matrix is either adjacency, Laplacian, or signless Laplacian.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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