# A STUDY ON BIPOLAR VAGUE IDEALS OF GAMMA-NEAR RINGS

V. P. VINEELA KORADA[1], S. RAGAMAYI[1], AIYARED IAMPAN[2,*]

[1]Department of Engineering Mathematics, Koneru Lakshmaiah Education Foundation, Guntur-522502, Andhra Pradesh, India

[2]Department of Mathematics, School of Science, University of Phayao, Mae Ka, Mueang, Phayao 56000, Thailand

[*]Corresponding author: aiyared.ia@up.ac.th

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Abstract. We inspect the abstraction of bipolar vague ideals (BVIs) of $\Gamma$-near rings (GNRs) and scrutinize its different attributes. Let $\mathscr{M}$ be a GNR. A bipolar vague set (BVS) is a BVI of $\mathscr{M}$ if and only if the bipolar vague cut set is an ideal of $\mathscr{M}$. In addition, we establish that the characteristic set is a BVI of $\mathscr{M}$ if and only if the crisp set is an ideal of $\mathscr{M}$. Then, we confirm that the intersection of BVIs is also a BVI of $\mathscr{M}$.

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## 1. Introduction

The fuzzy sets were first promoted by Zadeh [19] in 1965. The extensions of fuzzy set theory include interval-valued, intuitionistic, vague sets, etc. The philosophy of vague sets, the generality of fuzzy sets, was brought in by Gau and Buehrer [17]. Many researchers introduced and studied vague ideals and normal vague ideals in semirings and $\Gamma$-semirings. Further, Ragamayi and Eswarlal [5–10,12,15] made acquaintance with vague and normal vague ideals in GNRs. Lee [16] promoted bipolar-valued fuzzy sets, the extension of fuzzy sets. This route is intended to enhance research in various spheres such as algebraic structures, medical science, decision-making, machine theory, graph theory, etc. In this paper, we acquainted ourselves with the notation of BVIs of GNRs and cultivated some of their attributes.

## 2. Preliminaries

**Definition 2.1.** [18] A GNR is $(\mathscr{M}, +, \Gamma)$ so as
(i) $(\mathscr{M}, +)$ is a group,
(ii) $(\mathscr{M}, +, \dot{\alpha})$ is a near-ring for all binary operator $\dot{\alpha} \in \Gamma$,

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(iii) $\varkappa\dot{\alpha}(\vartheta\dot{\beta}\varpi) = (\varkappa\dot{\alpha}\vartheta)\dot{\beta}\varpi$ for all $\varkappa, \vartheta, \varpi \in \mathscr{M}$ and $\dot{\alpha}, \dot{\beta} \in \Gamma$.

**Remark 2.2.** [13] All over the paper, $\mathscr{M}$ denotes a zero-symmetric GNR (right), i.e., $\varkappa\dot{\alpha}0 = 0$ for each $\varkappa \in \mathscr{M}$ and $\dot{\alpha} \in \Gamma$ with at least two elements.

**Definition 2.3.** [14] The set $\mathscr{D}$ is said to be a vague set in the universe of discourse $\mathscr{U}$ for $(\mathscr{T}_{\mathscr{D}}, \mathscr{F}_{\mathscr{D}})$, where $\mathscr{T}_{\mathscr{D}} : \mathscr{U} \to [0,1]$ and $\mathscr{F}_{\mathscr{D}} : \mathscr{U} \to [0,1]$ are mappings so as $\mathscr{T}_{\mathscr{D}}(\varkappa) + \mathscr{F}_{\mathscr{D}}(\varkappa) \leq 1, \forall \varkappa \in \mathscr{U}$. The function $\mathscr{T}_{\mathscr{D}}$ is true membership, and the other function $\mathscr{F}_{\mathscr{D}}$ is false membership. Here, $\mathscr{V}_{\mathscr{D}} = (\mathscr{T}_{\mathscr{D}}, 1 - \mathscr{F}_{\mathscr{D}})$ implies a vague set.

**Definition 2.4.** [15,17] For the vague sets $\mathscr{D}$ and $\mathscr{E}$ in $\mathscr{U}$, $\mathscr{D} \subseteq \mathscr{E}$ if $\mathscr{V}_{\mathscr{D}}(\varkappa) \leq \mathscr{V}_{\mathscr{E}}(\varkappa)$, i.e., $\mathscr{T}_{\mathscr{D}}(\varkappa) \leq \mathscr{T}_{\mathscr{E}}(\varkappa)$ and $1 - \mathscr{F}_{\mathscr{D}}(\varkappa) \leq 1 - \mathscr{F}_{\mathscr{E}}(\varkappa), \forall \varkappa \in \mathscr{U}$.

**Definition 2.5.** [15,17] The vague sets $\mathscr{D}$ and $\mathscr{E}$ in $\mathscr{U}$ are equal, $\mathscr{D} = \mathscr{E}$, if $\mathscr{D} \subseteq \mathscr{E}$ and $\mathscr{E} \subseteq \mathscr{D}$, i.e., $\mathscr{V}_{\mathscr{D}}(\varkappa) \leq \mathscr{V}_{\mathscr{E}}(\varkappa)$ and $\mathscr{V}_{\mathscr{E}}(\varkappa) \leq \mathscr{V}_{\mathscr{D}}(\varkappa), \forall \varkappa \in \mathscr{U}$.

**Definition 2.6.** [15,17] The union of the vague sets $\mathscr{D}$ and $\mathscr{E}$ in $\mathscr{U}$ is a vague set $\mathscr{C}$, represented as $\mathscr{C} = \mathscr{D} \cup \mathscr{E}$, with membership functions $\mathscr{T}_{\mathscr{C}} = \max\{\mathscr{T}_{\mathscr{D}}, \mathscr{T}_{\mathscr{E}}\}$ and $1 - \mathscr{F}_{\mathscr{C}} = \max\{1 - \mathscr{F}_{\mathscr{D}}, 1 - \mathscr{F}_{\mathscr{E}}\}$.

**Definition 2.7.** [15,17] The intersection of the vague sets $\mathscr{D}$ and $\mathscr{E}$ in $\mathscr{U}$ is a vague set $\mathscr{C}$, represented as $\mathscr{C} = \mathscr{D} \cap \mathscr{E}$, with membership functions $\mathscr{T}_{\mathscr{C}} = \min\{\mathscr{T}_{\mathscr{D}}, \mathscr{T}_{\mathscr{E}}\}$ and $1 - \mathscr{F}_{\mathscr{C}} = \min\{1 - \mathscr{F}_{\mathscr{D}}, 1 - \mathscr{F}_{\mathscr{E}}\}$.

**Definition 2.8.** [16] Consider a set $\mathscr{D}$ over the universal set $\mathscr{U}$ defined by the positive and negative membership functions, $\mu^{+}_{\mathscr{D}} : \mathscr{U} \to [0,1]$ and $\mu^{-}_{\mathscr{D}} : \mathscr{U} \to [-1,0]$. Then $\mathscr{D}$ is claimed to be a bipolar fuzzy set (BFS) of $\mathscr{U}$, and represented as $\mathscr{D} = (\mu^{+}_{\mathscr{D}}, \mu^{-}_{\mathscr{D}})$.

**Definition 2.9.** [11,16] The bipolar vague set (BVS) in $\mathscr{U}$ is represented as $D = ((\mathscr{T}^{+}_{\mathscr{D}}, \mathscr{F}^{+}_{\mathscr{D}}), (\mathscr{T}^{-}_{\mathscr{D}}, \mathscr{F}^{-}_{\mathscr{D}}))$, where $0 \leq \mathscr{T}^{+}_{\mathscr{D}}(\varkappa) + \mathscr{F}^{+}_{\mathscr{D}}(\varkappa) \leq 1$ and $-1 \leq \mathscr{T}^{-}_{\mathscr{D}}(\varkappa) + \mathscr{F}^{-}_{\mathscr{D}}(\varkappa) \leq 0, \forall \varkappa \in \mathscr{U}$. Here, $\mathscr{V}^{+}_{\mathscr{D}} = (\mathscr{T}^{+}_{\mathscr{D}}, 1 - \mathscr{F}^{+}_{\mathscr{D}})$ and $\mathscr{V}^{-}_{\mathscr{D}} = (-1 - \mathscr{F}^{-}_{\mathscr{D}}, \mathscr{T}^{-}_{\mathscr{D}})$ indicate vague sets.

**Definition 2.10.** [4] A BFS $B = (\mu^{+}_{B}, \mu^{-}_{B})$ of a GNR $\mathscr{M}$ is a bipolar fuzzy ideal (BFI) of $\mathscr{M}$ if for all $\varkappa, \vartheta, \varpi \in \mathscr{M}, \dot{\alpha} \in \Gamma$,

(i) $\mu^{+}_{B}(\varkappa - \vartheta) \geq \min\{\mu^{+}_{B}(\varkappa), \mu^{+}_{B}(\vartheta)\}$,

(ii) $\mu^{+}_{B}(\varkappa + \vartheta - \varkappa) \geq \mu^{+}_{B}(\vartheta)$,

(iii) $\mu^{+}_{B}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq \mu^{+}_{B}(\varpi)$,

(iv) $\mu^{+}_{B}(\varkappa\dot{\alpha}\vartheta) \geq \mu^{+}_{B}(\varkappa)$,

(v) $\mu^{-}_{B}(\varkappa - \vartheta) \leq \max\{\mu^{-}_{B}(\varkappa), \mu^{-}_{B}(\vartheta)\}$,

(vi) $\mu^{-}_{B}(\varkappa + \vartheta - \varkappa) \leq \mu^{-}_{B}(\vartheta)$,

(vii) $\mu^{-}_{B}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq \mu^{-}_{B}(\varpi)$,

(viii) $\mu^{-}_{B}(\varkappa\dot{\alpha}\vartheta) \leq \mu^{-}_{B}(\varkappa)$.

## 3. Bipolar vague ideals of Γ-near rings

Here, we bring in and inspect the notion of bipolar vague ideals of GNRs and their attributes.

**Definition 3.1.** A BVS $\mathscr{D} = ((\mathscr{T}^+_\mathscr{D}, \mathscr{F}^+_\mathscr{D}), (\mathscr{T}^-_\mathscr{D}, \mathscr{F}^-_\mathscr{D}))$ in $\mathscr{M}$ is called a bipolar vague ideal (BVI) of $\mathscr{M}$ if for all $\varkappa, \vartheta, \varpi \in \mathscr{M}, \dot\alpha \in \Gamma$,

(i) $\mathscr{V}^+_\mathscr{D}(\varkappa - \vartheta) \geq \min\{\mathscr{V}^+_\mathscr{D}(\varkappa), \mathscr{V}^+_\mathscr{D}(\vartheta)\}$,
(ii) $\mathscr{V}^+_\mathscr{D}(\varkappa + \vartheta - \varkappa) \geq \mathscr{V}^+_\mathscr{D}(\vartheta)$,
(iii) $\mathscr{V}^+_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \geq \mathscr{V}^+_\mathscr{D}(\varpi)$,
(iv) $\mathscr{V}^+_\mathscr{D}(\varkappa\dot\alpha\vartheta) \geq \mathscr{V}^+_\mathscr{D}(\varkappa)$,
(v) $\mathscr{V}^-_\mathscr{D}(\varkappa - \vartheta) \leq \max\{\mathscr{V}^-_\mathscr{D}(\varkappa), \mathscr{V}^-_\mathscr{D}(\vartheta)\}$,
(vi) $\mathscr{V}^-_\mathscr{D}(\varkappa + \vartheta - \varkappa) \leq \mathscr{V}^-_\mathscr{D}(\vartheta)$,
(vii) $\mathscr{V}^-_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \leq \mathscr{V}^-_\mathscr{D}(\varpi)$,
(viii) $\mathscr{V}^-_\mathscr{D}(\varkappa\dot\alpha\vartheta) \leq \mathscr{V}^-_\mathscr{D}(\varkappa)$,

i.e.,

(i) $\mathscr{T}^+_\mathscr{D}(\varkappa - \vartheta) \geq \min\{\mathscr{T}^+_\mathscr{D}(\varkappa), \mathscr{T}^+_\mathscr{D}(\vartheta)\}$ and $1 - \mathscr{F}^+_\mathscr{D}(\varkappa - \vartheta) \geq \min\{1 - \mathscr{F}^+_\mathscr{D}(\varkappa), 1 - \mathscr{F}^+_\mathscr{D}(\vartheta)\}$,
(ii) $\mathscr{T}^+_\mathscr{D}(\varkappa + \vartheta - \varkappa) \geq \mathscr{T}^+_\mathscr{D}(\vartheta)$ and $1 - \mathscr{F}^+_\mathscr{D}(\varkappa + \vartheta - \varkappa) \geq 1 - \mathscr{F}^+_\mathscr{D}(\vartheta)$,
(iii) $\mathscr{T}^+_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \geq \mathscr{T}^+_\mathscr{D}(\varpi)$ and $1 - \mathscr{F}^+_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \geq 1 - \mathscr{F}^+_\mathscr{D}(\varpi)$,
(iv) $\mathscr{T}^+_\mathscr{D}(\varkappa\dot\alpha\vartheta) \geq \mathscr{T}^+_\mathscr{D}(\varkappa)$ and $1 - \mathscr{F}^+_\mathscr{D}(\varkappa\dot\alpha\vartheta) \geq 1 - \mathscr{F}^+_\mathscr{D}(\varkappa)$,
(v) $\mathscr{T}^-_\mathscr{D}(\varkappa - \vartheta) \leq \max\{\mathscr{T}^-_\mathscr{D}(\varkappa), \mathscr{T}^-_\mathscr{D}(\vartheta)\}$ and $-1 - \mathscr{F}^-_\mathscr{D}(\varkappa - \vartheta) \leq \max\{-1 - \mathscr{F}^-_\mathscr{D}(\varkappa), -1 - \mathscr{F}^-_\mathscr{D}(\vartheta)\}$,
(vi) $\mathscr{T}^-_\mathscr{D}(\varkappa + \vartheta - \varkappa) \leq \mathscr{T}^-_\mathscr{D}(\vartheta)$ and $-1 - \mathscr{F}^-_\mathscr{D}(\varkappa + \vartheta - \varkappa) \leq -1 - \mathscr{F}^-_\mathscr{D}(\vartheta)$,
(vii) $\mathscr{T}^-_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \leq \mathscr{T}^-_\mathscr{D}(\varpi)$ and $-1 - \mathscr{F}^-_\mathscr{D}(\varkappa\dot\alpha(\vartheta + \varpi) - \varkappa\dot\alpha\vartheta) \leq -1 - \mathscr{F}^-_\mathscr{D}(\varpi)$,
(viii) $\mathscr{T}^-_\mathscr{D}(\varkappa\dot\alpha\vartheta) \leq \mathscr{T}^-_\mathscr{D}(\varkappa)$ and $-1 - \mathscr{F}^-_\mathscr{D}(\varkappa\dot\alpha\vartheta) \leq -1 - \mathscr{F}^-_\mathscr{D}(\varkappa)$.

**Example 3.2.** Let $\mathscr{M} = \mathbb{R}$ be a set of real numbers, which is a GNR. Let $\mathscr{D} = ((\mathscr{T}^+_\mathscr{D}, \mathscr{F}^+_\mathscr{D}), (\mathscr{T}^-_\mathscr{D}, \mathscr{F}^-_\mathscr{D}))$ be a BVS in $\mathscr{M}$, defined as

| | $\varkappa = 0$ | $\varkappa > 0$ | $\varkappa < 0$ |
|---|---|---|---|
| $\mathscr{V}^+_\mathscr{D}$ | $(0.4, 0.3)$ | $(0.5, 0.2)$ | $(0.5, 0.2)$ |
| $\mathscr{V}^-_\mathscr{D}$ | $(-0.4, -0.1)$ | $(-0.6, -0.2)$ | $(-0.6, -0.2)$ |

Thus, $\mathscr{D}$ is a BVI of $\mathscr{M}$.

**Remark 3.3.** The BVS $\mathscr{D} = ((\mathscr{T}^+_\mathscr{D}, \mathscr{F}^+_\mathscr{D}), (\mathscr{T}^-_\mathscr{D}, \mathscr{F}^-_\mathscr{D}))$ in $\mathscr{M}$ is a BVI of $\mathscr{M}$ if and only if $\mathscr{T}^+_\mathscr{D}, 1 - \mathscr{F}^+_\mathscr{D}, \mathscr{T}^-_\mathscr{D}, -1 - \mathscr{F}^-_\mathscr{D}$ are fuzzy ideals of $\mathscr{M}$.

**Definition 3.4.** For $\rho^+, \varsigma^+ \in [0,1]$ with $\rho^+ \leq \varsigma^+$, and $\rho^-, \varsigma^- \in [-1, 0]$ with $\rho^- \geq \varsigma^-$, the $((\rho^+, \varsigma^+), (\rho^-, \varsigma^-))$-cut or bipolar vague cut of the BVS $\mathscr{D} = ((\mathscr{T}^+_\mathscr{D}, \mathscr{F}^+_\mathscr{D}), (\mathscr{T}^-_\mathscr{D}, \mathscr{F}^-_\mathscr{D}))$ is the crisp subset of $\mathscr{U}$ given by
$\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))} = \{\varkappa \in \mathscr{U} \mid \mathscr{V}^+_\mathscr{D}(\varkappa) \geq (\rho^+, \varsigma^+), \mathscr{V}^-_\mathscr{D}(\varkappa) \leq (\rho^-, \varsigma^-)\}$, i.e.,
$\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))} = \{\varkappa \in \mathscr{U} \mid \mathscr{T}^+_\mathscr{D}(\varkappa) \geq \rho^+, 1 - \mathscr{F}^+_\mathscr{D}(\varkappa) \geq \varsigma^+, \mathscr{T}^-_\mathscr{D}(\varkappa) \leq \rho^-, -1 - \mathscr{F}^-_\mathscr{D}(\varkappa) \leq \varsigma^-\}$.

**Theorem 3.5.** *A BVS $\mathscr{D} = ((\mathscr{T}_{\mathscr{D}}^{+}, \mathscr{F}_{\mathscr{D}}^{+}), (\mathscr{T}_{\mathscr{D}}^{-}, \mathscr{F}_{\mathscr{D}}^{-}))$ in $\mathscr{M}$ is a BVI of $\mathscr{M}$ if and only if for all $\rho^+, \varsigma^+ \in [0,1], \rho^-, \varsigma^- \in [-1,0]$, the bipolar vague cut $\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ is an ideal of $\mathscr{M}$.*

*Proof.* Let $\mathscr{D} = ((\mathscr{T}_{\mathscr{D}}^{+}, \mathscr{F}_{\mathscr{D}}^{+}), (\mathscr{T}_{\mathscr{D}}^{-}, \mathscr{F}_{\mathscr{D}}^{-}))$ is a BVI of $\mathscr{M}$.

Let $\varkappa, \vartheta, \varpi \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ for $\rho^+, \varsigma^+ \in [0,1], \rho^-, \varsigma^- \in [-1,0]$. Then

$\mathscr{T}_{\mathscr{D}}^{+}(\varkappa) \geq \rho^+, \mathscr{T}_{\mathscr{D}}^{+}(\vartheta) \geq \rho^+, \mathscr{T}_{\mathscr{D}}^{+}(\varpi) \geq \rho^+,$

$1 - \mathscr{F}_{\mathscr{D}}^{+}(\varkappa) \geq \varsigma^+, 1 - \mathscr{F}_{\mathscr{D}}^{+}(\vartheta) \geq \varsigma^+, 1 - \mathscr{F}_{\mathscr{D}}^{+}(\varpi) \geq \varsigma^+$ and

$\mathscr{T}_{\mathscr{D}}^{-}(\varkappa) \leq \rho^-, \mathscr{T}_{\mathscr{D}}^{-}(\vartheta) \leq \rho^-, \mathscr{T}_{\mathscr{D}}^{-}(\varpi) \leq \rho^-,$

$-1 - \mathscr{F}_{\mathscr{D}}^{-}(\varkappa) \leq \varsigma^-, -1 - \mathscr{F}_{\mathscr{D}}^{-}(\vartheta) \leq \varsigma^-, -1 - \mathscr{F}_{\mathscr{D}}^{-}(\varpi) \leq \varsigma^-.$

Let $\dot{\alpha} \in \Gamma$. Then

$\mathscr{V}_{\mathscr{D}}^{+}(\varkappa - \vartheta) \geq \min\{\mathscr{V}_{\mathscr{D}}^{+}(\varkappa), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta)\} = \min\{(\rho^+,\varsigma^+),(\rho^+,\varsigma^+)\} = (\rho^+,\varsigma^+),$

$\mathscr{V}_{\mathscr{D}}^{+}(\varkappa + \vartheta - \varkappa) \geq \mathscr{V}_{\mathscr{D}}^{+}(\vartheta) = (\rho^+,\varsigma^+),$

$\mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_{\mathscr{D}}^{+}(\varpi) = (\rho^+,\varsigma^+),$

$\mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_{\mathscr{D}}^{+}(\varkappa) = (\rho^+,\varsigma^+),$

$\mathscr{V}_{\mathscr{D}}^{-}(\varkappa - \vartheta) \leq \max\{\mathscr{V}_{\mathscr{D}}^{-}(\varkappa), \mathscr{V}_{\mathscr{D}}^{-}(\vartheta)\} = \max\{(\rho^-,\varsigma^-),(\rho^-,\varsigma^-)\} = (\rho^-,\varsigma^-),$

$\mathscr{V}_{\mathscr{D}}^{-}(\varkappa + \vartheta - \varkappa) \leq \mathscr{V}_{\mathscr{D}}^{-}(\vartheta) = (\rho^-,\varsigma^-),$

$\mathscr{V}_{\mathscr{D}}^{-}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_{\mathscr{D}}^{-}(\varpi) = (\rho^-,\varsigma^-),$

$\mathscr{V}_{\mathscr{D}}^{-}(\varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_{\mathscr{D}}^{-}(\varkappa) = (\rho^-,\varsigma^-).$

So $\varkappa - \vartheta, \varkappa + \vartheta - \varkappa, \varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta, \varkappa\dot{\alpha}\vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$.

Hence, $\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ is an ideal of $\mathscr{M}$.

Conversely, assume $\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ is an ideal of $\mathscr{M}$ for $\rho^+, \varsigma^+ \in [0,1], \rho^-, \varsigma^- \in [-1,0]$.

(1) Let $\varkappa, \vartheta \in \mathscr{M}$. Suppose $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa - \vartheta) < \min\{\mathscr{V}_{\mathscr{D}}^{+}(\varkappa), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta)\}$. Choose $\rho^+, \varsigma^+ \in [0,1]$ such that $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa - \vartheta) < (\rho^+,\varsigma^+) < \min\{\mathscr{V}_{\mathscr{D}}^{+}(\varkappa), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta)\}$. Then $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa) > (\rho^+,\varsigma^+), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta) > (\rho^+,\varsigma^+)$ and $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa - \vartheta) < (\rho^+,\varsigma^+)$. Thus $\varkappa, \vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ but $\varkappa - \vartheta \notin \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ which is a contradiction. Hence, $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa - \vartheta) \geq \min\{\mathscr{V}_{\mathscr{D}}^{+}(\varkappa), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta)\}$. Similarly, we can prove that $\mathscr{V}_{\mathscr{D}}^{-}(\varkappa - \vartheta) \leq \max\{\mathscr{V}_{\mathscr{D}}^{-}(\varkappa), \mathscr{V}_{\mathscr{D}}^{-}(\vartheta)\}$.

(2) Let $\varkappa, \vartheta \in \mathscr{M}, \mathscr{V}_{\mathscr{D}}^{+}(\varkappa) = (\rho^+,\varsigma^+), \mathscr{V}_{\mathscr{D}}^{+}(\vartheta) = (\rho^+,\varsigma^+), \mathscr{V}_{\mathscr{D}}^{-}(\varkappa) = (\rho^-,\varsigma^-), \mathscr{V}_{\mathscr{D}}^{-}(\vartheta) = (\rho^-,\varsigma^-)$. For $\varkappa, \vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}, \varkappa + \vartheta - \varkappa \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$. Therefore, $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa + \vartheta - \varkappa) \geq (\rho^+,\varsigma^+) \Rightarrow \mathscr{V}_{\mathscr{D}}^{+}(\varkappa + \vartheta - \varkappa) \geq \mathscr{V}_{\mathscr{D}}^{+}(\vartheta)$, and $\mathscr{V}_{\mathscr{D}}^{-}(\varkappa + \vartheta - \varkappa) \leq (\rho^-,\varsigma^-) \Rightarrow \mathscr{V}_{\mathscr{D}}^{-}(\varkappa + \vartheta - \varkappa) \leq \mathscr{V}_{\mathscr{D}}^{-}(\vartheta)$.

(3) Let $\varkappa, \vartheta, \varpi \in \mathscr{M}, \dot{\alpha} \in \Gamma, \mathscr{V}_{\mathscr{D}}^{+}(\varpi) = (\rho^+,\varsigma^+), \mathscr{V}_{\mathscr{D}}^{-}(\varpi) = (\rho^-,\varsigma^-)$. For $\varkappa, \vartheta, \varpi \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}, \varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$. Therefore, $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq (\rho^+,\varsigma^+) \Rightarrow \mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_{\mathscr{D}}^{+}(\varpi)$, and $\mathscr{V}_{\mathscr{D}}^{-}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq (\rho^-,\varsigma^-) \Rightarrow \mathscr{V}_{\mathscr{D}}^{-}(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_{\mathscr{D}}^{-}(\varpi)$.

(4) Let $\varkappa, \vartheta \in \mathscr{M}, \dot{\alpha} \in \Gamma, \mathscr{V}_{\mathscr{D}}^{+}(\varkappa) = (\rho^+,\varsigma^+), \mathscr{V}_{\mathscr{D}}^{-}(\varkappa) = (\rho^-,\varsigma^-)$. For $\varkappa, \vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}, \varkappa\dot{\alpha}\vartheta \in \mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$. Therefore, $\mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}\vartheta) \geq (\rho^+,\varsigma^+) \Rightarrow \mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_{\mathscr{D}}^{+}(\varkappa)$, and $\mathscr{V}_{\mathscr{D}}^{-}(\varkappa\dot{\alpha}\vartheta) \leq (\rho^-,\varsigma^-) \Rightarrow \mathscr{V}_{\mathscr{D}}^{+}(\varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_{\mathscr{D}}^{-}(\varkappa)$.

Therefore, $\mathscr{D}_{((\rho^+,\varsigma^+),(\rho^-,\varsigma^-))}$ is a BVI of $\mathscr{M}$. □

**Theorem 3.6.** *The characteristic set of a non-empty subset $\chi$ of $\mathscr{M}$, $\mathscr{V}_\chi = ((\mathscr{T}_\chi^+, \mathscr{F}_\chi^+), (\mathscr{T}_\chi^-, \mathscr{F}_\chi^-))$ is a BVI of $\mathscr{M}$ if and only if $\chi$ itself is an ideal of $\mathscr{M}$.*

*Proof.* Presume that $\mathscr{V}_\chi = ((\mathscr{T}_\chi^+, \mathscr{F}_\chi^+), (\mathscr{T}_\chi^-, \mathscr{F}_\chi^-))$ is a BVI of $\mathscr{M}$. Let $\varkappa, \vartheta, \varpi \in \chi, \dot{\alpha} \in \Gamma$. Then

(1) $\mathscr{V}_\chi^+(\varkappa - \vartheta) \geq \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\} = (1,1)$.

(2) $\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) \geq \mathscr{V}_\chi^+(\vartheta) = (1,1)$.

(3) $\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_\chi^+(\varpi) = (1,1)$.

(4) $\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) \geq \mathscr{V}_\chi^+(\varkappa) = (1,1)$.

(5) $\mathscr{V}_\chi^-(\varkappa - \vartheta) \leq \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\} = (-1,-1)$.

(6) $\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) \leq \mathscr{V}_\chi^-(\vartheta) = (-1,-1)$.

(7) $\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_\chi^-(\varpi) = (-1,-1)$.

(8) $\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) \leq \mathscr{V}_\chi^-(\varkappa) = (-1,-1)$.

That implies $\varkappa - \vartheta, \varkappa + \vartheta - \varkappa, \varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta, \varkappa\dot{\alpha}\vartheta \in \chi$. Hence, $\chi$ is an ideal of $\mathscr{M}$.

Conversely, suppose that $\chi$ is an ideal of $\mathscr{M}$. Let $\varkappa, \vartheta, \varpi \in \mathscr{M}, \dot{\alpha} \in \Gamma$.

(1) If $\varkappa, \vartheta, \varpi \in \chi$, then $\varkappa - \vartheta, \varkappa + \vartheta - \varkappa, \varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta, \varkappa\dot{\alpha}\vartheta \in \chi$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) = (1,1) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) = (-1,-1) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (-1,-1) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(2) If $\varkappa, \vartheta, \varpi \notin \chi$, then $\mathscr{V}_\chi^+(\varkappa) = \mathscr{V}_\chi^+(\vartheta) = \mathscr{V}_\chi^+(\varpi) = (0,0), \mathscr{V}_\chi^-(\varkappa) = \mathscr{V}_\chi^-(\vartheta) = \mathscr{V}_\chi^-(\varpi) = (0,0)$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) = (1,1) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) = (-1,-1) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (-1,-1) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(3) If $\varkappa, \vartheta \in \chi, \varpi \notin \chi$, then $\mathscr{V}_\chi^+(\varkappa) = \mathscr{V}_\chi^+(\vartheta) = (1,1), \mathscr{V}_\chi^+(\varpi) = (0,0), \mathscr{V}_\chi^-(\varkappa) = \mathscr{V}_\chi^-(\vartheta) = (-1,-1), \mathscr{V}_\chi^-(\varpi) = (0,0)$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) = (1,1) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \geq (0,0) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) = (-1,-1) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (-1,-1) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) \leq (0,0) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(4) If $\varkappa, \varpi \in \chi, \vartheta \notin \chi$, then $\mathscr{V}_\chi^+(\varkappa) = \mathscr{V}_\chi^+(\varpi) = (1,1), \mathscr{V}_\chi^+(\vartheta) = (0,0), \mathscr{V}_\chi^-(\varkappa) = \mathscr{V}_\chi^-(\varpi) = (-1,-1), \mathscr{V}_\chi^-(\vartheta) = (0,0)$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) \geq (0,0) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) \geq (0,0) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) \leq (0,0) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) \leq (0,0) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(5) If $\vartheta, \varpi \in \chi, \varkappa \notin \chi$, then $\mathscr{V}_\chi^+(\vartheta) = \mathscr{V}_\chi^+(\varpi) = (1,1), \mathscr{V}_\chi^+(\varkappa) = (0,0), \mathscr{V}_\chi^-(\vartheta) = \mathscr{V}_\chi^-(\varpi) = (-1,-1), \mathscr{V}_\chi^-(\varkappa) = (0,0)$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) \geq (0,0) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) \leq (0,0) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (-1,-1) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(6) If $\varkappa \in \chi, \vartheta, \varpi \notin \chi$, then $\mathscr{V}_\chi^+(\varkappa) = (1,1), \mathscr{V}_\chi^+(\vartheta) = \mathscr{V}_\chi^+(\varpi) = (0,0), \mathscr{V}_\chi^-(\varkappa) = (-1,-1), \mathscr{V}_\chi^-(\vartheta) = \mathscr{V}_\chi^-(\varpi) = (0,0)$. Thus

$\mathscr{V}_\chi^+(\varkappa - \vartheta) \geq (0,0) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (0,0) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) \geq \mathscr{V}_\chi^+(\varpi)$,

$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) \leq (0,0) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (0,0) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (-1,-1) \leq \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) = \mathscr{V}_\chi^-(\varkappa)$.

(7) If $\vartheta \in \chi, \varkappa, \varpi \notin \chi$, then $\mathscr{V}_\chi^+(\vartheta) = (1,1), \mathscr{V}_\chi^+(\varkappa) = \mathscr{V}_\chi^+(\varpi) = (0,0), \mathscr{V}_\chi^-(\vartheta) = (-1,-1), \mathscr{V}_\chi^-(\varkappa) = \mathscr{V}_\chi^-(\varpi) = (0,0)$. Thus
$\mathscr{V}_\chi^+(\varkappa - \vartheta) \geq (0,0) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (0,0) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) = (1,1) \geq \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) \leq (0,0) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) = (1,1) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (0,0) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) = (-1,-1) \leq \mathscr{V}_\chi^-(\varkappa)$.

(8) If $\varpi \in \chi, \varkappa, \vartheta \notin \chi$, then $\mathscr{V}_\chi^+(\varpi) = (1,1), \mathscr{V}_\chi^+(\varkappa) = \mathscr{V}_\chi^+(\vartheta) = (0,0), \mathscr{V}_\chi^-(\varpi) = (-1,-1), \mathscr{V}_\chi^-(\varkappa) = \mathscr{V}_\chi^-(\vartheta) = (0,0)$. Thus
$\mathscr{V}_\chi^+(\varkappa - \vartheta) = (0,0) = \min\{\mathscr{V}_\chi^+(\varkappa), \mathscr{V}_\chi^+(\vartheta)\}$,
$\mathscr{V}_\chi^+(\varkappa + \vartheta - \varkappa) \geq (0,0) = \mathscr{V}_\chi^+(\vartheta)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^+(\varpi)$,
$\mathscr{V}_\chi^+(\varkappa\dot{\alpha}\vartheta) \geq (0,0) = \mathscr{V}_\chi^+(\varkappa)$,
$\mathscr{V}_\chi^-(\varkappa - \vartheta) = (0,0) = \max\{\mathscr{V}_\chi^-(\varkappa), \mathscr{V}_\chi^-(\vartheta)\}$,
$\mathscr{V}_\chi^-(\varkappa + \vartheta - \varkappa) \leq (0,0) = \mathscr{V}_\chi^-(\vartheta)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}(\vartheta + \varpi) - \varkappa\dot{\alpha}\vartheta) = (1,1) = \mathscr{V}_\chi^-(\varpi)$,
$\mathscr{V}_\chi^-(\varkappa\dot{\alpha}\vartheta) \leq (0,0) = \mathscr{V}_\chi^-(\varkappa)$.

Hence, $\mathscr{V}_\chi$ is a BVI of $\mathscr{M}$. □

**Theorem 3.7.** *Let $\mathscr{D} = ((\mathscr{T}_\mathscr{D}^+, \mathscr{F}_\mathscr{D}^+), (\mathscr{T}_\mathscr{D}^-, \mathscr{F}_\mathscr{D}^-))$ and $\mathscr{E} = ((\mathscr{T}_\mathscr{E}^+, \mathscr{F}_\mathscr{E}^+), (\mathscr{T}_\mathscr{E}^-, \mathscr{F}_\mathscr{E}^-))$ be BVIs of $\mathscr{M}$, then $\mathscr{D} \cap \mathscr{E}$ is also a BVI of $\mathscr{M}$.*

*Proof.* Let $\varkappa, \vartheta, \varpi \in \mathscr{M}$ and $\dot{\alpha} \in \Gamma$. Then

$$\begin{aligned}
\mathscr{V}_{(\mathscr{D}\cap\mathscr{E})}^+(\varkappa - \vartheta) &= \min\{\mathscr{V}_\mathscr{D}^+(\varkappa - \vartheta), \mathscr{V}_\mathscr{E}^+(\varkappa - \vartheta)\} \\
&\geq \min\{\min\{\mathscr{V}_\mathscr{D}^+(\varkappa), \mathscr{V}_\mathscr{D}^+(\vartheta)\}, \min\{\mathscr{V}_\mathscr{E}^+(\varkappa), \mathscr{V}_\mathscr{E}^+(\vartheta)\}\} \\
&= \min\{\min\{\mathscr{V}_\mathscr{D}^+(\varkappa), \mathscr{V}_\mathscr{E}^+(\varkappa)\}, \min\{\mathscr{V}_\mathscr{D}^+(\vartheta), \mathscr{V}_\mathscr{E}^+(\vartheta)\}\} \\
&= \min\{\mathscr{V}_{(\mathscr{D}\cap\mathscr{E})}^+(\varkappa), \mathscr{V}_{(\mathscr{D}\cap\mathscr{E})}^+(\vartheta)\},
\end{aligned}$$

$$\begin{aligned}\mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa-\vartheta) &= \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa-\vartheta), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa-\vartheta)\}\\ &\leq \max\{\max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa), \mathscr{V}^{-}_{\mathscr{D}}(\vartheta)\}, \max\{\mathscr{V}^{-}_{\mathscr{E}}(\varkappa), \mathscr{V}^{-}_{\mathscr{E}}(\vartheta)\}\}\\ &= \max\{\max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa)\}, \max\{\mathscr{V}^{-}_{\mathscr{D}}(\vartheta), \mathscr{V}^{-}_{\mathscr{E}}(\vartheta)\}\}\\ &= \max\{\mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa), \mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\vartheta)\},\end{aligned}$$

$$\begin{aligned}\mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa+\vartheta-\varkappa) &= \min\{\mathscr{V}^{+}_{\mathscr{D}}(\varkappa+\vartheta-\varkappa), \mathscr{V}^{+}_{\mathscr{E}}(\varkappa+\vartheta-\varkappa)\}\\ &\geq \min\{\mathscr{V}^{+}_{\mathscr{D}}(\vartheta), \mathscr{V}^{+}_{\mathscr{E}}(\vartheta)\}\\ &= \mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\vartheta),\end{aligned}$$

$$\begin{aligned}\mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa+\vartheta-\varkappa) &= \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa+\vartheta-\varkappa), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa+\vartheta-\varkappa)\}\\ &\leq \max\{\mathscr{V}^{-}_{\mathscr{D}}(\vartheta), \mathscr{V}^{-}_{\mathscr{E}}(\vartheta)\}\\ &= \mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\vartheta),\end{aligned}$$

$$\begin{aligned}&\mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta)\\ &= \min\{\mathscr{V}^{+}_{\mathscr{D}}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta), \mathscr{V}^{+}_{\mathscr{E}}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta)\}\\ &\geq \min\{\mathscr{V}^{+}_{\mathscr{D}}(\varpi), \mathscr{V}^{+}_{\mathscr{E}}(\varpi)\}\\ &= \mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\varpi),\end{aligned}$$

$$\begin{aligned}&\mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta)\\ &= \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa\dot{\alpha}(\vartheta+\varpi)-\varkappa\dot{\alpha}\vartheta)\}\\ &\leq \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varpi), \mathscr{V}^{-}_{\mathscr{E}}(\varpi)\}\\ &= \mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varpi),\end{aligned}$$

$$\begin{aligned}\mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa\dot{\alpha}\vartheta) &= \min\{\mathscr{V}^{+}_{\mathscr{D}}(\varkappa\dot{\alpha}\vartheta), \mathscr{V}^{+}_{\mathscr{E}}(\varkappa\dot{\alpha}\vartheta)\}\\ &\geq \min\{\mathscr{V}^{+}_{\mathscr{D}}(\varkappa), \mathscr{V}^{+}_{\mathscr{E}}(\varkappa)\}\\ &= \mathscr{V}^{+}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa),\end{aligned}$$

$$\begin{aligned}\mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa\dot{\alpha}\vartheta) &= \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa\dot{\alpha}\vartheta), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa\dot{\alpha}\vartheta)\}\\ &\leq \max\{\mathscr{V}^{-}_{\mathscr{D}}(\varkappa), \mathscr{V}^{-}_{\mathscr{E}}(\varkappa)\}\\ &= \mathscr{V}^{-}_{(\mathscr{D}\cap\mathscr{E})}(\varkappa).\end{aligned}$$

Hence, $\mathscr{D}\cap\mathscr{E}$ is a BVI of $\mathscr{M}$. □

**Theorem 3.8.** *If $\{\mathscr{D}_\iota\}_{\iota\in\Delta}$ is a family of BVIs of $\mathscr{M}$, then $\cap\mathscr{D}_\iota$ is also a BVI of $\mathscr{M}$, where $\cap\mathscr{D}_\iota = ((\wedge\mathscr{T}^+_{\mathscr{D}_\iota}, 1 - \wedge\mathscr{F}^+_{\mathscr{D}_\iota}), (\vee\mathscr{T}^-_{\mathscr{D}_\iota}, -1 - \vee\mathscr{F}^-_{\mathscr{D}_\iota}))$ defined by*

$$\wedge\mathscr{T}^+_{\mathscr{D}_\iota}(\varkappa) = \inf\{\mathscr{T}^+_{\mathscr{D}_\iota}(\varkappa) \mid \iota \in \Delta, \varkappa \in \mathscr{M}\},$$

$$\wedge\mathscr{F}^+_{\mathscr{D}_\iota}(\varkappa) = \inf\{\mathscr{F}^+_{\mathscr{D}_\iota}(\varkappa) \mid \iota \in \Delta, \varkappa \in \mathscr{M}\},$$

$$\vee\mathscr{T}^-_{\mathscr{D}_\iota}(\varkappa) = \sup\{\mathscr{T}^-_{\mathscr{D}_\iota}(\varkappa) \mid \iota \in \Delta, \varkappa \in \mathscr{M}\},$$

$$\vee\mathscr{F}^-_{\mathscr{D}_\iota}(\varkappa) = \sup\{\mathscr{F}^-_{\mathscr{D}_\iota}(\varkappa) \mid \iota \in \Delta, \varkappa \in \mathscr{M}\}.$$

**Theorem 3.9.** *If $\mathscr{D} = ((\mathscr{T}^+_{\mathscr{D}}, \mathscr{F}^+_{\mathscr{D}}), (\mathscr{T}^-_{\mathscr{D}}, \mathscr{F}^-_{\mathscr{D}}))$ is a BVI of $\mathscr{M}$, then $D^c = ((1 - \mathscr{T}^+_{\mathscr{D}}, 1 - \mathscr{F}^+_{\mathscr{D}}), (-1 - \mathscr{T}^-_{\mathscr{D}}, -1 - \mathscr{F}^-_{\mathscr{D}}))$ is also a BVI of $\mathscr{M}$.*

## 4. Conclusion

Right through this paper, we conceptualised BVIs of GNRs and investigated the concepts of intersection and characterization of BVIs of GNRs. In the near future, our work will be on the bipolar vague bi-ideals of GNRs which constitute a crucial part of the ideal theory of each algebraic structure.

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## Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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