Asia Pac. J. Math. 2024 **11**:39

RAINBOW CONNECTION NUMBER ON GENERALIZED FAREY GRAPH

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Received Jan. 12, 2024

Abstract. Originated from a well-known Farey sequence, the generalized Farey graph $G_{m,t}$ where $m \geq 1$ and $t \geq 1$ has been studied in both on network and combinatorial aspects. In this work, we show that the diameter of $G_{m,t}$ is t. Furthermore, the rainbow connection number of graph $G_{1,t}$ is equal to its diameter which is the smallest possible among the graphs with the same diameter. We also show that the rainbow connection number of $G_{m,t}$ is $t+1$ for $m > 1$ and $t > 1$.

2020 Mathematics Subject Classification. 05C15; 05C82.

Key words and phrases. generalized Farey graph; rainbow coloring; rainbow connection number

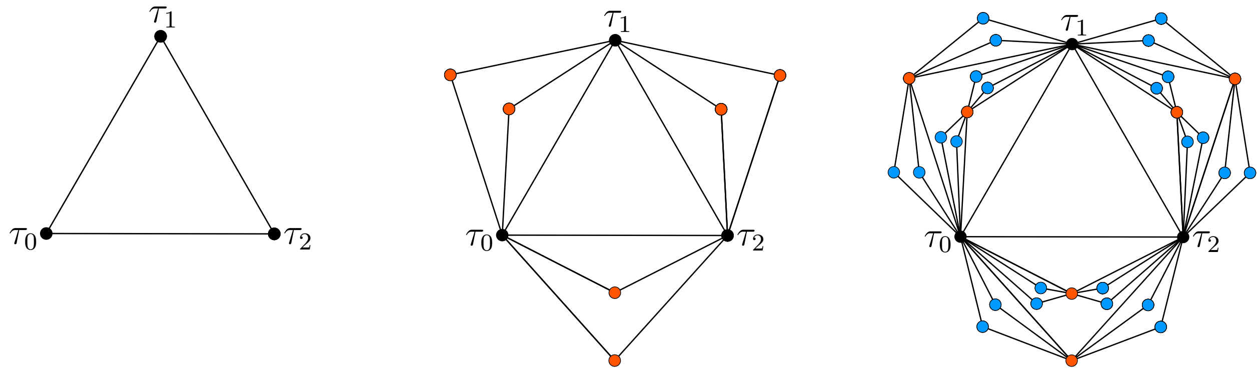
1. Introduction

A generalized Farey graph is characterized as a small-world network graph, and its properties have been extensively studied. In this work, we improve upon an existing result on a generalized Farey graph by obtaining the exact value of its diameter. Furthermore, we find its rainbow connection number whose definition and relation to diameter are described in this section.

Let $G = (V, E)$ be a graph with an edge-coloring c. A subgraph H of G is *rainbow* if $c(e_1) \neq c(e_2)$ for each pair of distinct $e_1, e_2 \in E(H)$. A path is *rainbow* if none of its edges have the same color. A graph G is *rainbow connected* if a rainbow (u, v)-path exists for each pair of distinct $u, v \in V(G)$. The *rainbow connection number* of a graph G, denoted by $\mathrm{rc}(G)$, is the minimum number required for G to be rainbow connected. The notion of rainbow coloring is introduced by Chartrand et al. [1]. Its bound $\mathrm{diam}(G) \leq \mathrm{rc}(G) \leq |E(G)|$ is obvious.

In 2013, Li et al. [2] illustrated an application in a security network in which the rainbow connection number represents the minimum codes required to secure the network. Subsequently, they raised an

DOI: 10.28924/APJM/11-39



(A) $G_{2,1}$ (B) $G_{2,2}$ (C) $G_{2,3}$

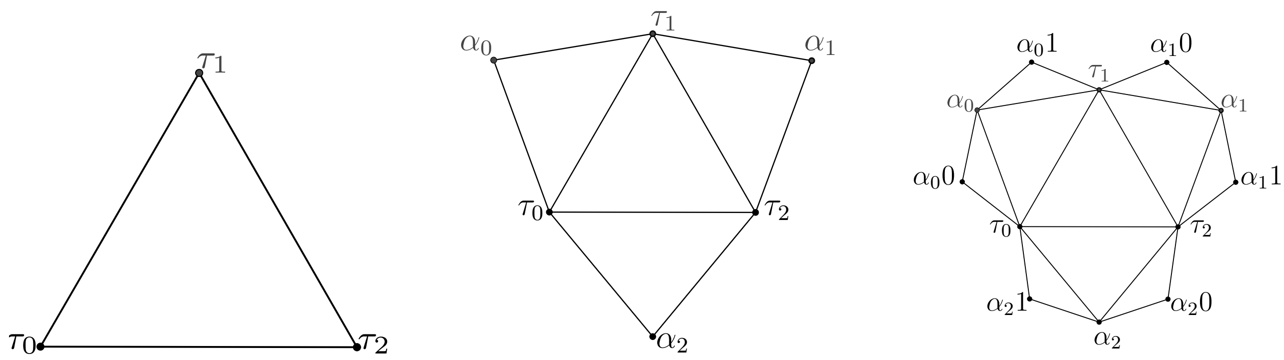
FIGURE 1. Drawings of $G_{2,t}$

interesting problem of characterizing a graph G with $\text{rc}(G) = \text{diam}(G)$. It is known that computing $\text{rc}(G)$ is NP-Hard and even deciding whether $\text{rc}(G) = 2$ is NP-Complete [3]. This may be a reason why not many results appeared for a graph with $\text{rc}(G) = \text{diam}(G)$. Unit interval graphs [4] and certain maximal outer-planar graphs constructed by Deng et al. [5] were shown to have $\text{rc}(G) = \text{diam}(G)$ and arbitrarily large diameters.

A *small-world Farey graph* $\mathcal{F}(t)$ [6] is constructed recursively from a path of length one as the initial graph $\mathcal{F}(0)$. For $\mathcal{F}(t)$ where $t \geq 1$, we add a vertex w and two edges uw and vw to $\mathcal{F}(t-1)$ for each edge uv that first appears in $\mathcal{F}(t-1)$. For $m, t \in \mathbb{N}$, a *generalized Farey graph* $G_{m,t}$ [7] is defined with a recursive construction similar to $\mathcal{F}(t)$ with the initial condition $G_{m,1} = K_3$ where K_3 is a triangle. For $G_{m,t}$ where $t \geq 2$, we add m new vertices and $2m$ edges connecting those new vertices with u and v for each edge uv that first appears in $G_{m,t-1}$ (see examples in Figure 1). Both graphs are characterized as small-world network graphs and their network properties have been investigated in many aspects [6,8].

Various coloring properties of the two graphs were obtained as follows. A small-world Farey graph $\mathcal{F}(t)$ has its chromatic number equal to 3 when $t \geq 1$ [6], and its δ-chromatic number is 2^t when $t > 2$ [9]. Zhang and Comellas [6] showed that $\text{diam}(\mathcal{F}(t)) = t$ when $t \geq 1$. Jiang et. al. [10] gave a shortest path (also called *geodesic path*) between each pair of vertices in each of these two graphs. It should be noted that a geodesic path is not necessarily a rainbow path resulting from edge-coloring. In 2018, Jiang et. al. [7] obtained the bound $\text{diam}(G_{m,t}) \leq 2t + 3$.

In 2022, the *rainbow vertex-connection number*, the minimum number of colors required for each pair of vertices to be connected by a path with internal vertices of distinct colors, of $\mathcal{F}(t)$ is $\text{diam}(\mathcal{F}(t)) = t - 1$ [11] which is the lowest possible among the graphs with the same diameter. So, a similar problem arises for the rainbow connection number of $\mathcal{F}(t)$. In Theorem 5, we improve the aforementioned result on $\text{diam}(G_{m,t})$ by showing that $\text{diam}(G_{m,t}) = \text{diam}(G(t)) = \text{diam}(\mathcal{F}_t) = t$ for $m \geq 1$. We also give unique geodesic paths in $G_{m,t}$ for $m \geq 1$ and $t \geq 1$. Finally, we show that, for $t \geq 1$,



(A) $G(1)$ labeling (B) $G(2)$ labeling (C) $G(3)$ labeling

FIGURE 2. Vertex labelings of $G(1)$, $G(2)$ and $G(3)$

$\mathrm{rc}(\mathcal{F}(t)) = \mathrm{rc}(G_{1,t}) = \mathrm{diam}(G_{1,t}) = t$ and, for $m > 1$ and $t > 1$, $\mathrm{rc}(G_{m,t}) = \mathrm{diam}(G_{m,t}) = t + 1$. As a consequence, the rainbow connection numbers of $G_{1,t}$ and $\mathcal{F}(t)$ are the lowest among the graphs with the same diameter.

2. Generalized Farey graph and its properties

Recall that, for $m, t \in \mathbb{N}$, a generalized Farey graph $G_{m,t}$ [7] is defined with a recursive construction similar to $\mathcal{F}(t)$ with the initial condition $G_{m,1} = K_3$ where K_3 is a triangle. Let τ_0, τ_1, τ_2 be the label of such K_3 (See Figure 2(A)). For the purpose of comparison with the recursive step in a small-world Farey graph, our initial condition starts with $t = 1$ while that of in the definition given by Jiang et al. [7] started with $t = 0$. We use notation $G(t) = G_{1,t}$.

Next, we establish notations and terminologies that will be used in this work. If a vertex $u \in V(G_{m,t})$ first appears in step i, then the *level* of u, denoted by $l(u)$, is i for $i = 1, \ldots, t$. Similarly, if an edge e first appears in step i, then the *level* of e, denoted by $l(e)$, is i. We note that the level of the vertices in $G_{m,t}$ begins with level one. In case $m = 1$, a symmetric drawing and vertex labeling of $G(t) = G_{1,t}$ in this paper are as in Figure 2. We can draw a graph so that w lies between its bases. The method of vertex labeling is explained explicitly in the next section.

For each pair of adjacent vertices $x, y \in V(G_{m,t})$ and a vertex $u \in V(G_{m,t})$ such that $l(u) \geq 2$, if u is added to $G_{m,t}$ correspondingly to the edge xy, then u is a *direct descendant* of x and y. If u is a direct descendant of x and y, then x and y are *bases* of u. We define a *descendant* recursively as follows. We say v is a descendent of u if v is a direct descendent of u or there is z such that v is a descendent of z and z is a descendant of u. We note that for each edge xy in $G_{m,t}$ with $l(xy) < t$, there are m direct descendants of x and y. For an edge xy, an (x, y)-*bundle* $B_{(x,y)}$ is the induced subgraph of $G_{m,t}$ consists of x, y and all of their descendants. We say that $\{x, y\}$ is the *origin* of $B_{(x,y)}$. We note that $B_{(x,y)} = B_{(y,x)}$. We also note that if x is a base of u, then $l(x) < l(u)$. Furthermore, for a vertex u with $l(u) > 2$, there are two bases of u and their levels are distinct in which exactly one of them has level $l(u) - 1$. The other base of

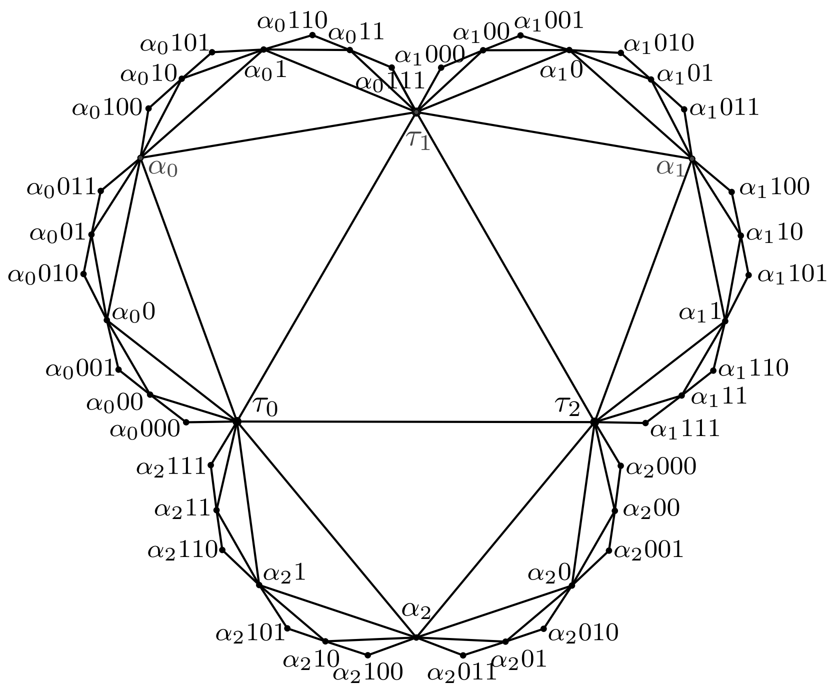


FIGURE 3. Symmetric drawing and vertex labeling of $G(5)$

u has level less than $l(u) - 1$. In 2018, Jiang et. al. [7] stated that $\text{diam}(G_{m,t}) \leq 2t + 1$. We improve such a statement in case $m = 1$ in Theorem 5.

In Lemma 1, we show that, for each vertex, the levels of the vertices in its geodesic paths to its origins are decreasing.

Lemma 1. *Let $u, u', x, y \in V(G_{m,t})$ be such that $l(x) \geq 2$, $u \in B_{(x,y)}$ and $u' \in \{x, y\}$. If $P = u_1 \ldots u_n$ is a geodesic (u, u')-path where $u_1 = u$ and $u_n = u'$ is the only vertex in $\{x, y\}$, then $l(u_i) > l(u_{i+1})$ for $i = 1, \ldots, n-1$.*

Proof. Suppose to the contrary that there exists a path P with a smallest i_0 such that $l(u_{i_0}) \leq l(u_{i_0+1})$. Since distinct vertices of the same level are not adjacent when their level is at least two, it follows that $l(u_{i_0}) < l(u_{i_0+1})$. So u_{i_0} is a base of u_{i_0+1}. Let i_1 be the maximum number such that $l(u_i) < l(u_{i+1})$ for all $i = i_0, \ldots, i_1 - 1$. If $i_1 = i_0 + 1$, then $u_{i_1+1} = u_{i_0+2}$ is a base of u_{i_1} and $u_{i_0+2} \neq u_{i_0}$. Thus, deleting a vertex u_{i_0+1} form P and adding an edge $u_{i_0}u_{i_0+2}$ giving a (u, v)-path with a shorter distance. If $i_1 \geq i_0 + 2$, then $u_{i_0+2} \notin B_{(u_{i_0}, u_{i_0+1})}$; otherwise, the path $u_1 P u_{n_1}$ must go through u_{i_0} or u_{i_0+1} twice. Similarly, we can conclude that $u_{i+2} \notin B_{(u_i, u_{i+1})}$ for all $i = i_0, \ldots, i_1 - 2$. Let w be a base of u_{i_0+1} where $w \neq u_{i_0}$. We have that w is also a base of u_i for $i = i_0, \ldots, i_1$. Since $l(u_{i_1}) > l(u_{i_1+1})$ and $u_{i_1+1} \neq u_{i_0}$, it follows that $w = u_{i_1+1}$. Hence, deleting $u_{i_0+1} P u_{i_1}$ from P and adding an edge $u_{i_0}u_{i_1+1}$ give a shorter (u, v)-path. This completes the proof. $\square$

Remark 2. *Let $u, u', x, y \in V(G_{m,t})$ be such that $l(x) \geq 2$, $u \in B_{(x,y)}$ and $u' \in \{x, y\}$. If $P = u_1 \ldots u_n$ is a geodesic (u, u')-path where $u_1 = u$ and $u_n = u'$ is the only vertex in $\{x, y\}$, then u_1 is a descendant of u_i for $i = 2, \ldots, n$, and u_j is a descendant of u_n for $j = 1, \ldots, n-1$.*

Remark 3. *Let $u, v, x, y \in V(G_{m,t})$ be such $B_{(x,y)}$ is the minimal bundle containing u and v. If P is a geodesic (u, v)-path, then $P \subseteq B_{(x,y)}$.*

The property appears in Lemma 4 is needed to find the diameter of $G_{m,t}$ in Theorem 5.

Lemma 4. *Let $u \in V(G_{m,t})$ be such that $l(u) \geq 2$. If $u \in B_{(\tau_i,\tau_j)}$ for $i \neq j$, then $d(u, u') \leq \frac{t}{2}$ for all $u' \in \{\tau_i, \tau_j\}$.*

Proof. Suppose there exists $\tau_{i_0} \neq \tau_{j_0}$ and $u_0 \in B_{(\tau_0,\tau_j)}$ such that $d(u_0, u'_0) > \frac{t}{2}$ for some $u'_0 \in \{\tau_{i_0}, \tau_{j_0}\}$. Since $B_{(\tau_{i_0},\tau_{j_0})}$ and $B_{(\tau_{i_1},\tau_{j_1})}$ are isomorphic for $\{i_0, j_0\} \neq \{i_1, j_1\}$, there exists $u_1 \in V(B_{(\tau_{i_1},\tau_{j_1})})$ and $u'_1 \in \{\tau_{i_1}, \tau_{j_1}\}$ where $d(u_1, u'_1) = d(u_0, u'_0) > \frac{t}{2}$. Since $V(B_{(\tau_{i_0},\tau_{j_0})}) \cap V(B_{(\tau_{i_1},\tau_{j_1})}) \subseteq \{\tau_0, \tau_1, \tau_2\}$, it follows that $d(u_0, u_1) > t$. Since $\{i_0, j_0\} \neq \{i_1, j_1\}$, there exists a subgraph $H \cong \mathcal{F}(t)$ of $G_{m,t}$ containing u_0, u'_0, u_1, u'_1. Thus $d_{G_{m,t}}(u_0, u_1) \leq d_H(u_0, u_1) \leq t$ contradiction. □

Theorem 5. *For $m \geq 1$ and $t \geq 1$,* $\mathrm{diam}(G_{m,t}) = t$.

Proof. The result can be easily verified when $t \leq 2$. Suppose $t > 2$. Let $u, v \in V(G_{m,t})$ and let $x, y \in V(G_{m,t})$ be such that $B_{(x,y)}$ is the minimal bundle containing u and v. If $x, y \in \{\tau_0, \tau_1\}$, then $B_{(x,y)} \cong \mathcal{F}(t)$. Let P be a geodesic (u, v)-path. By Remark 3, the path $P \subseteq B_{(x,y)}$. Thus $\mathrm{diam}(G_{m,t}) \geq \mathrm{diam}(\mathcal{F}(t)) = t$. In addition $d_{G_{m,t}}(u, v) \leq d_{\mathcal{F}(t)}(u, v) \leq t$ in this case.

We may now assume that $l(x) \geq 2$ and $l(u) \geq 3$. Let P be a geodesic (u, v)-path. If u and v have the same bases, then such bases are x and y. The only geodesic (u, v)-path are uxv and uyv. Consider the case that the bases of u and v are different. If there exists a direct descendant z of x and y where u and v are both descendants of z, then P contains x, y or z; otherwise, P contains x or y. In order to contain a vertex in $V(G_{m,t}) \backslash V(B_{(x,y)})$, the path P must exit $B_{(x,y)}$ at x or y and then enter back to the bundle again which adding a non-necessary length to the path; hence P contains no vertex in $V(G_{m,t}) \backslash V(B_{(x,y)})$. Let P_1 and P_2 be subpaths of P where $P = P_1 P_2$, $P_1 = u_1 \ldots u_{n_1}$, $P_2 = v_{n_2} \ldots v_1$ such that u_{n_1} and v_{n_2} are the only vertices in $\{x, y\}$ (or $\{x, y, z\}$ if such z exists). We note that it is possible that $u_{n_1} = v_{n_2}$. We construct subgraphs H_1 and H_2 of $B_{(x,y)}$ where $u \in V(H_1)$ and $v \in V(H_2)$ by choosing the vertices in H_1 and H_2 via the same recursive construction of a small-world Farey graph with initial condition xy. We note that all the descendants of x and y that lead to u are in H_1, and those that lead to v are in H_2. Since $x \notin \{\tau_0, \tau_1, \tau_2\}$, it follows that $H_i \cong \mathcal{F}(s_i)$ for some $s_i = 1, \ldots, t-1$ where $i = 1, 2$. We note that the origins of H_1 and H_2 are x and y. By the construction of H_1, H_2 and Remark 2, we have that $P_1 \subseteq H_1$ and $P_2 \subseteq H_2$. If there exists a direct descendant z of x and y

such that both u and v are descendants of z, then H_1 and H_2 can be chosen so that $H_1 = H_2$. Thus $d_{G_{m,t}}(u,v) = d_{H_1}(u,v) \le \max\{s_1, s_2\} \le t$. If there is no such z, then $H_1 \ne H_2$ and $1 \le s_1 \le t-1$ and $1 \le s_2 \le t-1$. Hence $d_{G_{m,t}}(u,v) \le d_{H_1}(u,u_{n_1}) + d_{H_2}(v,v_{n_2}) + 1 \le \text{diam}(\mathcal{F}(s_1)) + \text{diam}(\mathcal{F}(s_2)) + 1 \le t$ by Lemma 4. Therefore $\text{diam}(G_{m,t}) = t$. □

3. Property of $G_{m,t}$

In the first part of this section, we give a vertex labeling and some properties of $G(t)$. We then extend the results on $G(t)$ to $G_{m,t}$ later in this section. Several types of vertex labeling of a small-world Farey graph and a generalized Farey graph appear in [10,12,13]. In this work, we label each vertex with a concatenation of a special character with a string in $\{0,1\}^*$, possibly empty. The label used here can be associated with a binary representation of the label of a small-world Farey graph that appeared in [10].

We first label the vertices in $G(t) = G_{1,t}$ and then consider each vertex of $G_{m,t}$ as a copy of a vertex in $G(t)$. A *word* is a label of a vertex in $G(t)$. Recall that we label the vertices in $G(1)$ by τ_0, τ_1, τ_2. Let $\Sigma = \{0,1\}$ and Σ^* be the set of strings of finite length of elements in Σ including an empty string. For a string $s \in \Sigma^*$, the length of s is denoted by $|s|$. We label the vertices in $V(G(t))$ with level q for $2 \le q \le t$ by the words in

$$\mathcal{L}_q = \{\alpha_k\rho : \rho \in \Sigma^*, |\rho| = q-2 \text{ and } k = 0,1,2\}.$$

Hence, the set of labels of the vertices in $V(G(t))$ with level at least two is

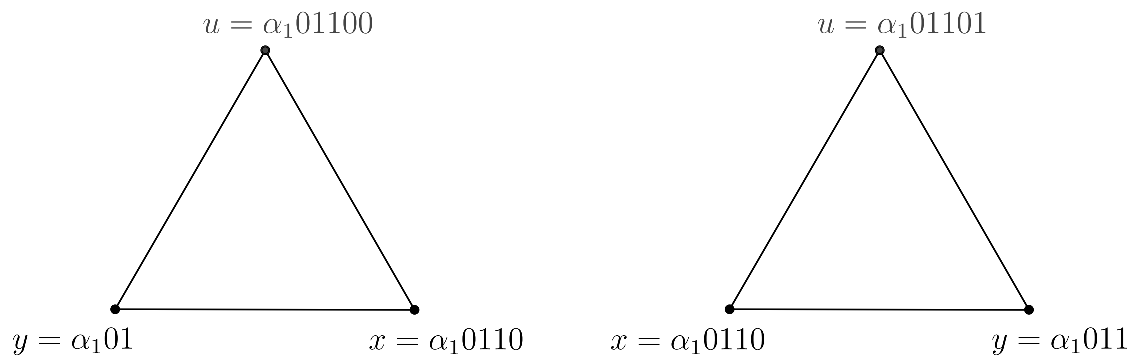
$$\mathcal{L} = \{\alpha_k\rho : \rho \in \Sigma^*, |\rho| = q-2 \text{ for } 2 \le q \le t \text{ and } k = 0,1,2\}.$$

For a word $w \in \mathcal{L}$, we denote the length of w by $|w|$. Hence $|w| = l(w) - 1$ for all $w \in \mathcal{L}$. Suppose $w = \eta_1 \dots \eta_{|w|}$. We denote a subword $w[i,j] = \eta_i \dots \eta_j$ and we write $w[i] = w[i,i]$ for $1 \le i \le j \le |w|$. A *block* β_i in w is the i-th maximal subword of consecutive identical elements. If w consists of p blocks, then we write $w = \beta_1 \dots \beta_p$. We use notation $w_\beta[i,j] = \beta_i \dots \beta_j$ and $w_\beta[i] = w_\beta[i,i]$ for $1 \le i \le j \le p$, and let $|w|_\beta$ be the number of blocks in w. For example, if $w = \alpha_0 0010$, then $|w| = 5$, $|w|_\beta = 4$, $\beta_1 = \alpha_0$, $\beta_2 = 00$, $\beta_3 = 1$ and $\beta_4 = 0$.

Next, we assign an explicit label to a vertex in $G(t)$. In level two, we label a direct descendant of τ_0 and τ_1 by α_0, a direct descendant of τ_1 and τ_2 by α_1, and a direct descendant of τ_0 and τ_2 by α_2. We then recursively label the vertices in $G(t)$. For a vertex $u \in V(G(t))$ where $l(u) \ge 3$ and its bases are x and y such that $l(x) > l(y)$, we label u by $w = \eta_1 \dots \eta_{l(u)-1} \in \mathcal{L}_{l(u)}$ where $\eta_j = x[j]$ for $j < l(u) - 1$ and

$$\eta_{l(u)-1} = \begin{cases} 0 & \text{if } u \text{ is on the counter-clockwise side of } x, \\ 1 & \text{if } u \text{ is on the clockwise side of } x. \end{cases}$$

An example of vertex labeling appears in Figure 4. For each $\epsilon \in \{0,1\}$, we denote $\overline{\epsilon} = 1 - 0$, and $\overline{w} = \eta_1 \overline{\eta}_2 \dots \overline{\eta}_{|w|}$.



(A) u is on the counter-clockwise side of x (B) u is on the clockwise side of x

FIGURE 4. Direct descendant labeling examples when $l(x) > l(y)$ and $l(x) \geq 2$ in $G(t)$

From here on, we may use the labels to represent the vertices. For a vertex u with $l(u) > 2$, the base of u with level $l(u) - 1$ is $u[1, l(u) - 2] = u[1, |u| - 1]$. For each pair of adjacent vertices u, v where $uv \notin E(G(1))$ and $1 \leq l(v) < l(u) < t$, the direct descendent of u and v in $G(t)$ is uniquely determined. We define $u \oplus v = w$ to be the direct descendant of u and v. Thus $w = w[1, |w| - 1] \oplus v$ for some $v \in V(G(t))$ with $l(v) < l(w) - 1$. Moreover, for each vertex u with $l(u) \geq 2$, there is exactly one direct descendant on the clockwise and counter-clockwise of u in $G(t)$. It implies that a vertex with level at least three has exactly one base on each side. Let A_0, A_1 and A_2 be a (τ_0, τ_1)-bundle, a (τ_1, τ_2)-bundle and a (τ_0, τ_2)-bundle respectively. We note that $\alpha_k \in V(A_k)$ for $k = 0, 1, 2$. For $t \geq 2$, let $G_2 = G_2(t)$ be the induced subgraph of $G(t)$ such that $V(G_2) = \{\tau_0, \tau_1, \tau_2, \alpha_0, \alpha_1, \alpha_2\}$, i.e., the subgraph generated in the second step of $G(t)$.

Now, we consider labeling in $G_{m,t}$ where $m > 1$. For a subgraph H of $G_{m,t}$ where $H \cong G(t)$, we inherit the label of $G(t)$ to H with a superscript H to indicate the subgraph that such vertex is contained. For example, a vertex $w \in V(H)$ which is a copy of vertex $\alpha_0 01 \in V(G(t))$ is labeled $w = (\alpha_0 01)^H$. We note that τ_0, τ_1, τ_2 are in all subgraph that is isomorphic to $G(t)$ of $G_{m,t}$. Thus we omit superscript for τ_0, τ_1, τ_2, and let G_1 be a triangle $\tau_0\tau_1\tau_2$. Let G_2^H be an induced subgraph of $G_{m,t}$ such that its vertex set consists of τ_0, τ_1, τ_2 and their direct descendants in H, i.e., $V(G_2^H) = \{\tau_0, \tau_1, \tau_2, \alpha_0^H, \alpha_1^H, \alpha_2^H\}$.

The following lemma compute explicit bases of each vertex in $G(t)$.

Lemma 6. [10] *In $G(t)$, let $w \in V(A_k)$ be such that $l(w) = n \geq 3$, and let $q = |\beta_{|w|_\beta}|$ for $k \in \{0, 1, 2\}$. If v is a base of w with $l(v) < l(w) - 1$, then*

$$v = \begin{cases} \tau_j & \text{if } w = \alpha_k \epsilon^{n-2} \text{ and } j \equiv k + \epsilon \pmod 3 \text{ for some } \epsilon \in \{0, 1\}, \\ w\,[1, n - q - 2] & \text{if } |w|_\beta \geq 3. \end{cases}$$

Remark 7. *In $G(t)$, for a vertex $w = \alpha_k \epsilon_1 \ldots \epsilon_n$ where $n > 1$, if $\epsilon_{n-1} \neq \epsilon_n$, then $w = w[1, n-1] \oplus w[1, n-2]$.*

For a path P and distinct vertices $u, v \in V(P)$, we let uPv be a subpath of P from u to v.

In 2018, Jiang et. al. [10] gave a shortest path between each pair of vertices in a generalized Farey graph. Lemmas 8, 9 and 11 show that each pair of vertices in these lemmas has a unique geodesic path.

Lemma 8. *In $G(t)$, for a fixed $k \in \{0,1,2\}$, let $w \in A_k$ be such that $l(w) = n$ where n is an even number that $2 \le n \le t$. If $w = \alpha_k(\epsilon\bar{\epsilon})^{\frac{n-2}{2}}$ where $\epsilon \in \{0,1\}$, then $d(w,\alpha_k) = \frac{n-2}{2}$. Moreover, there is a unique geodesic path which is $w_n w_{n-2} \dots w_2$ where $w_i = \alpha_k(\epsilon\bar{\epsilon})^{\frac{i-2}{2}}$ for a positive even number $i \le n$.*

Proof. Let $w_i = \alpha_k(\epsilon\bar{\epsilon})^{\frac{i-2}{2}}$ and $u_i = \alpha_k(\epsilon\bar{\epsilon})^{\frac{i-4}{2}}\epsilon$ for a positive even number $i \le n$. It is obvious that $d(w_2,\alpha_k) = 0$, $d(w_4,\alpha_k) = 1$ and there is a unique geodesic (w_4,α_k)-path w_4w_2. By Lemma 6, we have $w_i = u_i \oplus w_{i-2}$ for $i = 4,\dots,n$. Suppose that $d(w_i,\alpha_k) = \frac{i-2}{2}$ with a unique geodesic (w_i,α_k)-path $w_i w_{i-2}\dots w_2$ for $i = 2,4,\dots,n-2$.

Suppose that there exists a (w_n,α_k)-path P with length less than $d(w_{n-2},\alpha_k)+1$. Then $u_n \in V(P)$, $w_{n-2} \notin V(P)$ and $d(u_n,\alpha_k) < d(w_{n-2},\alpha_k)$. Hence $w_{n-2}u_nP\alpha_k$ is a (w_{n-2},α_k)-path of length $d(w_{n-2},\alpha_k)$ contrary to the uniqueness of the geodesic (w_{n-2},α_k)-path. Thus $d(w_n,\alpha_k) \ge d(w_{n-2},\alpha_k)+1 = \frac{n-2}{2}$. Since w_n and w_{n-2} are adjacent, it follows that $d(w_n,\alpha_k) = \frac{n-2}{2}$.

Next we show that the geodesic (w_n,α_k)-path is unique. Let Q be a path $w_nw_{n-2}\dots w_2$. Suppose there exists a geodesic (w_n,α_k)-path $Q' \ne Q$. We have $u_n \in V(Q')$, $w_{n-2} \notin V(Q')$ and $|E(u_nQ'\alpha_k)| = d(u_n,\alpha_k) = d(w_{n-2},\alpha_k) = |E(w_{n-2}Q\alpha_k)|$. By Lemma 6, we have $u_i = w_{i-2} \oplus u_{i-2}$ for $i = 4,6\dots,n$. Since $w_{n-2} \notin V(Q')$, it follows that $u_nu_{n-2} \in E(Q')$.

Let $i_0 = \max\{i : w_i \in V(Q')\}$. We note that $i_0 \ge 2$.

Claim $d(u_n,u_{i_0+2}) = \frac{n-i_0-2}{2}$.

It is obvious that $d(u_{i_0+4},u_{i_0+2}) = 1$. Suppose that $d(u_i,u_{i_0+2}) = \frac{i-i_0-2}{2}$ for $i = 6,\dots,n-2$. We have $d(u_n,u_{i_0+2}) \le d(u_{n-2},u_{i_0+2}) + d(u_n,u_{n-2}) = \frac{n-2-i_0-2}{2} + 1 = \frac{n-i_0-2}{2}$. If there exists a (u_n,u_{i_0+2})-path P of length less than $\frac{n-i_0-2}{2}$, then $u_{n-2} \notin V(P)$. Hence $w_{n-2} \in V(P)$ and $|E(w_{n-2}Pu_{i_0+2})| \le \frac{n-i_0-6}{2}$. Since u_{n-2} and w_{n-2} are adjacent, it follows that $|E(u_{n-2}w_{n-2}Pu_{i_0+2})| = \frac{n-i_0-4}{2} < d(u_{n-2},u_{i_0+2})$, a contradiction. Therefore $d(u_n,u_{i_0+2}) = \frac{n-i_0-2}{2}$ as claimed.

By claim, we have $|E(Q')| \ge |u_nQ'u_{i_0+2}| + d(w_{i_0},\alpha_k) + |E(u_{i_0+2}w_{i_0})| \ge \frac{n-i_0+2}{2} + \frac{i_0-2}{2} + 1 = \frac{n+2}{2}$, a contradiction. Therefore the geodesic (w_n,α_k)-path is unique and it is $w_nw_{n-2}\dots w_2$ with $d(w_n,\alpha_k) = \frac{n-2}{2}$. □

Lemma 9. *In $G(t)$, for a fixed $k \in \{0,1,2\}$, let $w \in A_k$ be such that $l(w) = n$ where n is odd and $3 \le n \le t$. If $w = \alpha_k(\epsilon\bar{\epsilon})^{\frac{n-3}{2}}\epsilon$, then $d(w,\tau_j) = \frac{n-1}{2}$ when $j \equiv k+\epsilon \pmod 3$. Moreover, there is a unique geodesic path which is $w_nw_{n-2}\dots w_3\tau_j$ where $w_i = \alpha_k(\epsilon\bar{\epsilon})^{\frac{i-3}{2}}\epsilon$ for a positive odd number $i = 3,5,\dots,n$.*

Proof. Similar to Proposition 8. □

Lemma 10. *In $G(t)$, for $k \in \{0,1,2\}$, let $w \in A_k$ be such that $w = \alpha_k(\epsilon\bar{\epsilon})^{\frac{n-3}{2}}\epsilon$ for some odd n where $3 \le n \le t$. If $j_1 \equiv k+\epsilon \pmod 3$ and $j_1 \ne j_2$, then $d(w,\tau_{j_1}) + 1 = d(w,\tau_{j_2})$ when $j_1 \equiv k+\epsilon \pmod 3$ and $j_1 \ne j_2$.*

Proof. Since $w[1,2] = \alpha_k\epsilon$, the vertex w is contained in bundle $B_{(\alpha_k,\tau_{j_1})}$. Let P be a geodesic (w, τ_{j_2})-path. Since $\tau_{j_2} \notin B_{(\alpha_k,\tau_{j_1})}$, to exit $B_{(\alpha_k,\tau_{j_1})}$, the path P must contain α_k or τ_{j_1}. Suppose to the contrary that $d(w, \tau_{j_1}) \geq d(w, \tau_{j_2})$. We have $\tau_{j_1} \notin V(P)$. Hence $\alpha_k \in V(P)$ and $d(w, \tau_{j_1}) \geq d(w, \tau_{j_2}) = 1 + d(w, \alpha_k)$. Combining paths $wP\alpha_k$ and $\alpha_k\tau_{j_1}$ yields a geodesic (w, τ_{j_1})-path that is different from the one obtained by Proposition 9. This contradicts the uniqueness of the geodesic (w, τ_{j_1})-path. Hence $d(w, \tau_{j_1}) < d(w, \tau_{j_2})$. Since τ_{j_1} and τ_{j_2} are adjacent, it follows that $d(w, \tau_{j_1}) + 1 = d(w, \tau_{j_2})$. □

Lemma 11. *In $G(t)$, for a fixed $k \in \{0, 1, 2\}$, let $w \in A_k$ be such that $l(w) = n$ where n is odd and $3 \leq n \leq t$. If there exists an odd number $i_0 \leq n$ such that $w = w[1, i_0](\epsilon\bar{\epsilon})^{\frac{n-i_0-1}{2}}$, then $d(w, w[1, i_0]) = \frac{n-i_0-1}{2}$ and the $(w, w[1, i_0])$-geodesic path is unique.*

Proof. The proof is similar to Proposition 8. □

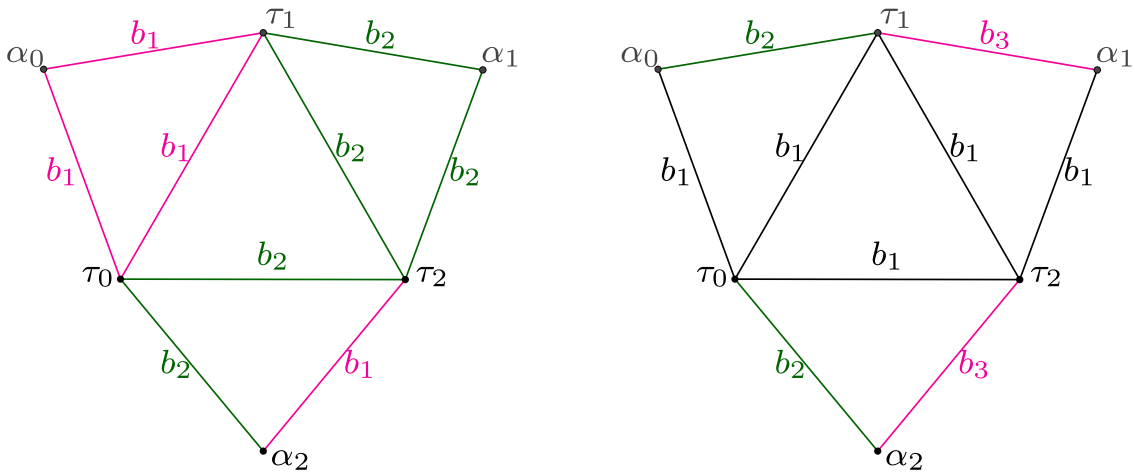
Lemma 12. *For $m > 1$ and $t \geq 1$, let $H \cong G(t)$ be a subgraph of $G_{m,t}$. Let $u, v \in V(H)$. If P is a unique geodesic (u, v)-path in H, then P is a unique geodesic (u, v)-path in $G_{m,t}$.*

Proof. Let $x, y \in V(H)$ be such that $B^H_{(x,y)}$ is the minimal bundle in H containing both u and v. By Remark 3, we have $P \subset B^H_{(x,y)}$. Let us recall that $B^H_{(x,y)}$ contains all the vertices having u or v as their descendant in $G_{m,t}$ with level at least $\max\{l(x), l(y)\}$. Suppose that there is a geodesic (u, v)-path $Q = u_1 \ldots u_n$ in $G_{m,t}$ where $u_1 = u$, $u_n = v$ and $P \neq Q$. Thus, there exists a maximum $i_0 \in \{2, \ldots, n-1\}$ such that $u_{i_0} \notin V(H)$. We have that u_{i_0+1} is a base of u_{i_0}, and $u_{i_0+1} \in V(H)$. Let $w \neq u_{i_0+1}$ be another base of u_{i_0}. We note that u and v are not descendants of u_{i_0}. In order to reach u and v, the path Q has to go back to a vertex in H which means Q has to go through w. Since w and u_{i_0+1} are adjacent and $|E(wQu_{i_0+1})| \geq 2$, replace the wQu_{i_0+1} path in Q with an edge wu_{i_0+1} yields a shorter (u, v)-path, contradiction. Therefore, if P is a geodesic (u, v)-path in H, then it is also a geodesic path in $G_{m,t}$. □

By Lemma 12, the paths given in Lemmas 8-11 are geodesic in $G_{m,t}$.

4. Rainbow connection number on a generalized small-world Farey graph

We investigate the rainbow connection number of $G(t)$ for $t \geq 1$, and later extend to $G_{m,t}$ for $m \geq 1$ at the end of this section. In $G(t)$, we give an ordering to the edges in the same level through the ordering of vertices in A_0. Let $\tau_0 < \alpha_0 < \tau_1$. The other vertices are ordered by the following process. In this ordering, we relabel a vertex labeled by w as $w\epsilon$, and let $0 < \epsilon < 1$. Then we order vertices by lexicographical ordering on the new labeling. For example, $\alpha_0 0\epsilon < \alpha_0\epsilon < \alpha_0 1\epsilon$, $\alpha_0 0\epsilon < \alpha_0 01\epsilon < \alpha_0\epsilon$. Outside this ordering, we still use the original labeling in the remaining of the paper. By this ordering, for any $x, y, z \in V(G(t))$ where x is a direct descendant of y and z such that $y < z$, we have that $y < x < z$. Let $u_1, u_2, v_1, v_2 \in V(A_0)$. For each pair of edges u_1v_1 and u_2v_2 that appear in the same level, where $u_i < v_i$ for $i = 1, 2$, we say that $u_1v_1 \prec u_2v_2$ if and only if $u_1 < u_2$.



(A) even t (B) odd t

FIGURE 5. The coloring of G_2-subgraph of $G_2(t)$

In $G(t)$, we define isomorphism functions $f_1 : B_{(\tau_0,\tau_1)} \to B_{(\tau_1,\tau_2)}$, $f_2 : B_{(\alpha_0,\tau_1)} \to B_{(\alpha_2,\tau_0)}$ and $f_3 : B_{(\alpha_1,\tau_1)} \to B_{(\alpha_2,\tau_2)}$ by

$$f_1(u) = \begin{cases} \tau_1 & \text{if } u = \tau_1, \\ \tau_2 & \text{if } u = \tau_0 \\ \alpha_1\overline{w} & \text{if } u = \alpha_0 w \in B_{(\tau_0,\tau_1)} \text{ for } w \in \{0,1\}^*, \end{cases}$$

$$f_2(u) = \begin{cases} \tau_0 & \text{if } u = \tau_1 \\ \alpha_2 w & \text{if } u = \alpha_0 w \in B_{(\alpha_0,\tau_1)} \text{ for } w \in \{0,1\}^*, \end{cases}$$

and

$$f_3(u) = \begin{cases} \tau_2 & \text{if } u = \tau_1 \\ \alpha_2 w & \text{if } u = \alpha_1 w \in B_{(\alpha_1,\tau_1)} \text{ for } w \in \{0,1\}^*. \end{cases}$$

For $uv \in E(G(t))$ and $i = 1,2,3$, let $f_i(uv) = f_i(u)f_i(v)$ where applicable. We note that f_i also preserves the level of the vertices and edges for $i = 1,2,3$. We order the edges in level $i \geq 2$ in A_0 as an increasing sequence $\{e^i_j\}_{j=0}^{2^{i-1}-1}$. The following statements are true:

- $e^i_0 = (\tau_0, \alpha_0 0^{i-2})$,
- for $j = 0, \ldots, 2^{i-2} - 1$, if an endpoint of e^i_j is in $\{\tau_0, \tau_1\}$, such endpoint is τ_0,
- for $j = 2^{i-2}, \ldots, 2^{i-1} - 1$, if an endpoint of e^i_j is in $\{\tau_0, \tau_1\}$, such endpoint is τ_1.

We color $G_2(t)$ according to the parity of t as in Figures 5a and 5b.

Let $h_t : E(G(t)) \to \{b_1, \ldots, b_t\}$ be an edge-coloring such that $h_t|_{G_2}$ is the coloring appeared in 5a and 5b. Now, we color the edges in A_0 that are not contained in G_2 by

$$h_t(e^i_j) = \begin{cases} b_{i-1} & \text{where } j \equiv 0, 3 \pmod 4, \\ b_i & \text{where } j \equiv 1, 2 \pmod 4). \end{cases}$$

Next, we color the edges in A_1. For any $e \in E(A_1) \backslash E(G_2)$, we define

$$h_t(e) = \begin{cases} h_t(f_1^{-1}(e)) + 1 & \text{if } h_t(f_1^{-1}(e)) \text{ and } t \text{ have different parities,} \\ h_t(f_1^{-1}(e)) - 1 & \text{if } h_t(f_1^{-1}(e)) \text{ and } t \text{ have the same parity.} \end{cases}$$

Next, we color $e \in E(A_2) \backslash E(G_2)$ by

$$h_t(e) = \begin{cases} h_t(f_2^{-1}(e)) & \text{for } e \in B_{(\alpha_2, \tau_0)}, \\ h_t(f_3^{-1}(e)) & \text{for } e \in B_{(\alpha_2, \tau_2)}. \end{cases}$$

By the definition of h_t, for $t \geq 3$ and $i \geq 3$, the first edge in level i is $(\tau_0, \alpha_0 0^{i-2})$. We color the edges in A_0 periodically by colors b_{i-1}, b_i, b_i and b_{i-1} starting at $(\tau_0, \alpha_0 0^{i-2})$. Then, we use the isomorphism functions to color the edges in A_1 and A_2. We say that an edge e is *odd* if $h_t(e) = b_i$ for some odd number i, and e is *even* if i is even. For an edge e with level at least three, if $e \in E(A_0)$, then the parities of the indices of the colors $h_t(e)$ and $h_t(f_1(e))$ are different. If $e \in E(B_{(\alpha_0,\tau_1)}) \cup E(B_{(\alpha_1,\tau_1)})$, then the parity of $h_t(e)$ and $h_t(f_j(e))$ is the same for $j = 2, 3$. For any distinct $u, v \in V(G(t))$, we say that a (u, v)-path $u_1 \ldots u_n$ where $u_1 = u$ and $u_n = v$ is an *odd-colored path* if its edges are all odd and $l(u_i) > l(u_{i+1})$ for $i \leq n - 1$. Similarly, a (u, v)-path $u_1 \ldots u_n$ where $u_1 = u$ and $u_n = v$ is an *even-colored path* if its edges are all even and $l(u_i) > l(u_{i+1})$ for $i \leq n - 1$.

Lemma 13. *In $(G(t), h_t)$, let $x, y \in V(A_0)$ where $l(x) \geq 3$, $l(y) \geq 2$ and y is a base of x. Then $l(x) = l(y) + 1$ if and only if $h_t(xy) = b_{l(x)}$.*

Proof. Let $w, z \in V(A_0)$ be the bases of y where $w < z$, and let w' and z' be the direct descendants of w, y and y, z respectively. It follows that $w < w' < y < z' < z$. The edges in level $l(y) + 1$ with both endpoints in $\{y, z, w, z', w'\}$ consists of $ww', w'y, yz', z'z$ ordered increasingly. Since y is a base of x and $l(x) = l(y) + 1$, it follows that $xy \in \{w'y, z'y\}$. If $w'w$ is the first edge of level $l(y) + 1$, then $h_t(w'w) = b_{l(y)} = h_t(z'z)$ and $h_t(w'y) = b_{l(y)+1} = h_t(w'z)$. Since each vertex in A_0 with level $l(y)$ gives four corresponding edges in level $l(y) + 1$ in such ordering. The lemma is true by the periodicity of the coloring h_t in A_0. □

Lemma 14 and 15 gives an existence of a rainbow path of the same parity of a vertex to one of its origins which later use to construct a rainbow path in $G(t)$ in Theorem 16.

Lemma 14. *Let $x, y, z \in V(G(t))$ with an edge-coloring h_t be such that $x = y \oplus z$. There exists a rainbow (u, u')-path with all edges of the same parity where u' is the only vertex in $\{x, y, z\}$ for all $u \in B_{(y,z)}$.*

Proof. Consider A_0. It can be easily verified when $u \in V(G_2)$. Suppose $u \notin V(G_2)$. By the definition of h_t, each $u \in V(A_0)\backslash V(G_2)$ is incident to one odd and one even edge connecting u to its bases. Hence, we are able to construct a path by consecutively choosing either odd or even edges to a base of a new vertex in the current path until it reaches x, y or z. Let $P = u_1 \dots u_n$ be the constructed path. By the construction, we have $l(u_i) > l(u_{i+1})$ for $i < n$. Thus, the path P is either an odd-colored path or an even-colored path where $u_n = u'$ is the only vertex in $\{x, y, z\}$. We note that P does not contain any edge in G_2.

Next, we show that P is a rainbow path. For a fixed $i_0 \le n-2$, we have $h_t(u_{i_0}u_{i_0+1}) \in \{b_{l(u_{i_0})}, b_{l(u_{i_0})-1}\}$ and $h_t(u_{i_0+1}u_{i_0+2}) \in \{b_{l(u_{i_0+1})}, b_{l(u_{i_0+1})-1}\}$. If $h_t(u_{i_0}u_{i_0+1}) \neq h_t(u_{i_0+1}u_{i_0+2})$, then we are done. Suppose to the contrary that $h_t(u_{i_0}u_{i_0+1}) = h_t(u_{i_0+1}u_{i_0+2})$. We have $h_t(u_{i_0}u_{i_0+1}) = h_t(u_{i_0+1}u_{i_0+2}) = b_{l(u_{i_0})-1} = b_{l(u_{i_0}+1)}$. Since $h_t(u_{i_0}u_{i_0+1}) = b_{l(u_{i_0})-1}$, it follows that $l(u_{i_0+1}) < l(u_{i_0}) - 1$ by Lemma 13. Hence $b_{l(u_{i_0+1})} \neq b_{l(u_{i_0})-1}$, a contradiction. Therefore P is a rainbow path.

We note that $h_t|_{A_1} = h_t \circ f_1|_{A_0}$ switches the parity of the pre-image edge in A_0 and its image in A_1. Moreover $h_t|_{B_{(\alpha_2,\tau_0)}} = h_t \circ f_2|_{B_{(\alpha_0,\tau_1)}}$ preserves the parity of the pre-image edge in A_0 and its image in A_2, while $h_t|_{B_{(\alpha_2,\tau_2)}} = h_t \circ f_3|_{B_{(\alpha_1,\tau_1)}}$ preserves the parity of the pre-image edge in A_1 and its image in A_2.

If $x, y, z \in V(A_1)$, then $f_1^{-1}(u), f_1^{-1}(x), f_1^{-1}(y), f_1^{-1}(z) \in V(A_0)$ where $f_1^{-1}(x) = f_1^{-1}(y) \oplus f_1^{-1}(z)$ and $f_1^{-1}(u) \in B_{(f_1^{-1}(y), f^{-1}(z))}$. Hence, there exists an odd-colored or even-colored rainbow $(f_1^{-1}(u), v)$-path P_1 where v is the only vertex contained in $\{f_1^{-1}(x), f_1^{-1}(y), f_1^{-1}(z)\}$. Thus $h_t(f_1(P_1))$ is an odd-colored or even-colored rainbow (u, u')-path where $u' = f_1(v)$ is the only vertex in $\{x, y, z\}$. Similarly, we have an even-colored or odd-colored rainbow (u, u')-path for $u \in V(A_2)$ by considering the preimages of f_2 and f_3. □

Lemma 15 is a direct result of Lemma 14.

Lemma 15. *Let $y, z \in V(G(t))$ with an edge-coloring h_t be such that their direct descendant is not in G_2. There exists a rainbow (u, u')-path with all edges of the same parity where u' is the only vertex in $\{y, z\}$ for all $u \in B_{(y,z)}$.*

For any $u \in V(A_i)\backslash V(G_2)$ and $v \in V(A_j)\backslash V(G_2)$, by Lemma 14, there exist an odd-colored rainbow (u, u')-path and an even-colored rainbow (v, v')-path where u' and v' are the only vertices in $V(G_2)$. Table 1 presents a rainbow (u, v)-path for all non-adjacent $u, v \in V(G_2)$. These paths are used to connect rainbow paths between A_i and A_j for $i \neq j$. We note that any pair of $u, v \in V(G_2)$ that is not presented in Table 1 is adjacent and we are able to use an edge uv to connect paths between A_i and A_j.

TABLE 1. List of a rainbow path in G_2

u'	v'	(u', v')-path	list of colors when t is even	list of colors when t is odd
α_0	α_1	$\alpha_0\tau_1\alpha_1$	b_1b_2	b_2b_3
α_0	α_2	$\alpha_0\tau_0\alpha_2$	b_1b_2	b_1b_2
α_0	τ_2	$\alpha_0\tau_1\tau_2$	b_1b_2	b_2b_1
α_1	α_2	$\alpha_1\tau_2\alpha_2$	b_2b_1	b_1b_3
α_1	τ_0	$\alpha_1\tau_1\tau_0$	b_2b_1	b_3b_1
α_2	τ_0	$\alpha_2\tau_0$ $\alpha_2\tau_2\tau_0$	b_2 b_1b_2	b_2 b_3b_1

TABLE 2. Parity of the chosen same-parity path when t is even

(1^{st}-bundle, parity of P_0)	(2^{nd}-bundle, parity of P_1)
(A_0, odd)	(A_1, even)
(A_0, odd)	(A_2, even)
(A_1, even)	(A_2, odd)

TABLE 3. Parity of the chosen same-parity path when t is odd

(1^{st}-bundle, parity of P_0)	(2^{nd}-bundle, parity of P_1)
(A_0, even)	(A_1, odd)
(A_0, odd)	($B_{(\alpha_2,\tau_0)}$, even)
(A_0, even)	($B_{(\alpha_2,\tau_2)}$, odd)
(A_1, odd)	($B_{(\alpha_2,\tau_0)}$, even)
(A_1, even)	($B_{(\alpha_2,\tau_2)}$, odd)

Theorem 16. *For a positive integer t, a graph $(G(t), h_t)$ is rainbow connected, and $\mathrm{rc}(G(t)) = t$.*

Proof. By Theorem 5, we have $\mathrm{rc}(G(t)) \geq t$. Next, we show that a graph $G(t)$ with coloring h_t is rainbow connected. This can easily be verified when $t = 1, 2$. Suppose $t \geq 3$. Let u and v be vertices in $V(G(t))$.

Case 1. $u \in V(A_i)$ and $v \in V(A_j)$ where $0 \leq i < j \leq 2$.

If u and v are non-adjacent vertices in $V(G_2)$, then we use the rainbow path in Table 1. Consider $u \in V(A_i)\backslash V(G_2)$ and $v \in V(A_j)\backslash V(G_2)$ for some $0 < i < j \leq 2$. By Lemma 14, there exist a same-parity-colored rainbow (u, u')-path P_0 and a same-parity-colored rainbow (v, v')-path P_1 when u', v' are the only vertices in G_2. The parities of P_0 and P_1 depend on the bundles A_j and A_j as appeared in

TABLE 4. Minimum even color b_{i_0} and odd color b_{j_0} that can appear in an even-colored (u, v)-path or an odd-colored (u, v)-path where v is the only vertex in $V(G_2(t))$

v	(bundle, b_{i_0}, b_{j_0}) when t is even	(bundle, b_{i_0}, b_{j_0}) when t is odd
α_0	(A_0, b_4, b_3)	(A_0, b_4, b_3)
α_1	(A_1, b_4, b_3)	(A_1, b_2, b_5)
α_2	$(B_{(\alpha_2,\tau_0)}, b_4, b_3)$ $(B_{(\alpha_2,\tau_2)}, b_4, b_3)$	$(B_{(\alpha_2,\tau_0)}, b_4, b_3)$ $(B_{(\alpha_2,\tau_2)}, b_2, b_5)$
τ_0	(A_0, b_2, b_3) (A_2, b_2, b_3)	(A_0, b_2, b_3) (A_2, b_2, b_3)
τ_1	(A_0, b_2, b_3) (A_1, b_4, b_1)	(A_0, b_2, b_3) (A_1, b_2, b_3)
τ_2	(A_1, b_4, b_1) (A_2, b_4, b_1)	(A_1, b_2, b_3) (A_2, b_2, b_3)

Tables 2 and 3. We note that the parity of colors of P_0 and P_1 are different. Tables 4 gives the smallest color that possibly appears in P_0 and P_1. Since there exists a path in Table 1 with colors less than those appear in P_0 and P_1, by combining the results in Tables 1 and 4, we are able to connect P_0 and $\overline{P_1}$ via the path in Table 1 if u' and v' are not adjacent. The combined path is a rainbow path. If either $u \in V(G_2)$ or $v \in V(G_2)$, then we use the same argument in which either P_0 or P_1 is trivial.

Case 2. $u, v \in V(A_i)$ for some $i = 0, 1, 2$.

Let $x, y \in V(G(t))$ be such that $B_{(x,y)} \subseteq A_i$ is the minimal bundle containing u and v and let z be the direct descendant of x and y.

Case 2.1. $B_{(x,y)} \neq A_i$ for all $0 \leq i \leq 2$.

If $\{x, y\} \neq \{\tau_i, \tau_j\}$ for some $0 \leq i < j \leq 2$, then there exists an odd-colored rainbow (u, u')-path P_0 and an even-colored rainbow (v, v')-path P_1 by Lemma 15. We note that changing the color of xy does not affect the result in Lemma 15. If P_0 and P_1 intersect, says at w, then $uP_0w\overline{P}_1v$ is a rainbow (u, v)-path. Now, we suppose that P_0 and P_1 do not intersect. Hence $u', v' \in \{x, y\}$ and $u' \neq v'$. Let $P_0 = u_1 \dots u_n$ and $P_1 = v_1 \dots v_s$ where $u_1 = u, u_n = u', v_1 = v, v_s = v'$ and u', v' are the only vertices in $\{x, y\}$. Suppose $u' = x$ and $v' = y$. If $h_t(u'v') \notin h_t(P_0) \cup h_t(P_1)$, then we are done. Suppose to the contrary that $h_t(u'v') \in h_t(P_0) \cup h_t(P_1)$. The only possible edge with color $h_t(xy) = h_t(u'v')$ in $E(B_{(x,y)}) \backslash \{xy\}$ is either xz or yz. Without loss of generality, we suppose that $h_t(xy) = h_t(xz)$ and $xz \in E(P_0)$. It follows that $u' = x$. Since P_0 and P_1 do not intersect, the vertex z is not in P_1 and $v' = y$. Hence $v \in V(B_{(y,z)})$ and $u \in V(B_{(x,z)})$. Let P'_0 and P'_1 be an even-colored (u, u'')-path and an odd-colored (v, v'')-path where u'' and v'' are the only vertices in $\{x, y\}$. If P_0' and P_1' intersect, then we also have a rainbow (u, v)-path. Suppose that P_0' and P_1' do not intersect. So $u'' \neq v''$. If $u' = v''$ and $v' = u''$, then P_0' and P_1' contain z which is not possible. Thus $u'' = u'$ and $v'' = v'$. Since the parity of the colors in P_0' and P_1' are different, we have that $xz \notin E(P_0')$. Hence $P_0'\overline{P}_1'$ is a rainbow (u, v)-path.

Case 2.2. $B_{(x,y)} = A_i$ for some $0 \leq i \leq 2$.

If $B_{(x,y)} = A_i$, then there exists an odd-colored rainbow (u, u')-path P_0 and an even-colored rainbow (v, v')-path P_1 by where u', v' are the only vertices in $\{x, y, z\}$ by Lemma 14. The similar argument in the case $B_{(x,y)} \neq A_i$ also leads a rainbow (u, v)-path in case $B_{(x,y)} = A_i$. Thus, there is a rainbow (u, v)-path.

Therefore $(G(t), h_t)$ is rainbow connected and hence $\mathrm{rc}(G(t)) = t$. □

Corollary 17. *For a positive integer t, we have $\mathrm{rc}(\mathcal{F}(t)) = \mathrm{diam}(\mathcal{F}) = t$.*

Next, we give a coloring that leads to a rainbow connected $G_{m,t}$. Let (H, h_t) be a subgraph of $G_{m,t}$ where $H \cong G(t)$. For each $e \in E(G_{m,t})$, let e^H be the copy of e in H. We define an edge-coloring $c_t : E(G_{m,t}) \to \{b_1, \ldots, b_{t+1}\}$ by

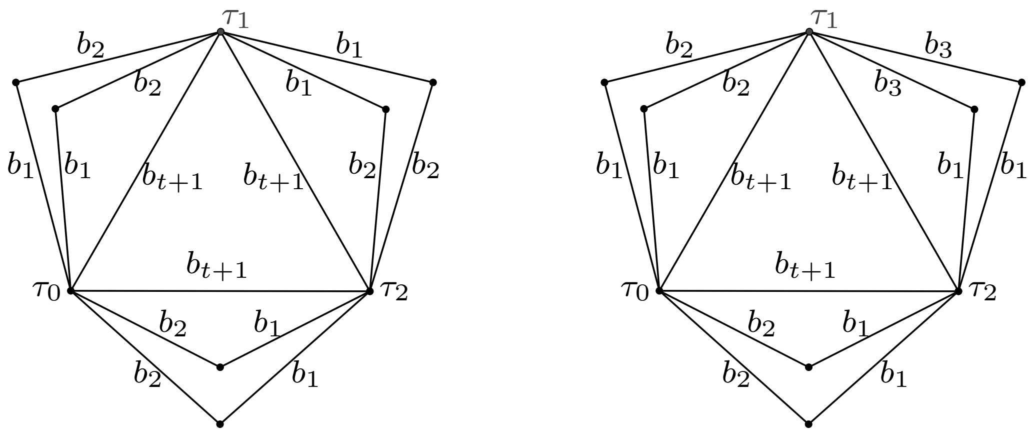
$$c_t(e) = \begin{cases} b_{t+1} & \text{if } e = \tau_i\tau_j \text{ for some } 0 \leq i < j \leq 2, \\ b_2 & \text{if } e^H = (\alpha_0\tau_1)^H \text{ and } t \text{ is even}, \\ b_1 & \text{if } e^H = (\alpha_1\tau_1)^H \text{ and } t \text{ is even}, \\ b_1 & \text{if } e^H = (\alpha_2\tau_2)^H \text{ and } t \text{ is odd}, \\ h_t(e^H) & \text{otherwise}, \end{cases}$$

for each $e \in E(G_{m,t})$. For a subgraph $H' \cong G(t)$ of $G_{m,t}$, we note that $c_t|_{H'}(e) = c_t|_H(e^H)$ for all $e \in E(H')$ where $l(e) \geq 3$.

By Theorem 16, we have $\mathrm{rc}(G_{1,t}) = t$ for $t \geq 1$. Since $G_{m,1}$ is a triangle, it follows that $\mathrm{rc}(G_{m,1}) = \mathrm{diam}(G_{m,1}) = 1$. In Theorem 18, we show that $\mathrm{rc}(G_{m,t}) = t + 1$ for $m > 1$ and $t > 1$.

Theorem 18. *For $m > 1$ and $t > 1$, we have $\mathrm{rc}(G_{m,t}) = t + 1$.*

Proof. Consider $G_{m,t}$ with the coloring c_t. Let $u, v \in V(G_{m,t})$. If $t = 2$, then let $\alpha_i^{(1)}$ and $\alpha_i^{(2)}$ be distinct direct descendants of τ_j and τ_k for some $i, j, k \in \{0, 1, 2\}$. If $\{u, v\} \neq \{\alpha_i^{(1)}, \alpha_i^{(2)}\}$, then a rainbow path between each pair of non-adjacent vertices appears in Table 5. If $\{u, v\} = \{\alpha_i^{(1)}, \alpha_i^{(2)}\}$, then we consider $u = \alpha_i^{(1)}$ and $v = \alpha_i^{(2)}$. Finding an $(\alpha_i^{(1)}, \alpha_i^{(2)})$-path is equivalent to finding a rainbow cycle in $G_{1,2}$ containing α_i. Since the color of the triangle $\tau_0\tau_1\tau_2$ is b_{t+1} and $h_t(\alpha_i\tau_j) \neq h_t(\alpha_i\tau_k)$ for $j \neq k$, a triangle $\alpha_i\tau_j\tau_k$ is a rainbow cycle. Thus there exists a rainbow $(\alpha_i^{(1)}, \alpha_i^{(2)})$-path. It can be easily verified that there is no coloring of 2 colors giving a rainbow connected $G_{m,2}$. Hence $\mathrm{rc}(G_{m,2}) = \mathrm{diam}(G_{m,2}) + 1 = 3$.



(A) even t (B) odd t

FIGURE 6. The coloring of $G_{(2,2)}$-subgraph of $G_{(2,t)}$

Suppose that $t > 2$. Let H_1 and H_2 be subgraphs of $G_{m,t}$ containing u and v, respectively, where $H_1 \cong G(t) \cong H_2$ ($H_1 = H_2$ if possible). If $H_1 = H_2$, then there exists a rainbow (u, v)-path by the same argument in Theorem 16 with an adjusted path in $G_2^{H_1}$ in Table 5. Now, we suppose that there is no H_1, H_2 where $H_1 = H_2$. Thus, there exist i_1, i_2 such that $u, v \in V(B_{(\tau_{i_1},\tau_{i_2})})$ where $0 \leq i_1 < i_2 \leq 2$. By Lemma 15, there exist an odd-colored rainbow (u, u')-path P_1 and an even-colored rainbow (v, v')-path P_2 where u' and v' are the only vertices in $V(G_2^{H_1})$ and $V(G_2^{H_2})$, respectively. If P_1 and P_2 intersect, says at x, then the $uP_1x\overline{P_2}v$ is a rainbow (u, v)-path. Suppose P_1 and P_2 are disjoint. We have that $u' \in \{\tau_{i_1}, \tau_{i_2}, \alpha_{i_3}^{H_1}\}$ and $v' \in \{\tau_{i_1}, \tau_{i_2}, \alpha_{i_3}^{H_2}\}$ where $\alpha_{i_3}^{H_1}$ and $\alpha_{i_3}^{H_2}$ are the direct descendants of τ_{i_1} and τ_{i_2} in H_1 and H_2, respectively. If $u', v' \in \{\tau_0, \tau_1, \tau_2\}$, then connecting P_1 and P_2 by $u'v'$ gives a rainbow (u, v)-path as the color of the triangle $\tau_0\tau_1\tau_2$ is b_{t+1}. Consider case $u' = \alpha_{i_3}^{H_1}$ and $v' = \tau_i$ for some $i = i_1, i_2$. Without loss of generality, we suppose that $v' = \tau_{i_1}$. If $c_t(\alpha_{i_3}^{H_1}\tau_{i_1}) \notin c_t(P_1) \cup c_t(P_2)$, then connecting P_1 and P_2 by $\alpha_{i_3}^{H_1}\tau_{i_1}$ gives a rainbow (u, v)-path. If $c_t(\alpha_{i_3}^{H_1}\tau_{i_1}) \in c_t(P_1) \cup c_t(P_2)$, then we connect P_1 and P_2 by $\alpha_{i_3}^{H_1}\tau_{i_2}\tau_{i_1}$ which gives a rainbow (u, v)-path by Table 4 and Figure 6. If $u' = \alpha_{i_3}^{H_1}$, $v' = \alpha_{i_3}^{H_2}$ and $\alpha_{i_3}^{H_1} \neq \alpha_{i_3}^{H_2}$, then we connect P_1 and P_2 by $\alpha_{i_3}^{H_1}\tau_{i_1}\tau_{i_2}\alpha_{i_3}^{H_2}$. Thus $G_{m,t}$ is rainbow-connected and $t \leq \mathrm{rc}(G_{m,t}) \leq t + 1$.

Next, we show that $\mathrm{rc}(G_{m,t}) \neq t$. Let c be an edge-coloring giving a rainbow connected $G_{m,t}$. Suppose $|c(G_{m,t})| = t$. Consider an even t. Let $H_3, H_4 \subset G_{m,t}$ be such that $H_3 \cong G(t) \cong H_4$ and $\alpha_k^{H_3} \neq \alpha_k^{H_4}$ for all $k = 0, 1, 2$. So $V(H_3) \cap V(H_4) = \{\tau_0, \tau_1, \tau_2\}$. Let $x = \alpha_0(01)^{\frac{t-2}{2}}$ and $y = \alpha_1(10)^{\frac{t-2}{2}}$. By Lemmas 8 and 12, there are a unique geodesic $(x^{H_i}, \alpha_0^{H_i})$-path $P_1^{H_i}$ and a unique geodesic $(y^{H_i}, \alpha_1^{H_i})$-path $P_2^{H_i}$ in $G_{m,t}$ for $i = 3, 4$. Since $B_{(\tau_0,\tau_1)}$ is the minimal bundle containing both x^{H_3} and x^{H_4}, a geodesic (x^{H_3}, x^{H_4})-path must contain τ_0 or τ_1, and $d(x^{H_3}, x^{H_4}) = t$ by Lemmas 8 and 12. Since $P_1^{H_3}$ and $P_1^{H_4}$ are the unique geodesic paths of length $\frac{t}{2} - 1$, the rainbow (x^{H_3}, x^{H_4})-path is $P_1^{H_3}\tau_0\tau_1\overline{P}_1^{H_4}$, or $P_1^{H_3}\tau_i\overline{P}_1^{H_4}$ for some $i = 0, 1$. Hence $c(P_1^{H_3}) \cap c(P_1^{H_4}) = \emptyset$. Thus, we need $t - 2$ colors to color $P_1^{H_3}$ and $P_1^{H_4}$. Now,

TABLE 5. List of a rainbow path in $G_2(t)$ where $0 \leq i < j \leq 2$

u'	v'	(u', v')-path	list of colors when t is even	lists of colors when t is odd
α_0	α_1	$\alpha_0\tau_1\alpha_1$	b_2b_1	b_2b_3
α_0	α_2	$\alpha_0\tau_0\alpha_2$	b_1b_2	b_1b_2
α_0	τ_2	$\alpha_0\tau_0\tau_2$	b_1b_{t+1}	b_1b_{t+1}
		$\alpha_0\tau_1\tau_2$		b_2b_{t+1}
α_1	α_2	$\alpha_1\tau_2\alpha_2$	b_2b_1	
		$\alpha_1\tau_2\tau_0\alpha_2$		$b_1b_{t+1}b_2$
		$\alpha_1\tau_1\tau_2\alpha_2$		$b_3b_{t+1}b_1$
α_1	τ_0	$\alpha_1\tau_1\tau_0$	b_1b_{t+1}	b_3b_{t+1}
α_2	τ_0	$\alpha_2\tau_0$	b_2	b_2
		$\alpha_2\tau_2\tau_0$	b_1b_{t+1}	b_1b_{t+1}
α_0	τ_1	$\alpha_0\tau_0\tau_1$	b_1b_{t+1}	b_1b_{t+1}
α_2	τ_1	$\alpha_2\tau_2\tau_1$	b_1b_{t+1}	b_1b_{t+1}
		$\alpha_2\tau_0\tau_1$	b_2b_{t+1}	b_2b_{t+1}
τ_i	τ_j	$\tau_i\tau_j$	b_{t+1}	b_{t+1}

we consider a rainbow (x^{H_i}, y^{H_4})-path for $i = 3, 4$. By Lemmas 8 and 12, a path $P_1^{H_i}\tau_1\overline{P}_2^{H_4}$ is a unique geodesic (x^{H_i}, y^{H_4})-path for $i = 3, 4$ with length t. Hence, we need at least $t-1$ colors to color $P_1^{H_3}\tau_1$ and $P_1^{H_4}\tau_1$, and $c(\tau_1 P_2^{H_4}) \cap (c(P_1^{H_3}\tau_1) \cup c(P_1^{H_4}\tau_1)) = \emptyset$. For $t > 2$, it follows that $|c(\tau_1 P_2^{H_4})| \geq 2$. Thus $|c(G_{m,t})| \geq t + 1$, a contradiction. By using a similar argument along with Lemmas 9, 10 and 12, we have that $\mathrm{rc}(G_{m,t}) \neq t$ when t is odd. Therefore $\mathrm{rc}(G_{m,t}) = t + 1$ for $m > 1$ and $t > 1$. □

5. Conclusion

In this work, we give a rainbow connection number of a generalized Farey graph $G_{m,t}$ for all $m \geq 1$ and $t \geq 1$. In case $m = 1$, the rainbow connection number of $G_{m,t}$ achieves the lowest possible value among the graph with the same diameter. We also show that $\mathrm{diam}(G_{m,t}) = t$ for $m \geq 1$ and $t \geq 1$. Several unique geodesic paths in $G_{m,t}$ are also given.

Acknowledgement

This research has received funding support from the NSRF via the Program Management Unit for Human Resources & Institutional Development, Research and Innovation under Grant [B05F640188].

Authors' Contributions

All authors have read and approved the final version of the manuscript.

The authors contributed to this work in the following ways:

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- K. Nakprasit: Conceptualization, Methodology, Formal analysis, Investigation, Writing - Original Draft, Supervision
- W. Tangjai: Conceptualization, Methodology, Validation, Formal analysis, Investigation, Resources, Writing - Original Draft, Supervision, Project administration, Funding acquisition

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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