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ON (τ_1, τ_2)-R_0 BITOPOLOGICAL SPACES

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Abstract. This paper is concerned with the concept of (τ_1, τ_2)-R_0 bitopological spaces. Furthermore, some characterizations of (τ_1, τ_2)-R_0 bitopological spaces are considered.

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1. Introduction

In 1943, Shanin [18] introduced the notion of R_0 topological spaces. Davis [8] introduced the concept of a separation axiom called R_1. These concepts are further investigated by Naimpally [16], Dube [12] and Dorsett [9]. Murdeshwar and Naimpally [15] and Dube [11] studied some of the fundamental properties of the class of R_1 topological spaces. As natural generalizations of the separations axioms R_0 and R_1, the concepts of semi-R_0 and semi-R_1 spaces were introduced and studied by Maheshwari and Prasad [14] and Dorsett [10]. In 2004, Caldas et al. [7] introduced and studied two new weak separation axioms called Λ_θ-R_0 and Λ_θ-R_1 by using the notions of (Λ, θ)-open sets and the (Λ, θ)-closure operator. In 2005, Cammaroto and Noiri [6] defined a weak separation axiom m-R_0 in m-spaces which are equivalent to generalized topological spaces due to Lugojan [13]. Noiri [17] introduced the notion of m-R_1 spaces and investigated several characterizations of m-R_0 spaces and m-R_1 spaces. Thongmoon and Boonpok [20] introduced and investigated the concept of (Λ, p)-R_1 topological spaces. In [1], the present authors introduced and studied the notions of $\delta s(\Lambda, s)$-R_0 spaces and $\delta s(\Lambda, s)$-R_1 spaces. Furthermore, several characterizations of Λ_p-R_0 spaces and (Λ, s)-R_0 spaces were established in [3] and [2], respectively. Recently, Thongmoon and Boonpok [19] introduced and studied the notion of

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sober $\delta p(\Lambda, s)$-R_0 spaces. In this paper, we introduce the concept of (τ_1, τ_2)-R_0 bitopological spaces. Moreover, some characterizations of (τ_1, τ_2)-R_0 bitopological spaces are discussed.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [5] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [5] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [5] of A and is denoted by $\tau_1\tau_2$-Int(A). A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)r$-*open* [21] (resp. $(\tau_1, \tau_2)s$-*open* [4], $(\tau_1, \tau_2)p$-*open* [4], $(\tau_1, \tau_2)\beta$-*open* [4]) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$).

Lemma 1. [5] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-*Cl*(A) *and* $\tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Cl*$(A)) = \tau_1\tau_2$-*Cl*(A).
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$-*Cl*$(A) \subseteq \tau_1\tau_2$-*Cl*$(B)$.
(3) $\tau_1\tau_2$-*Cl*(A) *is* $\tau_1\tau_2$-*closed.*
(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-*Cl*(A).
(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

Lemma 2. *For a subset A of a bitopological space (X, τ_1, τ_2), $x \in \tau_1\tau_2$-Cl(A) if and only if $U \cap A \neq \emptyset$ for every $\tau_1\tau_2$-open set U of X containing x.*

Definition 1. [5] *Let A be a subset of a bitopological space (X, τ_1, τ_2). The set $\cap\{G \mid A \subseteq G$ and G is $\tau_1\tau_2$-open$\}$ is called the $\tau_1\tau_2$-kernel of A and is denoted by $\tau_1\tau_2$-$ker(A)$.*

Lemma 3. [5] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-$ker(A)$.
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$-$ker(A) \subseteq \tau_1\tau_2$-$ker(B)$.
(3) *If A is* $\tau_1\tau_2$-*open, then* $\tau_1\tau_2$-$ker(A) = A$.
(4) $x \in \tau_1\tau_2$-$ker(A)$ *if and only if* $A \cap H \neq \emptyset$ *for every* $\tau_1\tau_2$-*closed set* H *containing* x.

3. Characterizations of (τ_1, τ_2)-R_0 spaces

In this section, we introduce the concept of (τ_1, τ_2)-R_0 spaces. Moreover, some characterizations of (τ_1, τ_2)-R_0 spaces are discussed.

Definition 2. *A bitopological space* (X, τ_1, τ_2) *is said to be* (τ_1, τ_2)-R_0 *if for each* $\tau_1\tau_2$*-open set* U *and each* $x \in U$, $\tau_1\tau_2$-$Cl(\{x\}) \subseteq U$.

Lemma 4. *Let* (X, τ_1, τ_2) *be a bitopological space and* x, y *be any points of* X*. Then, the following properties hold:*

(1) $y \in \tau_1\tau_2$-$ker(\{x\})$ *if and only if* $x \in \tau_1\tau_2$-$Cl(\{y\})$.
(2) $\tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-$ker(\{y\})$ *if and only if* $\tau_1\tau_2$-$Cl(\{x\}) = \tau_1\tau_2$-$Cl(\{y\})$.

Proof. (1) Let $x \notin \tau_1\tau_2$-Cl$(\{y\})$. Then, there exists a $\tau_1\tau_2$-open set U such that $x \in U$ and $y \notin U$. Thus, $y \notin \tau_1\tau_2$-$ker(\{x\})$. The converse is similarly shown.

(2) Suppose that $\tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-$ker(\{y\})$ for any points x, y. Since $x \in \tau_1\tau_2$-$ker(\{x\})$, $x \in \tau_1\tau_2$-$ker(\{y\})$ and by (1), $y \in \tau_1\tau_2$-Cl$(\{x\})$. By Lemma 1, we have $\tau_1\tau_2$-Cl$(\{y\}) \subseteq \tau_1\tau_2$-Cl$(\{x\})$. Similarly, we have $\tau_1\tau_2$-Cl$(\{x\}) \subseteq \tau_1\tau_2$-Cl$(\{y\})$ and hence $\tau_1\tau_2$-Cl$(\{x\}) = \tau_1\tau_2$-Cl$(\{y\})$. Next, suppose that $\tau_1\tau_2$-Cl$(\{x\}) = \tau_1\tau_2$-Cl$(\{y\})$. Since $x \in \tau_1\tau_2$-Cl$(\{x\})$, we have $x \in \tau_1\tau_2$-Cl$(\{y\})$ and by (1), $y \in \tau_1\tau_2$-$ker(\{x\})$. By Lemma 3, $\tau_1\tau_2$-$ker(\{y\}) \subseteq \tau_1\tau_2$-$ker(\tau_1\tau_2$-$ker(\{x\})) = \tau_1\tau_2$-$ker(\{x\})$. Similarly, we have $\tau_1\tau_2$-$ker(\{x\}) \subseteq \tau_1\tau_2$-$ker(\{y\})$ and hence $\tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-$ker(\{y\})$. □

Theorem 1. *For a bitopological space* (X, τ_1, τ_2)*, the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-R_0.
(2) *For each* $\tau_1\tau_2$*-closed set* F *and each* $x \in X - F$*, there exists a* $\tau_1\tau_2$*-open set* U *such that* $F \subseteq U$ *and* $x \notin U$.
(3) *For each* $\tau_1\tau_2$*-closed set* F *and each* $x \in X - F$, $\tau_1\tau_2$-$Cl(\{x\}) \cap F = \emptyset$.
(4) *For any distinct points* x, y *in* X, $\tau_1\tau_2$-$Cl(\{x\}) = \tau_1\tau_2$-$Cl(\{y\})$ *or* $\tau_1\tau_2$-$Cl(\{x\}) \cap \tau_1\tau_2$-$Cl(\{y\}) = \emptyset$.

Proof. (1) ⇒ (2): Let F be a $\tau_1\tau_2$-closed set and $x \in X - F$. Since $X - F$ is $\tau_1\tau_2$-open and by (1), $\tau_1\tau_2$-Cl$(\{x\}) \subseteq X - F$. Let $U = X - \tau_1\tau_2$-Cl$(\{x\})$. Then, we have U is $\tau_1\tau_2$-open, $F \subseteq U$ and $x \notin U$.

(2) ⇒ (3): Let F be a $\tau_1\tau_2$-closed set and $x \in X - F$. There exists a $\tau_1\tau_2$-open set U such that $F \subseteq U$ and $x \notin U$. By Lemma 2, $\tau_1\tau_2$-Cl$(\{x\}) \cap U = \emptyset$ and hence $\tau_1\tau_2$-Cl$(\{x\}) \cap F = \emptyset$.

(3) ⇒ (4): Let x, y be distinct points of X. Suppose that

$$\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) \neq \emptyset.$$

By (3), $x \in \tau_1\tau_2$-Cl$(\{y\})$ and $y \in \tau_1\tau_2$-Cl$(\{x\})$. By Lemma 1,

$$\tau_1\tau_2\text{-Cl}(\{x\}) \subseteq \tau_1\tau_2\text{-Cl}(\{y\}) \subseteq \tau_1\tau_2\text{-Cl}(\{x\}).$$

Thus, $\tau_1\tau_2$-Cl$(\{x\}) = \tau_2\tau_2$-Cl$(\{y\})$.

(4) $\Rightarrow$ (1): Let U be a $\tau_1\tau_2$-open set and $x \in U$. For any $y \notin U$, by Lemma 2, $\tau_1\tau_2\text{-Cl}(\{y\}) \cap U = \emptyset$ and hence $x \notin \tau_1\tau_2\text{-Cl}(\{y\})$. Therefore, $\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\})$. By (4), for each $y \notin U$,

$$\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) = \emptyset.$$

Since $X - U$ is $\tau_1\tau_2$-closed, $y \in \tau_1\tau_2\text{-Cl}(\{y\}) \subseteq X - U$ and

$$\cup_{y \in X-U} \tau_1\tau_2\text{-Cl}(\{y\}) = X - U.$$

Thus,

$$\begin{aligned}
\tau_1\tau_2\text{-Cl}(\{x\}) \cap (X - U) &= \tau_1\tau_2\text{-Cl}(\{x\}) \cap [\cup_{y \in X-U} \tau_1\tau_2\text{-Cl}(\{y\})] \\
&= \cup_{y \in X-U} [\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\})] \\
&= \emptyset
\end{aligned}$$

and hence $\tau_1\tau_2\text{-Cl}(\{x\}) \subseteq U$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Corollary 1. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if for each points x and y in X, $\tau_1\tau_2$-Cl$(\{x\}) \neq \tau_1\tau_2$-Cl$(\{y\})$ implies $\tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\}) = \emptyset$.*

Proof. This is obvious by Theorem 1 (4).

Conversely, let U be a $\tau_1\tau_2$-open set and $x \in U$. If $y \notin U$, then $\tau_1\tau_2\text{-Cl}(\{y\}) \cap U = \emptyset$. Therefore, $x \notin \tau_1\tau_2\text{-Cl}(\{y\})$ and $\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\})$. By the hypothesis, $\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) = \emptyset$ and hence $y \notin \tau_1\tau_2\text{-Cl}(\{x\})$. Thus, $\tau_1\tau_2\text{-Cl}(\{x\}) \subseteq U$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Theorem 2. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if for each points x and y in X, $\tau_1\tau_2$-$ker(\{x\}) \neq \tau_1\tau_2$-$ker(\{y\})$ implies $\tau_1\tau_2$-$ker(\{x\}) \cap \tau_1\tau_2$-$ker(\{y\}) = \emptyset$.*

Proof. Let (X, τ_1, τ_2) be (τ_1, τ_2)-R_0. Suppose that

$$\tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-}ker(\{y\}) \neq \emptyset.$$

Let $z \in \tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-}ker(\{y\})$. Then, $z \in \tau_1\tau_2\text{-}ker(\{x\})$ and by Lemma 4, $x \in \tau_1\tau_2\text{-Cl}(\{z\})$. Thus, $x \in \tau_1\tau_2\text{-Cl}(\{z\}) \cap \tau_1\tau_2\text{-Cl}(\{x\})$ and by Corollary 1, $\tau_1\tau_2\text{-Cl}(\{z\}) = \tau_1\tau_2\text{-Cl}(\{x\})$. Similarly, we have $\tau_1\tau_2\text{-Cl}(\{z\}) = \tau_1\tau_2\text{-Cl}(\{y\})$ and hence $\tau_1\tau_2\text{-Cl}(\{x\}) = \tau_1\tau_2\text{-Cl}(\{y\})$. By Lemma 4, $\tau_1\tau_2\text{-}ker(\{x\}) = \tau_1\tau_2\text{-}ker(\{y\})$.

Conversely, we show that sufficiency by using Corollary 1. Suppose that $\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\})$. By Lemma 4,

$$\tau_1\tau_2\text{-}ker(\{x\}) \neq \tau_1\tau_2\text{-}ker(\{y\})$$

and hence $\tau_1\tau_2\text{-}ker(\{x\}) \cap \tau_1\tau_2\text{-}ker(\{y\}) = \emptyset$. Therefore,

$$\tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-Cl}(\{y\}) = \emptyset.$$

In fact, assume $z \in \tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\})$. Then, $z \in \tau_1\tau_2$-Cl$(\{x\})$ implies $x \in \tau_1\tau_2$-$ker(\{z\})$ and hence

$$x \in \tau_1\tau_2\text{-}ker(\{z\}) \cap \tau_1\tau_2\text{-}ker(\{x\}).$$

By the hypothesis, $\tau_1\tau_2$-$ker(\{z\}) = \tau_1\tau_2$-$ker(\{x\})$ and by Lemma 4, $\tau_1\tau_2$-Cl$(\{z\}) = \tau_1\tau_2$-Cl$(\{x\})$. Similarly, we have

$$\tau_1\tau_2\text{-Cl}(\{z\}) = \tau_1\tau_2\text{-Cl}(\{y\})$$

and hence $\tau_1\tau_2$-Cl$(\{x\}) = \tau_1\tau_2$-Cl$(\{y\})$. This contradicts that

$$\tau_1\tau_2\text{-Cl}(\{x\}) \neq \tau_1\tau_2\text{-Cl}(\{y\}).$$

Therefore, $\tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-Cl$(\{y\}) = \emptyset$. Thus, (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Theorem 3. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-R_0.

(2) $x \in \tau_1\tau_2$-*Cl*$(\{y\})$ *if and only if* $y \in \tau_1\tau_2$-*Cl*$(\{x\})$.

Proof. $(1) \Rightarrow (2)$: Suppose that $x \in \tau_1\tau_2$-Cl$(\{y\})$. By Lemma 4, $y \in \tau_1\tau_2$-$ker(\{x\})$ and hence $\tau_1\tau_2$-$ker(\{x\}) \cap \tau_1\tau_2$-$ker(\{y\}) \neq \emptyset$. By Theorem 2, $\tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-$ker(\{y\})$ and hence $x \in \tau_1\tau_2$-$ker(\{y\})$. By Lemma 4, $y \in \tau_1\tau_2$-Cl$(\{x\})$. The converse is similarly shown.

$(2) \Rightarrow (1)$: Let U be a $\tau_1\tau_2$-open set and $x \in U$. If $y \notin U$, then $\tau_1\tau_2$-Cl$(\{y\}) \cap U = \emptyset$. Thus, $x \notin \tau_1\tau_2$-Cl$(\{y\})$ and $y \notin \tau_1\tau_2$-Cl$(\{x\})$. This implies that $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$. Therefore, (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Theorem 4. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-R_0.

(2) *For each nonempty set A of X and each $\tau_1\tau_2$-open set U such that $U \cap A \neq \emptyset$, there exists a $\tau_1\tau_2$-closed set F such that $A \cap F \neq \emptyset$ and $F \subseteq U$.*

(3) $F = \tau_1\tau_2$-$ker(F)$ *for each $\tau_1\tau_2$-closed set F.*

(4) $\tau_1\tau_2$-*Cl*$(\{x\}) \subseteq \tau_1\tau_2$-$ker(\{x\})$.

Proof. $(1) \Rightarrow (2)$: Let A be a nonempty set of X and U be a $\tau_1\tau_2$-open set such that $A \cap U \neq \emptyset$. Then, there exists $x \in A \cap U$ and hence $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$. Put $F = \tau_1\tau_2$-Cl$(\{x\})$, then F is $\tau_1\tau_2$-closed, $A \cap F \neq \emptyset$ and $F \subseteq U$.

$(2) \Rightarrow (3)$: Let F be a $\tau_1\tau_2$-closed set of X. By Lemma 3, we have $F \subseteq \tau_1\tau_2$-$ker(F)$. Next, we show $F \supseteq \tau_1\tau_2$-$ker(F)$. Let $x \notin F$. Then, $x \in X - F$ and $X - F$ is $\tau_1\tau_2$-open. By (2), there exists a $\tau_1\tau_2$-closed set K such that $x \in K$ and $K \subseteq X - F$. Now, put $U = X - K$. Then, U is $\tau_1\tau_2$-open, $F \subseteq U$ and $x \notin U$. Thus, $x \notin \tau_1\tau_2$-$ker(F)$ and hence $F \supseteq \tau_1\tau_2$-$ker(F)$.

(3) $\Rightarrow$ (4): Let $x \in X$ and $y \notin \tau_1\tau_2$-$ker(\{x\})$. There exists a $\tau_1\tau_2$-open set U such that $x \in U$ and $y \notin U$. Thus, $\tau_1\tau_2$-Cl$(\{y\}) \cap U = \emptyset$. By (3), $\tau_1\tau_2$-$ker(\tau_1\tau_2$-Cl$(\{y\})) \cap U = \emptyset$. Since $x \notin \tau_1\tau_2$-$ker(\tau_1\tau_2$-Cl$(\{y\}))$, there exists a $\tau_1\tau_2$-open set G such that $\tau_1\tau_2$-Cl$(\{y\}) \subseteq G$ and $x \notin G$. Therefore, $\tau_1\tau_2$-Cl$(\{x\}) \cap G = \emptyset$. Since $y \in G$, we have $y \notin \tau_1\tau_2$-Cl$(\{x\})$ and hence $\tau_1\tau_2$-Cl$(\{x\}) \subseteq \tau_1\tau_2$-$ker(\{x\})$. Moreover,

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(\{x\}) &\subseteq \tau_1\tau_2\text{-}ker(\{x\}) \\ &\subseteq \tau_1\tau_2\text{-}ker(\tau_1\tau_2\text{-Cl}(\{x\})) \\ &= \tau_1\tau_2\text{-Cl}(\{x\}).\end{aligned}$$

This shows that $\tau_1\tau_2$-Cl$(\{x\}) = \tau_1\tau_2$-$ker(\{x\})$.

(4) $\Rightarrow$ (5): This is obvious.

(5) $\Rightarrow$ (1): Let U be a $\tau_1\tau_2$-open set and $x \in U$. If $y \notin U$, then $\tau_1\tau_2$-Cl$(\{y\}) \cap U = \emptyset$ and $x \notin \tau_1\tau_2$-Cl$(\{y\})$. By Lemma 4, $y \notin \tau_1\tau_2$-$ker(\{x\})$ and by (5), $y \notin \tau_1\tau_2$-Cl$(\{x\})$. Thus, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ and hence (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Corollary 2. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if $\tau_1\tau_2$-$ker(\{x\}) \subseteq \tau_1\tau_2$-Cl$(\{x\})$ for each $x \in X$.*

Proof. This is obvious by Theorem 4.

Conversely, let $x \in \tau_1\tau_2$-Cl$(\{y\})$. Then by Lemma 4, $y \in \tau_1\tau_2$-$ker(\{x\})$ and hence $y \in \tau_1\tau_2$-Cl$(\{x\})$. Similarly, if $y \in \tau_1\tau_2$-Cl$(\{x\})$, then $x \in \tau_1\tau_2$-Cl$(\{y\})$. It follows from Theorem 3 that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Definition 3. *Let (X, τ_1, τ_2) be a bitopological space and $x \in X$. Then, $\langle x \rangle_{(\tau_1,\tau_2)}$ is defined by $\langle x \rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-$ker(\{x\})$.*

Corollary 3. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if $\langle x \rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\})$ for each $x \in X$.*

Proof. Let $x \in X$. By Theorem 4, $\tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-Cl$(\{x\})$.
Thus, $\langle x \rangle_{(\tau_1,\tau_2)} = \tau_1\tau_2$-Cl$(\{x\}) \cap \tau_1\tau_2$-$ker(\{x\}) = \tau_1\tau_2$-Cl$(\{x\})$.

Conversely, let $x \in X$. By the hypothesis,

$$\begin{aligned}\tau_1\tau_2\text{-Cl}(\{x\}) &= \langle x \rangle_{(\tau_1,\tau_2)} \\ &= \tau_1\tau_2\text{-Cl}(\{x\}) \cap \tau_1\tau_2\text{-}ker(\{x\}) \\ &\subseteq \tau_1\tau_2\text{-}ker(\{x\}).\end{aligned}$$

It follows from Theorem 4 that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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