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A REVERSE HILBERT-TYPE INTEGRAL INEQUALITY WITH THE GENERAL NONHOMOGENEOUS KERNEL

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Abstract. This paper introduces a reverse Hilbert-type integral inequality which has a nonhomogeneous kernel as $H(xv(y))$, where the real analysis methods and the weight functions were used. Several equivalent statements pertaining to the optimal constant factor and some parameters are presented. In terms of applications, we also explored the equivalent forms and certain corollaries pertaining to the homogeneous kernel.

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1. Introduction

Assuming that $0<\int_0^\infty f^2(x)\,dx<\infty$ and $0<\int_0^\infty g^2(y)\,dy<\infty$, there have the Hilbert integral inequality as follows(cf. [8], Theorem 316):

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)g(y)}{x+y}dxdy\\ &<\pi\left(\int_0^\infty f^2(x)dx\int_0^\infty g^2(y)dy\right)^{\frac{1}{2}}.\end{aligned}\tag{1}$$

where π representing the constant factor is the optimal value.

If $p>1, \frac{1}{p}+\frac{1}{q}=1, f(x), g(y)\geq 0, 0<\int_0^\infty f^p(x)\,dx<\infty$ and $0<\int_0^\infty g^q(y)\,dy<\infty, k(x,y)\,(\geq 0)$, $k_p:=\int_0^\infty k(u,1)\,u^{-\frac{1}{p}}du\in \mathbf{R}_+=(0,\infty)$, We have a Hilbert-type integral inequality with the general

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homogeneous kernel of degree -1 as $k(x,y)$ as follows(cf. [8], Theorem319):

$$\begin{aligned}&\int_0^\infty\int_0^\infty k(x,y)\,f(x)g(y)dxdy\\ <\ & k_p\left(\int_0^\infty f^p(x)dx\right)^{\frac{1}{p}}\left(\int_0^\infty g^q(y)dy\right)^{\frac{1}{q}},\end{aligned} \tag{2}$$

where k_p representing the constant factor is the optimal value. For $p=q=2, k(x,y)=\frac{1}{x+y}$, (2) reduces to (1).

In R_+, if $H(u)$ is a nonnegative measurable function and $p>1, \frac{1}{p}+\frac{1}{q}=1, \gamma=\sigma_1, \sigma\in\mathbf{R}=(-\infty,\infty)$, so that

$$k(\gamma):=\int_0^\infty H(u)\,u^{\gamma-1}du\ \in\mathbf{R}_+,$$

then a Hilbert-type integral inequality with a nonhomogeneous kernel is presented as follows(cf. [8], Theorem 350):

$$\begin{aligned}&\int_0^\infty\int_0^\infty H(xy)\,f(x)g(y)\,dxdy\\ <\ & k\left(\frac{1}{p}\right)\left\{\int_0^\infty x^{p-2}f^p(x)dx\right\}^{\frac{1}{p}}\left\{\int_0^\infty g^q(y)dy\right\}^{\frac{1}{q}}.\end{aligned} \tag{3}$$

where $K\left(\frac{1}{p}\right)$ representing the constant factor is the optimal value.

Moreover, an extension of (1) was given by the introduction a parameter $\lambda>0$, and is shown as follows [24]:

$$\begin{aligned}&\int_0^\infty\int_0^\infty \frac{f(x)\,g(y)}{(x+y)^\lambda}dxdy\\ <\ & B(\frac{\lambda}{2},\frac{\lambda}{2})\left(\int_0^\infty x^{1-\lambda}f^2(x)dx\int_0^\infty y^{1-\lambda}g^2(y)dy\right)^{\frac{1}{2}},\end{aligned} \tag{4}$$

where $B(\frac{\lambda}{2},\frac{\lambda}{2})$ representing the constant is the best possible, and the following beta function [18]

$$B(u,v):=\int_0^\infty\frac{t^{u-1}}{(1+t)^{u+v}}dt\ (u,v>0).$$

If $\lambda=1$, (4) also reduces to (1).

Recently, some theories and methods involving Hilbert integral inequalities were provided by Yang [22], [23], [25] and Hong [9], [12], some extension of (1)-(4) with general real homogeneous [1], [21] and nonhomogeneous kernels [10] [27], [28] were established. By introducing conjugate exponents and parameters [26], [19], and introducing linear operators and norms [2], [3], [5], a scientifically optimal expression is obtained for the generalized form with the optimal constant factor [7], [11], [13], [20]. Some similar works have been obtained for discrete and half-discrete [4], [6]. In 2023, Peng et. al [14] gave a reverse Mulholland-type inequality for half-discrete, at the same time, Liu et. al [17] gave a forward Hilbert inequality for integral, where $H(xv(y))$ was a general nonhomogeneous kernel relate to Beta function, weight function.

In the previous study, we have obtained a research result on the Hilbert-type integral inequality, $H\left(xv\left(y\right)\right)$ was a general nonhomogeneous kernel of the inequality [17]. In this article, utilizing real analysis techniques and weight functions, the reverse form of the inequality was proved. A few equivalent statements regarding the optimal constant factor and some parameters are supplied. As well as we can examine the equivalent forms and certain corollaries related to the homogeneous kernel.

2. Some lemmas

In the following, let us assume that $0<p<1$ $(q<0)$, $\frac{1}{p}+\frac{1}{q}=1$, $v(x)>0, v'(x)>0$ $(x\in\mathbf{R}_+)$, with $v(0^+)=0, v(\infty)=\infty$, in $\mathbf{R}_+, \sigma,\sigma_1\in\mathbf{R}$, $H(u)$ is a nonnegative measurable function, then

$$K(\gamma):=\int_0^\infty H(u)u^{\gamma-1}du\in\mathbf{R}_+\ (\gamma=\sigma,\sigma_1),$$

in $\mathbf{R}_+$, existing nonnegative measurable functions $f\left(x\right)$ and $g\left(y\right)$, meeting :

$$\begin{aligned} 0 &< \int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p\left(x\right)dx<\infty, \text{ and} \\ 0 &< \int_0^\infty \frac{\left[v\left(y\right)\right]^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{\left(v'\left(y\right)\right)^{q-1}} g^q(y)dy<\infty \end{aligned} \tag{5}$$

Lemma 1. *If we have a constant $\delta_0>0$, in a way that $K(\sigma\pm\delta_0)<\infty$, Subsequently the functions $K(\eta)$ is continuous in any $\eta\in(\sigma-\delta_0,\sigma+\delta_0)$.*

Proof. For any sequence $\{\delta_n\}_{n=1}^\infty\subset[\sigma-\delta_0-\eta,\sigma+\delta_0-\eta]$, $\delta_n\to 0\ (n\to\infty)$, we have $\sigma-\delta_0\le\eta+\delta_n\le\sigma+\delta_0$, $n\in\mathbf{N}=\{1,2,\cdots\}$ and

$$\begin{aligned} K(\eta+\delta_n) &= \int_0^1 H(u)u^{\eta+\delta_n-1}du+\int_1^\infty H(u)u^{\eta+\delta_n-1}du \\ &\le \int_0^1 H(u)u^{\sigma-\delta_0-1}du+\int_1^\infty H(u)u^{\sigma+\delta_0-1}du \\ &\le K(\sigma-\delta_0)+K(\sigma+\delta_0). \end{aligned}$$

We indicate the dominated function $F\left(u\right)$ as follows:

$$F\left(u\right):=\begin{cases} H(u)u^{\sigma-\delta_0-1}, u\in(0,1], \\ H(u)u^{\sigma+\delta_0-1}, u\in(1,\infty) \end{cases},$$

which follows that

$$\begin{aligned} f_n\left(u\right) &: = H(u)u^{\sigma+\delta_n-1}\le F\left(u\right), u\in(0,\infty), \\ 0 &< \int_0^\infty F\left(u\right)du\le K(\sigma-\delta_0)+K(\sigma+\delta_0)<\infty. \end{aligned}$$

Accord to the Lebesgue-dominated convergence theorem [15], we have

$$\begin{aligned} K(\eta+\delta_n) &= \int_0^\infty f_n(u)\,du = \int_0^\infty H(u)u^{\eta+\delta_n-1}du \\ &\to \int_0^\infty H(u)u^{\eta-1}du = K(\eta) \ \ (n\to\infty). \end{aligned}$$

Hence the function $K(\eta)$ is continuous in any $\eta\in(\sigma-\delta_0,\sigma+\delta_0)$. □

Lemma 2. *We have a reverse Hilbert-type integral inequality with a nonhomogeneous kernel, as shown below:*

$$\begin{aligned} I &:= \int_0^\infty\int_0^\infty H(xv(y))\,f(x)g(y)\,dxdy \\ &> K^{\frac{1}{p}}(\sigma)\,K^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}f^p(x)dx\right\}^{\frac{1}{p}} \\ &\times\left\{\int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}}g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \quad (6)$$

Proof. Given that $v'(y)>0$, with $v(0^+)=0$, $v(y)>0$ $(y\in\mathbf{R}_+)$ and then $v(y)$ is strictly increasing. The weight function is defined as follows:

$$\begin{aligned} \omega(\sigma,x) &:= x^\sigma\int_0^\infty H(xv(y))\,v^{\sigma-1}(y)\,v'(y)\,dy\ \ (x\in\mathbf{R}_+), \\ \varpi(\sigma_1,y) &:= (v(y))^{\sigma_1}\int_0^\infty H(xv(y))\,x^{\sigma_1-1}dx\ \ (y\in\mathbf{R}_+). \end{aligned} \quad (7)$$

For fixed $x>0$, setting $u=xv(y)$, we have

$$\omega(\sigma,x)=\int_0^\infty H(u)\,u^{\sigma-1}du=K(\sigma)\in\mathbf{R}_+. \quad (8)$$

In the same way, for fixed $y>0$, setting $u=xv(y)$, we have

$$\varpi(\sigma_1,y)=\int_0^\infty H(u)\,u^{\sigma_1-1}du=K(\sigma_1)\in\mathbf{R}_+. \quad (9)$$

By the reverse Hölder's inequality [16], Fubini theorem [15] and (7), we have

$$\begin{aligned} I &= \int_0^\infty\int_0^\infty H(xv(y))\,f(x)g(y)dxdy \\ &= \int_0^\infty\int_o^\infty H(xv(y))\left[\frac{(v(y))^{\frac{\sigma-1}{p}}(v'(y))^{\frac{1}{p}}}{x^{\frac{\sigma_1-1}{q}}}f(x)\right] \\ &\times\left[\frac{x^{\frac{\sigma_1-1}{q}}}{(v(y))^{\frac{\sigma-1}{p}}(v'(y))^{\frac{1}{p}}}g(y)\right]dxdy \\ &\geq \left[\int_0^\infty\int_0^\infty H(xv(y))\,\frac{(v(y))^{\sigma-1}v'(y)}{x^{(\sigma_1-1)(p-1)}}f^p(x)dydx\right]^{\frac{1}{p}} \\ &\times\left[\int_0^\infty\int_0^\infty H(xv(y))\,\frac{x^{\sigma_1-1}}{(v(y))^{(\sigma-1)(q-1)}(v'(y))^{q-1}}g^q(y)dxdy\right]^{\frac{1}{q}} \end{aligned}$$

$$= \left\{\int_0^\infty \omega(\sigma,x)x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}f^p(x)dx\right\}^{\frac{1}{p}}$$
$$\times\left\{\int_0^\infty \varpi(\sigma_1,y)\frac{(v\left(y\right))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'\left(y\right))^{q-1}}g^q(y)dy\right\}^{\frac{1}{q}} \tag{10}$$

If (10) maintains the equivalent form, there exist constants A and B [16], in a way that they are not both zero and

$$A\frac{[v\left(y\right)]^{\sigma-1}v'\left(y\right)}{x^{(\sigma_1-1)(p-1)}}f^p(x) = \frac{Bx^{\sigma_1-1}}{(v\left(y\right))^{(\sigma-1)(q-1)}(v'\left(y\right))^{q-1}}g^q\left(y\right)$$
$$\text{a.e. in } (0,\infty)\times(0,\infty).$$

We can find that if $A\neq 0$, for fixed a.e. $y\in(0,\infty)$, we have

$$x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}f^p(x)=\frac{Bg^q\left(y\right)}{A(v\left(y\right))^{(\sigma-1)q}(v'\left(y\right))^q}x^{\sigma_1-\sigma-1}\ a.e.\text{ in } (0,\infty).$$

Since $\int_0^\infty x^{\sigma_1-\sigma-1}dx=\infty$, The above expression can be expressed as

$$0<\int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}f^p\left(x\right)dx<\infty.$$

Therefore, by (8) and (9), we can get (6). □

Remark 1. *If $\sigma_1=\sigma$, accord to (5) and (6), we obtain*

$$0<\int_0^\infty x^{p(1-\sigma)-1}f^p\left(x\right)dx<\infty, 0<\int_0^\infty \frac{(v\left(y\right))^{q(1-\sigma)-1}}{(v'\left(y\right))^{q-1}}g^q(y)dy<\infty$$

and the reverse inequality as follows:

$$I = \int_0^\infty\int_0^\infty H\left(xv\left(y\right)\right)f(x)g\left(y\right)dxdy>K\left(\sigma\right)$$
$$\times\left[\int_0^\infty x^{p(1-\sigma)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{(v\left(y\right))^{q(1-\sigma)-1}}{(v'\left(y\right))^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}. \tag{11}$$

Lemma 3. *Assume there has a constant $\delta_0>0$, in a way that $K\left(\sigma\pm\delta_0\right)<\infty$, the constant factor $K\left(\sigma\right)$ in (11) is the optimal value.*

Proof. Since any $0<\varepsilon<p\delta_0$, we set

$$\widetilde{f}(x) := \begin{cases} 0, 0<x<1, \\ x^{\sigma-\frac{\varepsilon}{p}-1}, x\geq 1 \end{cases},$$
$$\widetilde{g}(y) := \begin{cases} (v\left(y\right))^{\sigma+\frac{\varepsilon}{q}-1}v'\left(y\right), 0<y\leq 1, \\ 0, y>1 \end{cases},$$

If there has a positive constant $M\ (\geq K(\sigma))$, such that (11) is valid when $K(\sigma)$ is replaced by M, particularly, we have

$$\begin{aligned}\widetilde{I}\ &:= \int_0^\infty \int_0^\infty H(xv(y))\,\widetilde{f}(x)\widetilde{g}(y)dxdy\\ &> M\left[\int_0^\infty x^{p(1-\sigma)-1}\widetilde{f}^P(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}}\widetilde{g}^q(x)dy\right]^{\frac{1}{q}}\\ &= M\left(\int_1^\infty x^{-\varepsilon-1}dx\right)^{\frac{1}{p}}\left[\int_0^1 (v(y))^{\varepsilon-1}v'(y)\,dy\right]^{\frac{1}{q}} = \frac{M}{\varepsilon}(v(1))^{\frac{\varepsilon}{q}}.\end{aligned}$$

By (7), for $\sigma_1 - \frac{\varepsilon}{p} = \sigma - \frac{\varepsilon}{p} \in (\sigma-\delta_0, \sigma+\delta_0)$, we have the following

$$\begin{aligned}\widetilde{I} &= \int_{y;0<y\leq 1}\left[(v(y))^{\sigma-\frac{\varepsilon}{p}}\int_{x;x\geq 1} H(xv(y))\,x^{\sigma-\frac{\varepsilon}{p}-1}dx\right](v(y))^{\varepsilon-1}v'(y)\,dy\\ &\leq \int_0^1\left[(v(y))^{\sigma-\frac{\varepsilon}{p}}\int_0^\infty H(xv(y))\,x^{\sigma-\frac{\varepsilon}{p}-1}dx\right](v(y))^{\varepsilon-1}v'(y)\,dy\\ &= \int_0^1 \omega\left(\sigma-\frac{\varepsilon}{p}, y\right)(v(y))^{\varepsilon-1}v'(y)\,dy\\ &= K\left(\sigma-\frac{\varepsilon}{p}\right)\int_0^1 (v(y))^{\varepsilon-1}v'(y)\,dy = \frac{1}{\varepsilon}K\left(\sigma-\frac{\varepsilon}{p}\right)v^{\varepsilon}(1).\end{aligned}$$

From the above results, one can conclude that

$$K\left(\sigma-\frac{\varepsilon}{p}\right)v^{\varepsilon}(1) \geq \varepsilon\widetilde{I} > Mv^{\frac{\varepsilon}{q}}(1).$$

For $\varepsilon \to 0^+$, according to Lemma 1, we have $K(\sigma) \geq M$. Therefore, $M = K(\sigma)$ is the optimal constant factor in (11). □

Remark 2. *Setting* $\widehat{\sigma} := \frac{\sigma}{p} + \frac{\sigma_1}{q} = \sigma - \frac{\sigma-\sigma_1}{q}$, *we can rewrite* (6) *as:*

$$\begin{aligned}&\int_0^\infty \int_0^\infty H(xv(y))\,f(x)g(y)\,dxdy\\ &> K^{\frac{1}{P}}(\sigma)\,K^{\frac{1}{q}}(\sigma_1)\left[\int_0^\infty x^{p(1-\widehat{\sigma})-1}f^p(x)dx\right]^{\frac{1}{p}}\\ &\times\left[\int_0^\infty \frac{(v(y))^{q(1-\widehat{\sigma})-1}}{(v'(y))^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}.\end{aligned} \tag{12}$$

By the reverse Hölder's inequality with weight [15], [16], *we have inequality as follows:*

$$\begin{aligned}K(\widehat{\sigma}) &= \int_0^\infty H(u)\left(u^{\frac{\sigma-1}{p}}\right)\left(u^{\frac{\sigma_1-1}{q}}\right)du\\ &\geq \left(\int_0^\infty H(u)\,u^{\sigma-1}du\right)^{\frac{1}{p}}\left(\int_0^\infty H(u)\,u^{\sigma_1-1}du\right)^{\frac{1}{q}}\\ &= K^{\frac{1}{P}}(\sigma)\,K^{\frac{1}{q}}(\sigma_1) > 0.\end{aligned} \tag{13}$$

If $\sigma - \sigma_1 \in (q\delta_0, -q\delta_0)$, *then* $\widehat{\sigma} \in (\sigma - \delta_0, \sigma + \delta_0)$, *and according to the proof of Lemma 1, we obtain*

$$\begin{aligned} K(\widehat{\sigma}) &= K\left(\sigma - \frac{\sigma - \sigma_1}{q}\right) \\ &\leq K(\sigma - \delta_0) + K(\sigma + \delta_0) < \infty. \end{aligned}$$

Lemma 4. *If there has a constant* $\delta_0 > 0$, *in a way that* $K(\sigma \pm \delta_0) < \infty$, *constant factor* $K^{\frac{1}{p}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ *in* (12) *or* (6) *is the optimal value, then for* $\sigma - \sigma_1 \in (q\delta_0, -q\delta_0)$ $(q < 0)$, *we can have* $\sigma_1 = \sigma$.

Proof. If a constant factor $K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ in (12) is the optimal value, then because of (11) (for $\sigma = \widehat{\sigma}$), we obtain:

$$K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \geq K(\widehat{\sigma}) \, (\in \mathbf{R}_+),$$

namely, (13) keeps the form of equality.

We notice that (13) maintains the equivalent form if and only if there exist constants A and B [16], in a way that they are not both zero and

$$Au^{\sigma-1} = Bu^{\sigma_1 - 1} \ a.e. \ in \ \mathbf{R}_+.$$

We can find that if $A \neq 0$, we have $u^{\sigma - \sigma_1} = \frac{B}{A}$ $a.e.\ in\ \mathbf{R}_+$, which follows that $\sigma - \sigma_1 = 0$, hence, $\sigma_1 = \sigma$. □

3. Main results

Theorem 1. *For* $0 < p < 1$, *the inequality* (6) *can be expressed as the following set of inequalities:*

$$\begin{aligned} J \ &:= \left[\int_0^\infty (v(y))^{p\left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right) - 1} v'(y) \left(\int_0^\infty H(xv(y)) f(x) dx\right)^p dy\right]^{\frac{1}{p}} \\ &> K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \left\{\int_0^\infty x^{p\left[1 - \left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right)\right] - 1} f^p(x) dx\right\}^{\frac{1}{p}}, \qquad (14) \\ J_1 \ &:= \left[\int_0^\infty x^{q\left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right) - 1} \left(\int_0^\infty H(xv(y)) g(y) dy\right)^q dx\right]^{\frac{1}{q}} \\ &> K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1) \left\{\int_0^\infty \frac{(v(y))^{q\left[1 - \left(\frac{\sigma}{p} + \frac{\sigma_1}{q}\right)\right] - 1}}{(v'(y))^{q-1}} g^q(y) dy\right\}^{\frac{1}{q}}. \qquad (15) \end{aligned}$$

The $K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1)$ *is the optimal constant factor in* (14) (*resp.* (15)) *if and only if the same constant factor in* (6) *is the optimal value.*

Specifically, for $\sigma_1 = \sigma$, if there has a constant $\delta_0 > 0$, such that $K(\sigma \pm \delta_0) < \infty$, subsequently, we obtain the following inequalities, which are equivalent to (11) and share the same optimal constant factor $K(\sigma)$:

$$\begin{aligned}
&\left[\int_0^\infty (v(y))^{p\sigma-1} v'(y)\left(\int_0^\infty H(xv(y)) f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\
> \ & K(\sigma)\left[\int_0^\infty x^{p(1-\sigma)-1} f^p(x)dx\right]^{\frac{1}{p}}, \qquad (16)\\
&\left[\int_0^\infty x^{q\sigma-1}\left(\int_0^\infty H(xv(y)) g(y)dy\right)^q dy\right]^{\frac{1}{q}} \\
> \ & K(\sigma)\left[\int_0^\infty \frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}} g^q(y)dy\right]^{\frac{1}{q}}. \qquad (17)
\end{aligned}$$

Proof. If (6) is valid, then the by the reverse Hölder's inquality [16], we have

$$\begin{aligned}
I &= \int_0^\infty \left[(v(y))^{\frac{-1}{p}+\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)}(v'(y))^{\frac{1}{P}}\int_0^\infty H(xv(y)) f(x)dx\right] \\
&\quad \times\left[(v(y))^{\frac{1}{p}-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)}(v'(y))^{-\frac{1}{P}} g(y)\right] dy \\
&\geq J\left\{\int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)\, dy\right\}^{\frac{1}{q}}. \qquad (18)
\end{aligned}$$

By (14), we have (6).

Conversely, if we assume the validity of (6) and set

$$g(y) := (v(y))^{p\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)-1} v'(y)\left(\int_0^\infty H(xv(y)) f(x)dx\right)^{p-1} dy \ \ (y>0).$$

Then it follows that

$$J^P = \int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy = I. \qquad (19)$$

If $J=\infty$, (14) is naturally valid, and if $J=0$, it is not possible to satisfy Equation (14)", that is $J>0$. Assume that $0<J<\infty$. According to (6), there have

$$\begin{aligned}
\infty \ &> \ J^P = \int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy = I \\
&> \ K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}} J^{P-1} > 0, \\
J \ &= \ \left\{\int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{P}} \\
&> \ K^{\frac{1}{P}}(\sigma) K^{\frac{1}{q}}(\sigma_1)\left\{\int_0^\infty x^{p\left[1-\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}},
\end{aligned}$$

hence, (14) follows, which is equivalent to (6).

In the same way, we can show that (15) is valid and it is also equivalent to (6). So (6), (14) and (15) are equivalent.

If the constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ in (6) is the optimal value, the same constant factor is also the optimal value in (14). Otherwise, by (18), there will be a contradiction that the same constant factor is not the optimal value in (6). On the other hand, if the constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ in (14) is the optimal value, then it is also the optimal value in (6). Otherwise, by (19), there will be a contradiction that the same constant factor in (14) is not the optimal value.

Similarly, we can show that the constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ is the optimal value in (6) if and only if the same constant factor in (15) is the optimal value. Therefore, the constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ in (14) (resp. (15)) is the optimal value if and only if the same constant factor in (6) is the optimal value. □

Theorem 2. *If there has a constant $\delta_0 > 0$, in a way that $K(\sigma \pm \delta_0) < \infty$, then $(i), (ii), (iii)$ and (iv) are equivalence statements:*

(i) Both $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ and $K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$ are independent of p, q;
(ii) $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1) = K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$;
(iii) for $\sigma - \sigma_1 \in (q\delta_0, -q\delta_0)$, we have $\sigma_1 = \sigma$;
(iv) The constant factor $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ in (6), (14) and (15) is the optimal value.

Proof. $(i) \Rightarrow (ii)$. There have

$$K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1) = \lim_{p\to 1^-}\lim_{q\to -\infty} K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1) = K(\sigma).$$

By Lemma 1, we find

$$\begin{aligned} K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right) &= \lim_{p\to 1^-}\lim_{q\to -\infty} K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right) \\ &= \lim_{q\to -\infty} K\left(\sigma + \frac{\sigma_1-\sigma}{q}\right) \\ &= K(\sigma) = K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1). \end{aligned}$$

$(ii) \Rightarrow (iii)$ If $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1) = K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$, then (13) keeps the form of equality. According to the proof in Lemma 4, for $\sigma - \sigma_1 \in (q\delta_0, -q\delta_0)$, we have $\sigma_1 = \sigma$.

$(iii) \Rightarrow (i)$ For $\sigma_1 = \sigma$, both $K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)$ and $K\left(\frac{\sigma}{p}+\frac{\sigma_1}{q}\right)$ are independent of p, q, which equal to $K(\sigma)$.

Hence, we have $(i) \Leftrightarrow (ii) \Leftrightarrow (iii)$.

$(iii) \Rightarrow (iv)$ For $\sigma_1 = \sigma$, by Lemma 3 and Theorem 1, the constant factor

$$K^{\frac{1}{p}}(\sigma)K^{\frac{1}{q}}(\sigma_1)\,(= K(\sigma))$$

is the optimal value in (6), (14) and (15).

$(iv) \Rightarrow (iii)$ Since $\sigma - \sigma_1 \in (q\delta_0, -q\delta_0)$, by Lemma 4, we have $\sigma_1 = \sigma$.

Hence, we have $(iii) \Leftrightarrow (iv)$.

Therefore, (i), (ii), (iii) and (iv) are equivalence statements. □

If $k_\lambda(x,y)\,(\geq 0)$ is a homogeneous function of degree $-\lambda$ in $\mathbf{R}_+$, then setting $H(u) = K_\lambda(1,u)$, replacing x by $\frac{1}{x}$and $x^{\lambda-2}f\left(\frac{1}{x}\right)$ by $f(x)$ in the Theorem 1 and Theorem 2, for $\sigma_1 = \lambda - \mu$,

$$K_\lambda(\gamma) := \int_0^\infty K_\lambda(1,u)\, u^{\gamma-1} du \in \mathbf{R}_+\,(\gamma = \sigma, \lambda - \mu),$$

we have

Corollary 1. *Assuming that*

$$\begin{aligned} 0 &< \int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)\,dx < \infty, \text{ and} \\ 0 &< \int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\sigma}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy < \infty \end{aligned}$$

the reverse equivalent Hilbert-type integral inequalities with the homogeneous kernel is valid in $\mathbf{R}_+$, as follows:

$$\begin{aligned} &\int_0^\infty \int_0^\infty K_\lambda(x, v(y))\, f(x) g(y)\, dxdy \\ > \;& K_\lambda^{\frac{1}{P}}(\sigma)\, K_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{\int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}} \\ &\times \left\{\int_0^\infty \frac{(v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{q}}, \end{aligned} \tag{20}$$

$$\begin{aligned} &\left[\int_0^\infty (v(y))^{p\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)-1} v'(y) \left(\int_0^\infty K_\lambda(x, v(y))\, f(x)dx\right)^p dy\right]^{\frac{1}{p}} \\ > \;& K_\lambda^{\frac{1}{P}}(\sigma)\, K_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{\int_0^\infty x^{p\left[1-\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)\right]-1} f^p(x)dx\right\}^{\frac{1}{p}}, \end{aligned} \tag{21}$$

$$\begin{aligned} &\left[\int_0^\infty x^{q\left(\frac{\lambda-\sigma}{p}+\frac{\mu}{q}\right)-1} \left(\int_0^\infty K_\lambda(x, v(y))\, g(y)dy\right)^q dx\right]^{\frac{1}{q}} \\ > \;& K_\lambda^{\frac{1}{P}}(\sigma)\, K_\lambda^{\frac{1}{q}}(\lambda-\mu) \left\{\frac{\int_0^\infty (v(y))^{q\left[1-\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)\right]-1}}{(v'(y))^{q-1}} g^q(y)dy\right\}^{\frac{1}{q}}. \end{aligned} \tag{22}$$

Furthermore, the constant factor $K_\lambda^{\frac{1}{P}}(\sigma)\, K_\lambda^{\frac{1}{q}}(\lambda-\mu)$ is the optimal value in (20) if and only if the same constant factor in (21) and (22) is the optimal value.

Specifically, for $\mu+\sigma=\lambda$, if there has a constant $\delta_0>0$, in a way that $K(\sigma\pm\delta_0)<\infty$, then we have the equivalent inequalities in $\mathbf{R}_+$ with the optimal constant factor $K_\lambda(\sigma), as follows:$

$$\begin{aligned}
&\int_0^\infty\int_0^\infty K_\lambda(x,v(y))\,f(x)g(y)\,dxdy\\
>\;& K_\lambda(\sigma)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}\left[\int_0^\infty \frac{(v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}, \qquad (23)
\end{aligned}$$

$$\begin{aligned}
&\left[\int_0^\infty (v(y))^{p\sigma-1}v'(y)\left(\int_0^\infty K_\lambda(x,v(y))\,f(x)dx\right)^p dy\right]^{\frac{1}{p}}\\
>\;& K_\lambda(\sigma)\left[\int_0^\infty x^{p(1-\mu)-1}f^p(x)dx\right]^{\frac{1}{p}}, \qquad (24)
\end{aligned}$$

$$\begin{aligned}
&\left[\int_0^\infty x^{q\mu-1}\left(\int_0^\infty K_\lambda(x,v(y))\,g(y)dy\right)^q dx\right]^{\frac{1}{q}}\\
>\;& K_\lambda(\sigma)\left[\frac{\int_0^\infty (v(y))^{q(1-\sigma)-1}}{(v'(y))^{q-1}}g^q(y)dy\right]^{\frac{1}{q}}. \qquad (25)
\end{aligned}$$

Corollary 2. *If there has a constant $\delta_0>0$, in a way that $K_\lambda(\sigma\pm\delta_0)<\infty$, then (I), (II). (III) and (IV) are equivalence statements as follows:*

(I) Both $K^{\frac{1}{P}}(\sigma)\,K^{\frac{1}{q}}(\lambda-\mu)$ and $K_\lambda\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)$ are independent of p,q;
(II) $K^{\frac{1}{P}}(\sigma)\,K^{\frac{1}{q}}(\lambda-\mu)=K_\lambda\left(\frac{\sigma}{p}+\frac{\lambda-\mu}{q}\right)$;
(III) for $\mu+\sigma-\lambda\in(q\delta_0,-q\delta_0)$, we have $\mu+\sigma=\lambda$;
(IV) The constant factor $K_\lambda^{\frac{1}{P}}(\sigma)\,K_\lambda^{\frac{1}{q}}(\lambda-\mu)$ in (20), (21) and (22) is the optimal value.

Example 1. *(i) If $H(u)=K_\lambda(1,u)=\frac{1}{(1+u)^\lambda}$ $(u>0;\lambda>0)$. Then*

$$\begin{aligned}
H(xv(y)) &= \frac{1}{(1+xv(y))^\lambda},\\
K_\lambda(x,v(y)) &= \frac{1}{(x+v(y))^\lambda}\ (x,y>0).
\end{aligned}$$

For $\gamma=\sigma,\sigma_1,\mu\in(0,\lambda)$, we obtain

$$K(\gamma)=K_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}}{(1+u)^\lambda}du=B(\gamma,\lambda-\gamma)\in\mathbf{R}_+,$$

and $\delta_0=\frac{\lambda-\sigma}{2}>0$.

(ii) If $H(u)=K_\lambda(1,u)=\frac{\ln u}{u^\lambda-1}$ $(u>0;\lambda>0)$. Then

$$\begin{aligned}
H(xv(y)) &= \frac{\ln xv(y)}{(xv(y))^\lambda-1},\\
K_\lambda(x,v(y)) &= \frac{\ln\frac{x}{v(y)}}{x^\lambda-(v(y))^\lambda}\ (x,y>0).
\end{aligned}$$

For $\gamma=\sigma,\sigma_1,\mu\in(0,\lambda)$, we obtain

$$K(\gamma) = K_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}\ln u}{u^\lambda-1}du = \frac{1}{\lambda^2}\int_0^\infty \frac{v^{(\gamma/\lambda)-1}\ln v}{v-1}dv=\left[\frac{\pi}{\lambda\sin(\pi\gamma/\lambda)}\right]^2\in\mathbf{R}_+,$$

and $\delta_0=\frac{\lambda-\sigma}{2}>0$.

(iii) If $H(u)=K_\lambda(1,u)=\frac{1}{(\max\{1,u\})^\lambda}$ $(u>0;\lambda>0)$. Then

$$H(xv(y)) = \frac{1}{(\max\{1,xv(y)\})^\lambda},$$
$$K_\lambda(x,v(y)) = \frac{1}{(\max\{x,v(y)\})^\lambda}\ (x,y>0).$$

For $\gamma=\sigma,\sigma_1,\mu\in(0,\lambda)$, we obtain

$$K(\gamma) = K_\lambda(\gamma)=\int_0^\infty \frac{u^{\gamma-1}}{(\max\{1,u\})^\lambda}du = \int_0^1 u^{\gamma-1}du+\int_1^\infty \frac{u^{\gamma-1}}{u^\lambda}du=\frac{\lambda}{\gamma(\lambda-\gamma)}\in\mathbf{R}_+,$$

and $\delta_0=\frac{\lambda-\sigma}{2}>0$.

By Theorem 1 and 2, Corollary 1 and 2, some equivalent reverse inequalities with the particular kernels and the optimal value constant factors can be obtained.

4. Conclusions

In this article, We obtain a reverse Hilbert-type integral inequality, by defining the following weight functions

$$\omega(\sigma,x) := x^\sigma\int_0^\infty H\left(xv\left(y\right)\right)v^{\sigma-1}\left(y\right)v'\left(y\right)dy\ \left(x\in\mathbf{R}_+\right),$$
$$\varpi(\sigma_1,y) := (v(y))^{\sigma_1}\int_0^\infty H\left(xv\left(y\right)\right)x^{\sigma_1-1}dx\ \left(y\in\mathbf{R}_+\right). \tag{26}$$

where $H\left(xv\left(y\right)\right)$ is the nonhomogeneous kernel. As well as the equivalent forms of (6) was constructed in Theorem 1. Several equivalent statements related to the optimal constant factor are provided in Theorem 2. By specifying certain parameters, we can obtain some corollaries for the homogeneous kernel and some examples for the particular kernels. An extensive account of this type of inequalities is obtained from the lemmas and theorems.

Authors' Contributions

T. L. carried out the mathematical studies, participated in the sequence alignment and drafted the manuscript. R. A. R. and B.Y. participated in the design of the study and per- formed the numerical analysis.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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