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CONVERGENCE THEOREM OF OSILIKE-BERINDE-G-NONEXPANSIVE MAPPINGS IN METRIC SPACES ENDOWED WITH GRAPH

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Abstract. In this paper, we also prove the strong and Δ-convergence theorems of the S-iteration process for Osilike-Berinde-G-nonexpansive mappings in metric spaces.

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1. Introduction

Let C be a nonempty subset of a metric space (X, d). A mapping $t : C \to C$ is called a contraction if there exists $\lambda \in [0, 1)$ such that

$$d(t(x), t(y)) \leq \lambda d(x, y) \text{ for all } x, y \in C. \tag{1.1}$$

If (1.1) is valid when $\lambda = 1$ then t is said to be nonexpansive. A point x in C is called a fixed point of t if $t(x) = x$.

Fixed point theory is an important tool for finding solutions of problems in the form of equations or inequalities. One of the fundamental and celebrated results in metric fixed point theory is the Banach contraction principle which stated that every contraction on a complete metric space always has a unique fixed point. This principle has been generalized in many directions, see, e.g., [12–14, 22, 28, 29] and references therein. Among other things, Osilike [22] generalized the concept of contractions to the following class of mappings: there exist $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$d(t(x), t(y)) \leq \lambda d(x, y) + L \cdot d(x, t(x)) \text{ for all } x, y \in C.$$

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In 2007, Berinde and Berinde [5] extended this concept to multi-valued mappings in the following manner: a multi-valued mapping $T : C \to CB(C)$ is called a weak contraction if there exist $\lambda \in [0, 1)$ and $L \in [0, \infty)$ such that

$$H(T(x), T(y)) \leq \lambda d(x, y) + L \cdot d(x, t(x)) \text{ for all } x, y \in C. \tag{1.2}$$

If (1.2) is valid when $\lambda = 1$, then T is called an Osilike-Berinde-nonexpansive mapping. A point $x \in C$ is a fixed point of T if $x \in T(x)$. We denote by $F(T)$ the set of all fixed points of T.

In 2019, Bunlue and Suantai [7] proved the existence of fixed points as well as the demi-closed principle for Osilike-Berinde-nonexpansive mappings in Banach spaces satisfying the Opial's condition. It was quickly noted by Klangpraphan and Panyanak [19] that the results in [7] can be extended to complete CAT(0) spaces.

On the other hand, Jachymski [15] combined the concepts of fixed point theory and graph theory to prove a generalization of the Banach contraction principle in a complete metric space endowed with a graph. In 2010, Beg et al. [4] extended Jachymski's result to the general setting of a multi-valued G-contraction. Later on, Alfuraidan and Khamsi [2] introduced the notion of multi-valued G-nonexpansive mappings and proved the existence of fixed points for such kind of mappings in hyperbolic metric spaces. Since then, the fixed point results in several kinds of metric spaces endowed with graphs have been developed and many papers have appeared, see, e.g., [3,6,8,11,16,23,25,32,35,37].

In 2009, Agarwal et al. [1] introduced the S-iteration following well-known iteration. For E a convex subset of a linear space X and t a mapping of E into itself. In 2012, Sokhuma and Akkasriworn [31] defined the S-iteration method scheme for a pair of single valued and multi-valued nonexpansive mappings. In 2020, Thabet et al. [33] introduce the modified Agarwal-O'Regan-Sahu iteration process (S-iteration) for finding endpoints of multi-valued nonexpansive mappings in the setting of Banach spaces. Under suitable conditions, some weak and strong convergence results of the iterative sequence generated by the proposed process are proved. Their results especially improve and unify some results of Panyanak [27]. In 2021, Kaewkhao, Klangpraphan and Panyanak [16] proved the strong and Δ-convergence theorems of the Ishikawa iteration process for the class of G-quasinonexpansive mappings which includes the class of Osilike-Berinde-G-nonexpansive mappings in metric spaces.

In this paper, motivated by the ideas of [1], [16], [27], [31] and [33], we prove the strong and Δ-convergence theorems of the S-iteration process for the class of G-quasinonexpansive mappings which includes the class of Osilike-Berinde-G-nonexpansive mappings in metric spaces.

2. Preliminaries

Throughout this paper, $\mathbb{N}$ stands for the set of natural numbers and $\mathbb{R}$ stands for the set of real numbers. Let G be a directed graph with a set of vertices $V(G)$ and a set of edges $E(G)$. In this paper, we assume that G contains all loops and has no parallel edges. Let $x, y \in V(G)$. We say that x dominates y if $(x, y) \in E(G)$. Let A and B be nonempty subsets of $V(G)$. We say that A dominates B if $(a, b) \in E(G)$ for all $a \in A$ and $b \in B$.

Let (X, d) be a metric space, C a nonempty subset of X and $G = (V(G), E(G))$ a directed graph such that $V(G) \subseteq C$. We denote by $CB(C)$ the family of nonempty closed bounded subsets of C and by $K(C)$ the family of nonempty compact subsets of C. The distance from a point x in X to a nonempty subset B of X is defined by

$$\text{dist}(x, B) := \inf\{d(x, b) : b \in B\}.$$

The Pompeiu-Hausdorff distance on $CB(C)$ is defined by

$$H(A, B) := \max\{\sup_{a\in A} \text{dist}(a, B), \sup_{b\in B} \text{dist}(b, A)\} \text{ for all } A,\ B \in CB(C).$$

A multi-valued mapping $T : C \to CB(C)$ is said to be edge-preserving if for each $(x, y) \in E(G)$, the following implication holds:

$$u \in T(x), v \in T(y) \Longrightarrow (u, v) \in E(G).$$

Let $\lambda \in [0, 1)$ and $L \geq 0$. The mapping T is said to be (λ, L)-G-contraction if it is edge-preserving and

$$H(T(x), T(y)) \leq \lambda d(x, y) + L \cdot \text{dist}(y, T(x)) \text{ for all } (x, y) \in E(G).$$

The existence of fixed points for (λ, L)-G-contraction is guaranteed by Tiammee and Suantai [34] in the following result.

Theorem 2.1. ([34]) *Let C be a nonempty closed subset of a complete metric space (X, d) and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Let $T : C \to CB(C)$ be a $(\lambda, L) - G$-contraction such that $C_T := \{x \in C : (x, y) \in E(G)$ for some $y \in T(x)\} \neq \emptyset$. Suppose that the following property holds:*

(*) for any sequence $\{x_n\}$ in C, if $x_n \to x$ and $(x_n, x_{n+1}) \in E(G)$ for all $n \in \mathbb{N}$, then there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Then T has a fixed point in C.

Definition 2.2. Let (X, d) be a metric space, C a nonempty subset of X and $G = (V(G), E(G))$ a directed graph such that $V(G) \subseteq C$. A multi-valued mapping $T : C \to CB(C)$ is said to be

(i) Osilike-Berinde-nonexpansive if there exists $L \geq 0$ such that

$$H(T(x), T(y)) \leq d(x, y) + L \cdot \text{dist}(x, T(x)) \text{ for all } x, y \in C;$$

(ii) Osilike-Berinde-G-nonexpansive if T is edge-preserving and there exists $L \geq 0$ such that

$$H(T(x), T(y)) \leq d(x, y) + L \cdot \text{dist}(x, T(x)) \text{ for all } (x, y) \in E(G);$$

(iii) quasinonexpansive if $F(T) \neq \emptyset$ and

$$H(T(x), T(y)) \leq d(x, y) \text{ for all } x \in C \text{ and } y \in F(T);$$

(iv) G-quasinonexpansive if T is edge-preserving such that $F(T) \neq \emptyset$ and

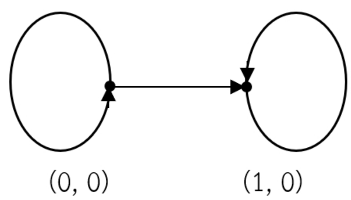
$$H(T(x), T(y)) \leq d(x, y) \text{ for all } (x, y) \in E(G) \text{ with } y \in F(T).$$

The following examples show that the class of Osilike-Berinde-nonexpansive mappings and the class of Osilike-Berinde-G-nonexpansive mappings are different.

Example 2.3. Let X be the Euclidean space $\mathbb{R}^2$ and $C = [0,1] \times [0,1]$ and let $G = (V(G), E(G))$ be such that $V(G) = \{(0,0), (1,0)\}$ and

$$E(G) = \{((0,0),(0,0)), ((0,0),(1,0)), ((1,0),(1,0))\}.$$

The graph G can be explained by the following diagram:



Let $T : C \to CB(C)$ be defined by

$$T(a, b) = \{(a, 1-b)\} \text{ for all } (a, b) \in C.$$

It follows from Example 2.1 of [25] that T is nonexpansive and hence Osilike-Berinde-nonexpansive. However, if we choose $x = (0,0), y = (1,0), u = (0,1)$ and $v = (1,1)$, then $(x, y) \in E(G)$, $u \in T(x)$ and $v \in T(y)$. But $(u, v) \notin E(G)$. This shows that T is not edge-preserving and hence is not Osilike-Berinde-G-nonexpansive.

Example 2.4. Let $X = \mathbb{R}$, $C = [0,1]$, $G = (V(G), E(G))$ be such that $V(G) = [0, \frac{1}{2}]$ and $E(G) = \{(x, y) : x, y \in V(G)\}$. Let $T : C \to CB(C)$ be defined by

$$T(x) = [0, x^2] \text{ for all } x \in C.$$

It is easy to see that T is edge-preserving. Let $(x, y) \in E(G)$. Then $0 \leq x, y \leq \frac{1}{2}$. Thus,

$$H(T(x), T(y)) = H([0, x^2], [0, y^2]) = \left| x^2 - y^2 \right| \leq |x - y| + \text{dist}(x, T(x)).$$

This shows that T is an Osilike-Berinde-G-nonexpansive mapping with $L = 1$. On the other hand, if $x = 1$ and $y = \frac{1}{2}$, then

$$H(T(x), T(y)) = H([0,1], [0, \frac{1}{4}]) = \left| 1 - \frac{1}{4} \right| > \left| 1 - \frac{1}{2} \right| = |x - y| + L \cdot \text{dist}(x, T(x)),$$

for all $L \geq 0$. This implies that T is not Osilike-Berinde-nonexpansive.

However, these two classes of mappings are identical under some additional conditions.

Proposition 2.5. ([16]) *Let C be a nonempty subset of a metric space and $T : C \to CB(C)$ a multi-valued mapping. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and $E(G) = C \times C$. Then the following statements hold:*

(i) *T is Osilike-Berinde-nonexpansive if and only if T is Osilike-Berinde-G-nonexpansive.*
(ii) *T is quasinonexpansive if and only if T is G-quasinonexpansive.*

The following proposition is an immediate consequence of Definition 2.2.

Proposition 2.6. ([16]) *The following statements hold:*

(i) *If T is Osilike-Berinde-nonexpansive and $F(T) \neq \emptyset$, then T is quasinonexpansive.*
(ii) *If T is Osilike-Berinde-G-nonexpansive and $F(T) \neq \emptyset$, then T is G-quasinonexpansive.*

The following example shows that the converses of (i) and (ii) in Proposition 2.6 do not hold.

Example 2.7. Let $X = \mathbb{R}$, $C = [0, 1]$, $G = (V(G), E(G))$ be such that $V(G) = C$ and $E(G) = \{(x, y) : x, y \in V(G)\}$. Let $T : C \to CB(C)$ be defined by

$$T(x) = \begin{cases} \left[0, \left|\frac{x}{x+1} \sin\left(\frac{1}{x}\right)\right|\right] & \text{if } x \neq 0; \\ \{0\} & \text{if } x = 0. \end{cases}$$

It is easy to see that $F(T) = \{0\}$. For $x \in (0, 1]$, we have

$$H(T(x), T(0)) = \left|\frac{x}{x+1} \sin\left(\frac{1}{x}\right)\right| \leq \left|\frac{x}{x+1}\right| \leq |x - 0|\,.$$

This implies that T is quasinonexpansive. On the other hand, for each $n \in \mathbb{N}$, we set $x_n := \frac{1}{2n\pi+(\pi/2)}$ and $y_n := \frac{1}{2n\pi}$. Then

$$\begin{aligned} \frac{H(T(x_n), T(y_n)) - |x_n - y_n|}{\text{dist}(x_n, T(x_n))} &= \left[\frac{x_n}{x_n + 1} - (y_n - x_n)\right]\left(\frac{x_n + 1}{x_n^2}\right) \\ &= \frac{1}{x_n} - \frac{(y_n - x_n)(x_n + 1)}{x_n^2} \\ &= (2n\pi + (\pi/2)) - \frac{(2n\pi + (\pi/2) + 1)}{4n} \to \infty. \end{aligned}$$

This implies that T is not Osilike-Berinde-nonexpansive. Since $V(G) = C$ and $E(G) = C \times C$, by Proposition 2.5 T is G-quasinonexpansive and is not Osilike-Berinde-G-nonexpansive.

The concept of uniformly convex hyperbolic spaces is introduced by Leuştean [21].

Definition 2.8. A hyperbolic space is a metric space (X, d) together with a function $W : X \times X \times [0, 1] \to X$ such that for all $x, y, z, w \in X$ and $t, s \in [0, 1]$, we have

(W1) $d(z, W(x, y, t)) \leq (1 - t)d(z, x) + td(z, y)$;

(W2) $d(W(x, y, t), W(x, y, s)) = |t - s|\, d(x, y)$;

(W3) $W(x, y, t) = W(y, x, 1 - t)$;

(W4) $d(W(x, z, t), W(y, w, t)) \leq (1 - t)d(x, y) + td(z, w)$.

For convenience, from now on, we will replace $W(x, y, t)$ by $(1 - t)x \oplus ty$. A nonempty subset C of X is said to be convex if $(1 - t)x \oplus ty \in C$ for all $x, y \in C$ and $t \in [0, 1]$. Let $G = (V(G), E(G))$ be a directed graph such that $V(G) \subseteq C$. We say that G is convex if for each $x, y, u, v \in C$ and $t \in [0, 1]$ such that (x, u) and (y, v) are in $E(G)$, we have $((1 - t)x \oplus ty, (1 - t)u \oplus tv) \in E(G)$.

The hyperbolic space (X, d) is said to be uniformly convex if for each $(r, \varepsilon) \in (0, \infty) \times (0, 2]$, there exists $\delta \in (0, 1]$ such that

$$d\left(\frac{1}{2}x \oplus \frac{1}{2}y, z\right) \leq (1 - \delta)r,$$

for all $x, y, z \in X$ with $d(x, z) \leq r, d(y, z) \leq r$ and $d(x, y) \geq r\varepsilon$.

A function $\eta : (0, \infty) \times (0, 2] \to (0, 1]$ providing such a $\delta := \eta(r, \varepsilon)$ is called a modulus of uniform convexity. In particular, if η is a nonincreasing function of r for every fixed ε, then we call η a monotone modulus of uniform convexity.

The concept of 2-uniformly convex hyperbolic spaces is introduced by Khamsi and Khan [17].

Definition 2.9. Let (X, d) be a uniformly convex hyperbolic space. For each $r \in (0, \infty)$ and $\varepsilon \in (0, 2]$, we define

$$\Psi(r, \varepsilon) := \inf\left\{\frac{1}{2}d^2(x, z) + \frac{1}{2}d^2(y, z) - d^2(\frac{1}{2}x \oplus \frac{1}{2}y, z)\right\},$$

where the infimum is taken over all $x, y, z \in X$ such that $d(x, z) \leq r, d(y, z) \leq r$, and $d(x, y) \geq r\varepsilon$. We say that (X, d) is 2-uniformly convex if

$$c_M := \inf\left\{\frac{\Psi(r, \varepsilon)}{r^2\varepsilon^2} : r \in (0, \infty), \varepsilon \in (0, 2]\right\} > 0.$$

In [20], the authors prove that

$$d^2((1 - t)x \oplus ty, z) \leq (1 - t)d^2(x, z) + td^2(y, z) - 4c_M t(1 - t)d^2(x, y), \tag{2.1}$$

for all $x, y, z \in X$ and $t \in [0, 1]$.

Example 2.10. (1) Every uniformly convex Banach space is a 2-uniformly convex hyperbolic space (see [36]).

(2) If X is a CAT(0) space, then it is a 2-uniformly convex hyperbolic space with $c_M = \frac{1}{4}$ (see [17]).

(3) If $\kappa > 0$ and X is a CAT(κ) space with diam$(X) \leq \frac{(\pi/2)-\varepsilon}{\sqrt{\kappa}}$ for some $\varepsilon \in (0, \pi/2)$, then by Lemma 2.3 of [24] we can conclude that

$$\Psi(r, \varepsilon) = \frac{r^2 \varepsilon^2 R}{8},$$

where $R = (\pi - 2\varepsilon)\tan(\varepsilon)$. This clearly implies that X is a 2-uniformly convex hyperbolic space with $c_M = \frac{R}{8}$.

From now on, X stands for a complete 2-uniformly convex hyperbolic space with a monotone modulus of uniform convexity. Let C be a nonempty subset of X and $\{x_n\}$ be a bounded sequence in X. The asymptotic radius of $\{x_n\}$ relative to C is defined by

$$r(C, \{x_n\}) := \inf \left\{ \limsup_{n\to\infty} d(x_n, x) : x \in C \right\}.$$

The asymptotic center of $\{x_n\}$ relative to C is the set

$$A(C, \{x_n\}) := \left\{ x \in C : \limsup_{n\to\infty} d(x_n, x) = r(C, \{x_n\}) \right\}.$$

It is known from [21] that if C is a nonempty closed convex subset of X, then $A(C, \{x_n\})$ consists of exactly one point. Now, we give the concept of Δ-convergence.

Definition 2.11. Let C be a nonempty closed convex subset of X and $x \in C$. Let $\{x_n\}$ be a bounded sequence in X. We say that $\{x_n\}$ Δ-converges to x if $A(C, \{u_n\}) = \{x\}$ for every subsequence $\{u_n\}$ of $\{x_n\}$. In this case we write $\Delta - \lim_{n\to\infty} x_n = x$.

It is known from [18] that every bounded sequence in X has a Δ-convergent subsequence. The following fact can be found in [10].

Lemma 2.12. *Let C be a nonempty closed convex subset of X and $\{x_n\}$ be a bounded sequence in X. If $A(C, \{x_n\}) = \{x\}$ and $\{u_n\}$ is a subsequence of $\{x_n\}$ with $A(C, \{u_n\}) = \{u\}$ and the sequence $\{d(x_n, u)\}$ converges, then $x = u$.*

In [34], Tiammee et al. introduce a property that is stronger than the condition (*) in Theorem 2.1.

Definition 2.13. Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$. Then C is said to have property G if for any sequence $\{x_n\}$ in C such that $\Delta - \lim_{n\to\infty} x_n = x \in C$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $(x_{n_k}, x) \in E(G)$ for all $k \in \mathbb{N}$.

Now, we introduce the demiclosed principle for Osilike-Berinde-G-nonexpansive mappings in complete uniformly convex hyperbolic spaces of Theorem 3.2 in [16].

Theorem 2.14. ([16]) *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a directed graph such that $V(G) = C$ and C has property G. Let $T : C \to K(C)$ be an Osilike-Berinde-G-nonexpansive mapping with $L \geq 0$. Then $I - T$ is demiclosed.*

We can obtain the following fixed point theorem of Theorem 3.3 in [16].

Theorem 2.15. ([16]) *Let C be a nonempty bounded closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) = C$ and C has property G. Let $T : C \to K(C)$ be a Osilike-Berinde-G-nonexpansive mapping. Suppose there exist $u \in C_T$ and $\mu \geq 0$ such that*

$$H(T(x), T(y)) \leq d(x, y) + \mu \cdot dist(y, \alpha u \oplus (1 - \alpha)T(x)), \tag{2.2}$$

for all $\alpha \in [0, 1]$ and $(x, y) \in E(G)$. Then T has a fixed point in C.

Notice that Theorem 2.15 is an extension of Theorem 4.2 in [19].

Theorem 2.16. ([19]) *Let E be a nonempty bounded closed convex subset of a Hadamard space (X, d) and $T : E \to K(E)$ be a B^2-nonexpansive mapping. Suppose there exist $u \in E$ and $L \geq 0$ such that*

$$H(T(x), T(y)) \leq d(x, y) + L \cdot dist(y, \alpha u \oplus (1 - \alpha)T(x)), \tag{2.3}$$

for all $x, y \in E$ and $\alpha \in [0, 1]$. Then T has a fixed point in E.

3. Convergence Theorem

In this section, we prove strong and Δ-convergence theorems of the S-iteration process for G-quasinonexpansive mappings and obtain the results for Osilike-Berinde-G-nonexpansive mappings as corollaries.

In 2009, Agarwal et al. [1] introduced the S-iteration following well-known iteration. For E a convex subset of a linear space X and t a mapping of E into itself, the iterative sequence $\{x_n\}$ of the S-iteration process is generated from $x_1 \in E$ and is defined by

$$y_n = (1 - \beta_n)x_n + \beta_n t x_n,$$
$$x_{n+1} = (1 - \alpha_n)t x_n + \alpha_n t y_n,$$

for all $n \in \mathbb{N}$, where $\{\alpha_n\}$, $\{\beta_n\}$ are sequence in $(0, 1)$ satisfying the condition:

$$\sum_{n=1}^{\infty} \alpha_n \beta_n (1 - \beta_n) = \infty.$$

In 2012, Sokhuma and Akkasriworn [31] defined the S-iteration method scheme for a pair of single valued and multi-valued nonexpansive mappings as follow:

Let E be a nonempty compact convex subset of a uniformly convex Banach space X, and $t : E \to E$ and $T : E \to KC(E)$ be a single valued nonexpansive mapping and a multi-valued nonexpansive mapping, respectively. Assume in addition that $F(t) \cap F(T) \neq \emptyset$ and $T(w) = \{w\}$ for all $w \in F(t) \cap F(T)$.

Suppose $\{x_n\}$ is generated iterative by $x_1 \in E$,

$$y_n = (1-\beta_n)x_n + \beta_n z_n,$$

$$x_{n+1} = (1-\alpha_n)z_n + \alpha_n t y_n,$$

for all $n \in \mathbb{N}$, where $z_n \in T(x_n)$ and $\{\alpha_n\}$, $\{\beta_n\}$ are sequences of positive numbers satisfying $0 < a \leq \alpha_n, \beta_n \leq b < 1$. Then the sequence $\{x_n\}$ converges strongly to a common fixed point of t and T.

In this paper, we present an iteration method modifying the above ones and call it the S-iteration.

Let C be a nonempty convex subset of X, and $\{\alpha_n\}$, $\{\beta_n\}$ be sequences in $[0,1]$, and $T : C \to CB(C)$ be a multi-valued mapping. The sequence of S-iteration is defined by $x_1 \in C$,

$$\begin{cases} y_n = (1-\beta_n)x_n \oplus \beta_n z_n, \\ x_{n+1} = (1-\alpha_n)z_n \oplus \alpha_n z'_n, \end{cases} \tag{3.1}$$

where $z_n \in T(x_n)$ and $z'_n \in T(y_n)$.

Lemma 3.1. *Let C be nonempty convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to CB(C)$ be an edge-preserving mapping. Let $\{y_n\}$ and $\{x_n\}$ be defined by (3.1). If x_1 dominates $p \in F(T)$, then x_n and y_n dominate p for all $n \in \mathbb{N}$.*

Proof. Since $(x_1, p) \in E(G)$ and T is edge-preserving, $(z_1, p) \in E(G)$. It follows from the convexity of G that $(y_1, p) \in E(G)$. Since T is edge-preserving, $(z'_1, p) \in E(G)$. By the convexity of G we have $(x_2, p) \in E(G)$. Continue in this way, we can show that $(y_n, p) \in E(G)$ and $(x_n, p) \in E(G)$ for all $n \geq 2$. □

Recall that a multi-valued mapping $T : C \to CB(C)$ is said to satisfy the endpoint condition [30] if $F(T) \neq \emptyset$ and $T(x) = \{x\}$ for all $x \in F(T)$. A sequence $\{x_n\}$ in X is said to be Fejér monotone with respect to C if

$$d(x_{n+1}, c) \leq d(x_n, c) \text{ for all } c \in C \text{ and } n \in \mathbb{N}.$$

The following lemma shows that the sequence of S-iteration defined by (3.1) is Fejér monotone with respect to the fixed point set of G-quasinonexpansive mapping.

Lemma 3.2. *Let C be a nonempty convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to CB(C)$ be a G-quasinonexpansive mapping satisfying the endpoint condition. Let $\{x_n\}$ be defined by (3.1). If x_1 dominates $F(T)$, then $\{x_n\}$ is Fejér monotone with respect to $F(T)$.*

Proof. Let $p \in F(T)$. By Lemma 3.1, $\{x_n\}$ and $\{y_n\}$ dominate p. Since T is G-quasi nonexpansive and satisfies the endpoint condition,

$$\begin{aligned} d(y_n, p) &\leq (1-\beta_n)d(x_n, p) + \beta_n d(z_n, p) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n H(T(x_n), T(p)) \\ &\leq (1-\beta_n)d(x_n, p) + \beta_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

This implies that

$$\begin{aligned} d(x_{n+1}, p) &\leq (1-\alpha_n)d(z_n, p) + \alpha_n d(z'_n, p) \\ &\leq (1-\alpha_n)H(T(x_n), T(p)) + \alpha_n H(T(y_n), T(p)) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(y_n, p) \\ &\leq (1-\alpha_n)d(x_n, p) + \alpha_n d(x_n, p) \\ &= d(x_n, p). \end{aligned}$$

Therefore, $\{x_n\}$ is Fejér monotone with respect to $F(T)$. □

The following lemmas are also needed.

Lemma 3.3. *Let C be a nonempty closed convex subset of X and $T : C \to CB(C)$ be a multivalued mapping. If $I - T$ is demiclosed, then $F(T)$ is closed in X.*

Proof. Let $\{x_n\}$ be a sequence in $F(T)$ such that $\lim_{n\to\infty} x_n = x$. Then $\text{dist}(x_n, T(x_n)) = 0$ for all $n \in \mathbb{N}$. It follows from the demiclosedness of $I - T$ that $x \in T(x)$, and hence $x \in F(T)$. This shows that $F(T)$ is closed in X. □

Lemma 3.4. *Let C be a nonempty closed convex subset of X and $T : C \to CB(C)$ be a multi-valued mapping such that $I - T$ is demiclosed. If $\{x_n\}$ is a bounded sequence in C such that $\lim_{n\to\infty} dist(x_n, Tx_n) = 0$ and $\{d(x_n, v)\}$ converges for all $v \in F(T)$, then $\omega_w(x_n) \subseteq F(T)$. Here $\omega_w(x_n) := \bigcup A(C, \{u_n\})$, where the union is taken over all subsequences $\{u_n\}$ of $\{x_n\}$. Moreover, $\omega_w(x_n)$ consists of exactly one point.*

Proof. Let $u \in \omega_w(x_n)$, then there exists a subsequence $\{u_n\}$ of $\{x_n\}$ such that $A(C, \{u_n\}) = \{u\}$. Since $\{u_n\}$ is bounded, there exists a subsequence $\{v_n\}$ of $\{u_n\}$ such that $\Delta - \lim_{n\to\infty} v_n = v \in C$. It follows from Proposition 2.5 and the demiclosedness of $I - T$ that $u = v \in F(T)$. This implies $\omega_w(x_n) \subseteq F(T)$. Next, we show that $\omega_w(x_n)$ consists of exactly one point. Let $\{u_n\}$ be a subsequence of $\{x_n\}$ with $A(C, \{u_n\}) = \{u\}$ and let $A(C, \{x_n\}) = \{x\}$. Since $u \in \omega_w(x_n) \subseteq F(T)$, $\{d(x_n, u)\}$ converges. By Proposition 2.5, $x = u$. This completes the proof. □

Now, we prove the Δ-convergence theorem.

Theorem 3.5. *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to CB(C)$ be a G-quasinonexpansive mapping satisfying the endpoint condition and $I - T$ is demiclosed. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be defined by (3.1) such that x_1 dominates $F(T)$. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Proof. Let $p \in F(T)$. It follows from (2.1) that

$$\begin{aligned} d^2(y_n, p) &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(z_n, p) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n H^2(T(x_n), T(p)) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &\leq (1-\beta_n)d^2(x_n, p) + \beta_n d^2(x_n, p) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &= d^2(x_n, p) - 4c_M\beta_n(1-\beta_n)d^2(x_n, z_n). \end{aligned}$$

This implies

$$\begin{aligned} d^2(x_{n+1}, p) &\leq (1-\alpha_n)d^2(z_n, p) + \alpha_n d^2(z'_n, p) - 4c_M\alpha_n(1-\alpha_n)d^2(z_n, z'_n) \\ &\leq (1-\alpha_n)H^2(T(x_n), T(p)) + \alpha_n H^2(T(y_n), T(p)) - 4c_M\alpha_n(1-\alpha_n)d^2(z_n, z'_n) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(y_n, p) \\ &\leq (1-\alpha_n)d^2(x_n, p) + \alpha_n d^2(x_n, p) - 4c_M\alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n) \\ &= d^2(x_n, p) - 4c_M\alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n). \end{aligned}$$

Thus

$$\sum_{n=1}^{\infty} a^2(1-b)d^2(x_n, z_n) \leq \sum_{n=1}^{\infty} \alpha_n\beta_n(1-\beta_n)d^2(x_n, z_n) < \infty. \tag{3.2}$$

So that $\lim_{n\to\infty} d^2(x_n, z_n) = 0$, and hence $\lim_{n\to\infty} \text{dist}(x_n, T(x_n)) = 0$. By Lemma 3.3, $\{d(x_n, v)\}$ converges for all $v \in F(T)$. By Lemma 3.4, $\omega_w(x_n)$ consists of exactly one point and is contained in $F(T)$. This shows that $\{x_n\}$ Δ-converges to an element of $F(T)$. □

As a consequence of Theorems 2.14 and Theorem 3.5, we can obtain the following corollary.

Corollary 3.6. *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) = C$ and C has property G. Let $T : C \to K(C)$ be an Osilike-Berinde-G-nonexpansive mapping satisfying the endpoint condition. Let α_n, $\beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be defined by (3.1) such that x_1 dominates $F(T)$. Then $\{x_n\}$ Δ-converges to a fixed point of T.*

Next, we will prove strong convergence theorems. Recall that a multi-valued mapping $T : C \to CB(C)$ is said to satisfy condition (IG) if $F(T) \neq \emptyset$ and there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for $r \in (0, \infty)$ such that $\text{dist}(x, T(x)) \geq f(\text{dist}(x, F(T)))$ for all x which dominates $F(T)$. The following fact can be found in [9].

Lemma 3.7. *Let E be a nonempty closed subset of X and $\{x_n\}$ a Fejér monotone sequence with respect to E. Then $\{x_n\}$ converges strongly to an element of E if and only if $\lim_{n\to\infty}$ dist$(x_n, E) = 0$.*

Theorem 3.8. *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to CB(C)$ be a G-quasinonexpansive mapping satisfying the endpoint condition. Suppose that T satisfies condition (IG) and $I - T$ is demiclosed. Let α_n, $\beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be defined by (3.1) such that x_1 dominates $F(T)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. By Lemma 3.3, $F(T)$ is closed in X. As in the proof of Theorem 3.5, we can show that $\lim_{n\to\infty} \text{dist}(x_n, T(x_n)) = 0$. Since T satisfies condition (IG), $\lim_{n\to\infty} \text{dist}(x_n, F(T)) = 0$. By Lemma 3.2, $\{x_n\}$ is Fejér monotone with respect to $F(T)$. The conclusion follows from Lemma 3.7. □

Corollary 3.9. *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$. Let $T : C \to CB(C)$ be an Osilike-Berinde-G-nonexpansive mapping satisfying the endpoint condition. Suppose that T satisfies condition (IG). Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$ and $\{x_n\}$ be defined by (3.1) such that x_1 dominates $F(T)$. Then $\{x_n\}$ converges strongly to a fixed point of T.*

The following example supports Theorem 3.5.

Example 3.10. Let $X = (\mathbb{R}, |\cdot|), C = [0, 1]$ and $G = (V(G), E(G))$ be such that $V(G) = [0, 1)$ and $E(G) = \{(x, y) : x, y \in V(G)\}$. Let $T : C \to CB(C)$ be defined by

$$T(x) = \begin{cases} [0, x^2] & \text{if } x \in [0, 1); \\ \{1\} & \text{if } x = 1. \end{cases}$$

It is easy to see that G is convex and T is edge-preserving. Notice also that $F(T) = \{0, 1\}$ and T satisfies the endpoint condition. If $x = 1$ and $y = \frac{1}{2}$, then

$$H(T(x), T(y)) = H(\{1\}, \left[0, \frac{1}{4}\right]) = 1 > \frac{1}{2} = |x - y|.$$

This implies that T is not quasinonexpansive. On the other hand, if $(x, y) \in E(G)$ such that $y \in F(T)$, then $y = 0$ and hence

$$H(T(x), T(y)) = H(\left[0, x^2\right], \{0\}) = x^2 \le x = |x - y|.$$

This shows that T is G-quasinonexpansive. Moreover, if $\{v_n\}$ is a sequence in C such that $\Delta - \lim_{n\to\infty} v_n = v$ and $\lim_{n\to\infty} \text{dist}(v_n, T(v_n)) = 0$, then either $v = 0$ or $v = 1$. This implies that $I - T$ is demiclosed. Let $\alpha_n, \beta_n \in [a, b] \subset (0, 1)$. By Theorem 3.5, for any starting point $x_1 \in [0, 1)$, the sequence $\{x_n\}$ defined by 3.1 converges to a point $x \in F(T)$. However, since $1 > x_1 \ge x_2 \ge ...$, it must be the case that $x = 0$.

Finally, we prove a strong convergence theorem for hemicompact mappings. Recall that a multi-valued mapping $T: C \to CB(C)$ is said to be hemicompact if for any sequence $\{x_n\}$ in C such that $\lim_{n\to\infty} \text{dist}(x_n, T(x_n)) = 0$, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ and $q \in C$ such that $\lim_{k\to\infty} x_{n_k} = q$. The following fact is also needed.

Lemma 3.11. ([26]) *Let $\{\alpha_n\}$, $\{\beta_n\}$ be two real sequences in $[0, 1)$ such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$. Let $\{\gamma_n\}$ be a nonnegative real sequence such that $\sum \alpha_n\beta_n(1 - \beta_n)\gamma_n < \infty$. Then $\{\gamma_n\}$ has a subsequence which converges to zero.*

Theorem 3.12. *Let C be a nonempty closed convex subset of X and $G = (V(G), E(G))$ be a convex directed graph such that $V(G) \subseteq C$ and C has property G. Let $T : C \to CB(C)$ be an Osilike-Berinde-G-nonexpansive mapping satisfying the endpoint condition. Let $\alpha_n, \beta_n \in [0, 1)$ be such that $\beta_n \to 0$ and $\sum \alpha_n\beta_n = \infty$ and $\{x_n\}$ be defined by (3.1) such that x_1 dominates $F(T)$. If T is hemicompact, then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. From (3.2) we get that

$$\sum_{n=1}^{\infty} \alpha_n\beta_n(1 - \beta_n)d^2(x_n, z_n) < \infty.$$

By Lemma 3.11, there exist subsequences $\{x_{n_k}\}$ and $\{z_{n_k}\}$ of $\{x_n\}$ and $\{z_n\}$ respectively, such that $\lim_{k\to\infty} d(x_{n_k}, z_{n_k}) = 0$, and hence $\lim_{k\to\infty} \text{dist}(x_{n_k}, T(x_{n_k})) = 0$. Since T is hemicompact and C has property G, by passing to a subsequence, we may assume that there exists $q \in C$ such that $\lim_{k\to\infty} x_{n_k} = q$ and $(x_{n_k}, q) \in E(G)$ for all $k \in \mathbb{N}$. Since T is Osilike-Berinde-G-nonexpansive, there exists $L \geq 0$ such that

$$H(T(x_{n_k}), T(q)) \leq d(x_n, q) + L \cdot \text{dist}(x_n, T(x_{n_k})) \text{ for all } k \in \mathbb{N}.$$

This implies that

$$\begin{aligned} \text{dist}(q, T(q)) &\leq d(q, x_{n_k}) + \text{dist}(x_{n_k}, T(x_{n_k})) + H(T(x_{n_k}), T(q)) \\ &\leq 2d(x_{n_k}, q) + (1 + L) \cdot \text{dist}(x_{n_k}, T(x_{n_k})) \to 0 \text{ as } k \to \infty. \end{aligned}$$

Thus $q \in T(q)$. By Lemma 3.2, $\lim_{n\to\infty} d(x_n, q)$ exists and hence q is the strong limit of $\{x_n\}$. □

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Conflicts of Interest

The author declare that there are no conflicts of interest regarding the publication of this paper.

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