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AN ANALYTIC TREATMENT OF EVOLUTION COPULAS

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Abstract. In this paper, we make clearer *evolution copulas* and explain how they describe phenomena which vary with respect to time. Elliptic and harmonic copulas solving a Cauchy problem in some classical spaces will serve as the prototype of the study. Semigroup theory is used to write explicitly the solution using the heat kernel that permits a deep understanding of the asymptotic behavior. Some classical properties and parameters, such as asymmetry and statistical coefficients, preserved along the evolution are studied.

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1. Introduction

The interest of copulas consists in their ability to describe dependence between two or more random variables, see the recent paper [13]. Sklar in [6] was the first initiator of linking margin distributions to joint one after an Fréchet attempt to reproduce contingency table from the margin distributions [5]. Henceforth, copulas gained an important interest in modeling dependence and quantifying correlation by the way of transcription of almost known statistical ratios such as τ of Kendal and ρ of Spearman.

Recently, a particular attention is paid to evolution problems and some aspects of interdependence of their components. Indeed, several realistic and deterministic but also stochastic evolution phenomena depend actually on time. The most suitable tool to model such problems is given by copulas that grow with time, baptized *evolution copulas* and denoted naturally $C(t, X)$. In this latter notion, $t \in \mathbb{R}^+$ connotes time and $X = (X_1, X_2, ..., X_n), n \geq 2$ is a $n-$entries random vector uniformly distributed on $\mathbb{I}^n = [0, 1]^n$ satisfying the following additional hypotheses:

(1) Boundary conditions: $\forall (t, X) \in [0, +\infty[$: $C(t, X) = 0$ as soon as there exists at least one entry X_i equals 0.

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(2) For all $x = (x_1, ..., x_n) \in \mathbb{I}^n$ et pour tout $j \in \{1, ..., d\}$, on a $C(x) = x_j$ if all entries of x are equal to 1 except probably for x_j.

(3) For all $n-$box $E = \Pi_{i=1}^n [a_i, b_i] \in \mathbb{I}^n$, the $C-$volume V_B of E is positive, where $V_B(E) = \sum_{s \in S} (-1)^{n(s)} C(s)$. Here S is the set of all vertices $S = (a_i)_i \cup (b_j)_j$ of $[a, b]$ and $n(s) = \big|\{s_k \in S; s_k = a_k\}\big|$.

Many formulations of these hypotheses exist. We have adopted here items (1), (2) and (3) above as introduced by Nelsen in his unmissable chef-d'oeuvre [7].

For the sake of simplicity and accessibility , we restrict ourselves, along the current paper, to the case $n = 2$. We hope as initiated in Ishimura and Yoshizawa [2] to study deeply evolution copulas, say those which depend on time as defined above. A reformulation in this simple framework leads to the following definition

Definition 1. *An evolution copula $C(t, ., .)$ is a bifunction on $\mathbb{I}^2$ into $\mathbb{I} = [0, 1]$ which satisfies the following conditions for all u, v, u_1, v_1, u_2, v_2 in $\mathbb{I}$ such that $u_1 \leq u_2$ and $v_1 \leq v_2$:*

(1) *Boundary conditions:* $\forall t \geq 0 : \ C(t, 0, v) = C(t, u, 0) = 0.$

(2) *Uniform margins:* $\forall t \geq 0 : \ C(t, 1, v) = v$ *and* $C(t, u, 1) = u.$

(3) *The 2-increasing property:*

$$C(t, u_2, v_2) - C(t, u_2, v_1) - C(t, u_1, v_2) + C(t, u_1, v_1) \geq 0.$$

Since, in practice, the parameter ($t \notin [0, +\infty[$), we start in this paper to examine evolution copulas associated with finite horizon, say $t \in [0, \tau]$ for some $\tau > 0$ and explain how we to extend the evolution to infinity using a continuation technique that preserves *copula proprety*.

It is worth to mention that the time t appears slightly like a general parameter for a usual indexed family of copulas. Indeed, in some situations, it is the case. But an indexed family may refer to a general collection of copulas eventually depending on at least two parameters such as Marshall-Olkin bivariate copulas. These ones are known to be a family $C_{a,b}$ of copulas given by

$$C_{a,b}(u, v) = \min(u^{1-a}v, uv^{1-b}), \quad \forall (u, v) \in [0, 1]^2.$$

To avoid any exception, we assume that any multi-parametrized family of copulas whereby at least one parameter is in $(0, +\infty)$ is also an evolution copula. When the parameter describing evolution is compelled to belong to finite horizon $[0, T]; T > 0$, we will give an elementary technique to extrapolate the dynamic to infinite horizon.

In this paper, we study how copulas evolve according to a specified dynamic, investigate asymptotic behavior and properties of the copula solution at the horizon in light of the initial or start copula. We treat as a prototype the evolution equation

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,x,y) = \Delta C(t,x,y) \\ C(0,,x,y) = C_0(x,y). \end{cases} \tag{1}$$

An efficient tool, according to our humble opinion for a suitable analytic treatment is a semigroup theory initiated by Kosaku Yosida [20], Hille-Philips [21], Pazy [10] and the unavoidable two references Naguel and Engel [8] and Arendt et.al [1] if one prefers to give just the most eminent references.

As a preamble, we recall basic results and relationship on copulas, semi groups and evolution families as main tools to bring up our topic. In the next step, we reformulate briefly the Ishimura's method for harmonic copulas. In the third one, we present a new method based on semigroup theory. At last, we give conclusions and examine how it will be possible to generalize our approach to similar cases.

2. Preliminary

2.1. **Copulas.** We restrict ourselves in this first study to bivariate copulas as recalled in definition (1). The crucial property relating to the order, although this one is partial, is boundedness of the set of all copulas $\mathcal{S}$ in the space of all continuous bivariate mappings $\mathcal{F}(I^2,\mathbb{R})$. Precisely we have

Proposition 1. *For all copula $C \in \mathcal{S}$, the double estimation holds:*

$$\forall (x,y) \in I^2 : \quad W(x,y) \leq C(x,y) \leq M(x,y),$$

where

- $W(x,y) = \max(x+y-1, 0)$
- $M(x,y) = \min(x,y)$.

The bifunctions W and M are the Hoeffding-Fréchet bounds. They are copulas in this particular case of dimension and W fails to be a copula for multivariate copulas defined on $\mathbb{I}^n$ for $n \geq 3$.

To describe the independence structure of two variable, the product function
$\Pi : (x,y) \mapsto xy$ is the copula which ensures, when linked to the vector (X,Y), that random variables X and Y are independent. It is very important to recall how the joint distribution F of two variables X and Y may be regenerated by their respective margin distribution F_X and G_Y. The bridge is established by the well known Sklar's theorem

Theorem 1 (Sklar's theorem)**.** *Let F be two-dimensional distribution function on a probability space (Ω, p) with marginal distribution functions F and G. Then there exists copula C such that*

$$\forall (x,y) \in \mathbb{I}^2 : \quad F(x,y) = C(F_X(x), F_Y(y)). \tag{2}$$

If, in addition F_X and F_Y are assumed to be continuous then the copula C is unique.

For our concern, copulas depending on one or two parameters are considered. One of them is seen as time and the copula, at a fixed parameter, describes the dependence between empirical data. A natural question immediately arises:

Question : How to enable the process $(C(t,.,.))_{t\geq 0}$ surviving to infinity when the copula property is restricted to finite horizon in time?

Fortunately, a classical method of truncation was suggested by Werner Hürlimann in [14] where the author treated problems related extension of FGM copulas. Therein, he proposed following extension, for the particular case of upper bound $M(u,v) = \min(u,v)$ and the α parametrized $FGM(\alpha) = uv + \alpha uv(1-u)(1-v)$ copula

Proposition 2. [14, *Theorem 3.1*] *Let M and $FGM(.,.)$ the copulas defined for all $(u,v) \in \mathbb{I}^2$ by $M(u,v) = \min(u,v)$ and $FGM(\alpha) = uv + \alpha uv(1-u)(1-v); -1 < \alpha < 1$. Then the bifunction*

$$C_\alpha(u,v) = \min(M(u,v); FGM(\alpha))$$

defines a copula for all $\alpha \geq 0$.

This result gives a meaning to evolution along time interval $[0,T]$ and stimulates a study of the behavior at infinite horizon, say when T tends to infinity. One of the pertinent and useful tool in mathematical analysis is the semigroup theory for which we recall the most important ingredients in the following subsection.

2.2. **Semigroups.** Let X be a Banach space. In our context, it is instructive to think X as being $C(\mathbb{I}^2)$ or one of its subspaces. We recall elementary results on semigroup theory. The references adopted here are [1], [8] and [10].

Definition 2. *A one parameter family $T(t)_{t\geq 0}$, of bounded linear operators from X into X is a semigroup of bounded linear operators on X if*

- $T(0) = I$, *(I is the identity operator on X).*
- $T(t+s) = T(t)T(s)$ *for every $t, s \geq 0$.*

A semigroup of bounded linear operators, $T(t)$, is uniformly continuous or continuous in norm, if $\lim\limits_{t\to 0} \|T(t) - I\| = 0$. In practical cases and for most PDEs, this kind of continuity is replaced by strong one. Namely, the semigroup is strongly continuous or of the class $\mathcal{C}_0$ if it satisfies

$$\forall x \in E: \quad \lim_{t\to 0} \|T(t)x - x\| = 0.$$

The infinitesimal generator of the semigroup $(T(t)_{t\geq 0})$ is the linear operator A defined by

$$D(A) = \{x \in X : \lim_{t\to 0} \frac{T(t)x - x}{t} = l \text{ exists}\}$$

and

$$\forall x \in D(A) \quad Ax = l = \frac{d}{dt} T(t)x|_{t=0}.$$

Here we summarize some facts on a given *continuous* semigroup $(T(t)_{t\geq 0})$ (i.e $\lim\limits_{t\to 0} T(t)x = x$) and its infinitesimal generator

(a). The generator defines uniquely the semigroup in the sense that if $(T(t))$ and $(S(t))_t$ are two semigroups such that that $\frac{d}{dt}T(t)x|_{t=0} = \frac{d}{dt}S(t)x|_{t=0}$, then $\forall t \geq 0,\ T(t) = S(t)$.

(b). There exists $M \geq 1$ and $\omega \in \mathbb{R}$ such that

$$\forall x \in D(A) \text{ and } t \geq 0, \quad \|T(t)\| \leq Me^{\omega t}.$$

(c). When the generator A is bounded, the semigroup is uniformly continuous and characterized entirely by the sum

$$T(t) = \sum_{n=0}^{\infty} \frac{t^n A^n}{n!} = e^{tA}. \tag{3}$$

Conversely, for any uniformly continuous semigroup $(T(t))_{t\geq 0}$, we can associate uniquely a bounded operator A such that (3) holds.

(d). For strongly semigroups, the generator is, in general unbounded, which means that is just defined on a subspace D of E and may not be continuous. Although these impurities, the following nice properties always hold

- A is a closed operator in the sense that its graph $\{(x, Ax), x \in D\}$ is a closed subspace of $D \times E$ when it is endowed with the graph norm.
- The domain D of the generator A is dense in E.
- The subset of analytic vectors $\{x \in E, T(t)x = \sum_{n=0}^{\infty} \frac{t^n A^n x}{n!}\}$ is dense in E.

Before clarification of the well known bridge between semigroups and partial differential equations, we recall the famous results on generation which give sufficient conditions on a given operator A to be a generator of a $\mathcal{C}_0-$ semigroup.

Theorem 2 (Hill-Yosida)**.** *A linear (unbounded) operator A is the infinitesimal generator of a $\mathcal{C}_0-$semigroup of contractions $(T(t)_{t\geq 0})$, if and only If*

(i) *A is closed and $\overline{D(A)} = E$.*

(ii) *The resolvent set $\rho(A) = \{\lambda \in \mathbb{C}, A - \lambda I_E$ is invertible in $\mathcal{L}(E)\}$ of A contains $\mathbb{R}^+$ and*

$$\forall \lambda > 0 \quad \|(\lambda I_E - A)^{-1}\| \leq \frac{1}{\lambda}$$

Hill-Yosida theorem (2) characterizes generators of contractive semigroups, for which ω, in the property b. above may taken equal zero. A general, but not practical, version of the theorem exists. The inconvenient is that the formula used therein involves all powers of the resolvent operator $R(\lambda, A) = (\lambda I_E - A)^{-1}$. Actually, one may encounter this apparent problem with a simple technique of rescaling: for a given semigroup $T_1(t)$ satisfying $\|T_1(t)\| \leq e^{\omega t}$, the semigroup $T_2(t) = e^{-\omega t}T_1(t)$ is of contractions and the concern of contractivity is immediately overcome.

The second theorem characterizing generators is the classical Lumer-Philips result for dissipative operators. Let us define the notion of dissipation

Definition 3. *A linear operator A is dissipative if for all $\lambda > 0$ and every $x \in D(A)$ one has $\|\lambda x - Ax\| \geq \lambda\|x\|$.*

In fact, the original definition involves the duality set of a given vector x in E as the set $\{z^\star \in E^\star, z^\star(z) = \|x\|_E^2 = \|z\|_{E^\star}^2\}$, however, for the sake of simplicity, and since this work is devoted to statistical evolution phenomena, we adopt the definition (3) which is originally a nice characterization of *dissipativity* property of the generator.

Dissipative operators A for which the translated actions $A - \lambda I_E, \lambda > 0$ have good analytic properties. The most important, according to point of view, is the generation of semi group as stated in the following theorem.

Theorem 3 (Lumer-Philips)**.** *Let A be a linear operator with dense domain $D(A)$ in E.*

i. *If A is dissipative and there is a $\lambda_0 > 0$ such that the range, $R(\lambda_0 I_E - A) = E$, then A is the infinitesimal generator of a $\mathcal{C}_0-$semigroup of contractions on E.*
ii. *If A is the infinitesimal generator of a $\mathcal{C}_0-$semigroup of contractions on E, then $R(\lambda I_E - A) = E$ for all $\lambda > 0$ and A is dissipative.*

At this stage, we are able to enunciate the bridge between the semigroup theory and well-posedness of evolution equation of the first order

Theorem 4 (Generation-wellposedness,102)**.** *Let A be a densely defined linear operator with a nonempty resolvent set $\rho(A)$. The initial value problem*

$$\begin{cases} \dot{u}(t) = Au(t),\ t > 0 \\ u(0) = x \end{cases}$$

has a unique solution $u(t)$, which is continuously differentiable on $[0, \infty[$, for every initial value $x \in D(A)$ if and only if A is the infinitesimal generator of a $\mathcal{C}_0$ semigroup $(T(t))_{t\geq 0}$.

When further regularity is required, the *holomorphic semigroups* play a central role to ensure analyticity of the solution. For more results and knowledge on semigroups and applications, mainly holomorphic ones, we refer to [8] or the most recent book [9].

3. Elliptic evolution copulas

3.1. **Classical Ishimura's approach.** (See [2] or [3])

As mentioned above, the equation to deal with is the evolution problem (5).
A suitable functional space to consider is one of the regular functions $C(.,.)$ on $[0,\infty[\times\mathbb{I}^2$, say differentiable functions with respect to time $t \in \mathbb{R}^+$ and of class $\mathcal{C}^2$ with respect to space. Actually, there is no reason to restrict the treatment to low dimensions. The unique concern is, in analytical point of view, to reduce calculation and in statistical one to avoid the problem of the lower Hoftding-fréchet bound of copulas (i.e W as a best counter monotonicity descriptor) which fails to be a copula in higher dimensions. Apart from this little problem, the analysis is the same.

The start point is the important result of Ishimura [2] where the solution of the problem 5 is given explicitly by

$$C(t,x,y) = uv + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t}\sin(m\pi x)\sin(n\pi y)K_{m,n}(C_0 - \Pi)$$

Where

$$K_{m,n}(C,C') = \iint_{I^2}\sin(m\pi u)\sin(n\pi v)\big(C_0(u,v) - C'(u,v)\big)dudv$$

Ishimura proved that $C(t,.,.)_{t\geq 0}$ is a deterministic process that solves (5) using, on one hand, positivity of the density function which ensures volume hypothesis H_3 of $C-$volume of the copula (see the introduction) and on the other one, the maximum principle to the kernel

$$p(t,x,y) = 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t}\sin(m\pi x)\sin(n\pi y)K_{m,n}\left(\frac{\partial^2 C_0}{\partial u\partial v}, \mathbf{1}_{\mathbb{I}^2}\right).$$

Here $\mathbf{1}_{\mathbb{I}^2}$ denotes the constant function equal to one everywhere on $\mathbb{I}^2$. The main analytic transformation was, although it does not preserve the *copula property*, the change of variable $\overline{C}(t,.,.) = C(t,.,.) - \Pi(.,.)$ which enabled the authors to see $\overline{C}(t,.,.)$ as a solution of the following evolution equation with boundary and initial condition

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,x,y) = \Delta C(t,x,y) \\ C(t,u,v) = 0 \quad \text{on } \partial\mathbb{I}^2\times[0,+\infty[\\ C(0,x,y) = C_0(x,y) - xy. \end{cases} \tag{4}$$

The most remarkable result of Ishimura analysis is the asymptotic behavior of the copula solution in the following proposition

Proposition 3. [2, *Theorem 2*] *The problem* (5) *has a unique solution* $C(t,.,.)$ *which satisfies*

$$\lim_{t\to\infty} C(t,.,.) = \Pi(.,.).$$

In the remainder of this paper, we pursue the analysis in the same spirit and we start with the natural evolution, namely when the start point C_0 describes the independence. So the problem (5) becomes

$$\begin{cases} \dfrac{\partial C}{\partial t}(t,x,y) = \Delta C(t,x,y) \\ C(0,,x,y) = \Pi(x,y) = xy. \end{cases} \tag{5}$$

The asymptotic behavior and uniqueness of the solution, as stated in proposition (3), above leads immediately to $C(t,.,.) = \Pi(.,.), \forall t > 0$. Henceforth, the copula $C(t.,.)$ is automatically harmonic. An analogous argument guarantees that the independence copula Π is the unique harmonic one which solves the problem (5). We can thus affirm that any probability measure ensuring independence is invariant under elliptical evolution. This remark is so fundamental in the following sense: consider a process $(X_t, Y_t) \in \mathbb{R}^2$. The obtained result says that in Cameron-Martin theorem, if there is a single time t_0 for which the vector (X_{t_0}, Y_{t_0}) is governed by Π copula, then it will be the same for any (X_s, Y_s) for any $s \geq t$.

Let us know discuss asymmetry problems of the solution. It is quasi obvious to see that if C_0 is symmetric (resp. radial symmetric), then $C(t,x,y)$ will remain symmetric (resp. radially symmetric).

We can therefore affirm that if two random phenomena are interchangeable and if their evolution is elliptical, they will remain so. The latter property fits with the behavior at infinity since at infinity, the limit Π describes also interchangeable phenomena.

Let us know summarize these results in

Theorem 5 (Characterization of harmonic copulas)**.**

a. *The unique harmonic bivariate copula is that of independence*$($*i.e the copula* $\Pi)$.
b. *There is no harmonic perturbation of any* C^2 *copula that solves the problem* (5) *in particular the independece one.*

Proof. (1) If the copula Ha is harmonic then, by uniqueness of the solution, Ha is the *constant* solution of the evolution equation:

$$\begin{cases} \partial_t C(t,x,y) = \Delta C(t,x,y) \\ C(0,x,y) = Ha(x,y). \end{cases}$$

Since the limit of the copula process is Π, then $Ha = \Pi$.

(2) Let Ha be such a perturbation, then $C_0 + Ha$ is, for the same reasons above, the unique solution of the evolution problem initialized at $C_0 + Ha$. The behavior at infinity immediately leads to $C_0 + Ha = C_0$. This result will fit with the deeper comment given below on harmonic polynomial copulas stating that Π is the unique harmonic $d-$homogeneous polynomial copula $(d \geq 2)$ that solves uniquely (5).

□

Let's go deeper into the analysis. One may be tempted to look for the polynomials polynomial Copulas which are candidate to solve the problem (5). The regularity of semigroup $(T(t)_{t\geq 0})$ and owing to theorem (3), if $C(t,.,.)$ is a such solution then $\Delta C(t,.,.) = \Delta(\lim_{t\to\infty} C(t,.,.)) = \Delta\Pi(t,.,.) = 0$. Then $C(t,.,.)$ is harmonic. we are therefore led to seek harmonic polynomials P_d of fixed degree d that satisfies $\Delta P_d = 0$. That is a classical but interesting question. We recall the elementary method to characterize such polynomials and the link with the copula property that they must satisfy.

Let $\mathbb{R}_d$ the space of all polynomial of degree at most d with N variables $x_1, x_2, \dot{;} x_N$, and $\mathbb{H}_d$ denote its subspace of harmonic polynomials of degree d. Consider the mapping

$$\begin{cases} \Phi : \mathbb{R}_d \to \mathbb{R}_d \\ P \mapsto \phi(P) = \Delta P. \end{cases}$$

It is easy to see that the range of Φ coincides with ϕ is exactly $\mathbb{R}_{d-2}$ and its kernel is $\mathbb{H}_d$. A classical theorem of the range allows a precise calculation of $\mathbb{H}_d$ dimension. By an elementary countability procedure, we obtain

$$dim(\mathbb{H}_d) = dim(\mathbb{R}_d) - dim(\mathbb{R}_{d-2}) = \begin{pmatrix} d+N-1 \\ d \end{pmatrix} - \begin{pmatrix} d+N-3 \\ d-2 \end{pmatrix}$$

which gives

$$dim(\mathbb{H}_d) = \frac{(N+2d-2)(N+d-3)!}{d!(N-2)!}.$$

This algebraic important result leads for our case, $(N = 2)$, to $dim(\mathbb{H}_d) = 2$. We thus retrieve the important result that Π is the unique regular harmonic copula. Indeed, according to the above result on dimension of $\mathbb{H}_d$, the only polynomial copulas candidate to be harmonic are $C(x,y) = x^2$, or $C(x,y) = y^2$, or $C(x,y) = xy$. Since the two first candidates do not satisfy the boundary conditions, the independence copula is the unique solution.

3.2. **Semigroup approach.** Here, we consider a more suitable functional framework that allows nice properties of the Laplacian operator. If one considers $L^2(\mathbb{R}^N)$, where $N \in \mathbb{N}$ (but we have restricted the treatment to $N = 2$), the operator Δ enjoys nice properties mainly generation property: Δ generates a $\mathcal{C}_0-$semigroup $T(z)_{t\geq 0}$ which is, in further, holomorphic on the right half plan and is given by (see for instance [11]) for all $z \in \mathbb{C}$, $\Re(z) > 0$,

$$T(z)(\vec{x}) = \frac{1}{(4\pi z)^{\frac{N}{2}}} \int_{\mathbb{R}^N} \exp(\frac{-\|\vec{x} - \vec{h}\|^2}{4z}) C_0(\vec{h}) d\vec{h}, \quad \vec{x} \in \mathbb{R}^N$$

It is worth to precise that for our treatment $N = 2$ and the vector $\vec{x} = (x,y)$ and when it will be necessary $\vec{h} = (h,k)$. So the formula above becomes for all $z \in \mathbb{C}$, $\Re(z) > 0$,

$$T(z)(x,y) = \frac{1}{4\pi z} \int_{\mathbb{R}^2} \exp\left(\frac{-(x-h)^2 - (y-k)^2}{4z}\right) C_0(h,k) dh dk, \ (x,y) \in \mathbb{R}^2. \tag{6}$$

On the way, it is worth to mention that $T(t)_{t\geq 0}$ as defined above is known to be Gaussian semigroup and may be written, in trems of kernels, as $T(t)(\vec{h}) = (k_t \star C_0)(\vec{h})$, where k_t denotes the heat kernel (or the Gaussian kernel) given by

$$k_t(\vec{h}) = \frac{1}{(4\pi z)^{\frac{N}{2}}} \exp(\vec{h}).$$

The fact that Laplacian operator generates such regular (holomorphic) semigroup allows immediately the existence of a unique solution of (5). In addition, we retrieve easily that

- The solution depends continuously on the start datum copula C_0.
- The solution is analytic with respect to time. This consequence is not automatic for general evolution problem as explained by Evans in [4, Remark in page 62]

A wise reader will certainly wonder why this recourse to Sobolev framework L^p. A natural answer is the excess of regularity when we restrict the study to $\mathcal{C}^2$ function according to this classical well known result that we resume as it is

Theorem 6 (Theorem 8,page 59 [4])**.** *If the solution $C(t,.,.)$ is of class $\mathcal{C}^2$, then it is of class $\mathcal{C}^\infty$, and this occurs even if $C(t,.,.)$ attains non-smooth boundary values.*

A surprising consequence of this latter theorem is the nonexistence of stochastic process $(X)_{t\geq 0}$ governed punctually by elliptic evolution copulas and converges to monotonicity $M(.,.) = \min(.,.)$ or counter-monotonicity state $W(.,.) = \max(. + . - 1, 0)$, since the two latter copulas are not of class $\mathcal{C}^2$.

Remark 1. *The integral formula (6) allows to overcome the problem of regularity excess imposed by* the copula property *that the function $C(t,.,.)$ should preserve at any time. Although the verification of boundary conditions is not easy to handle with this formula, one may encounter the calculus by invoking the density of $\mathcal{C}^2$ function in $L^2(I^2)$.*

Let us know finish the treatment by investigating the asymmetry and computing, whenever it is possible, some parameters characterizing the solution of (5). The following definition give an analytic expression of the well known Spearman's coefficient ρ in terms of copula governing the correlation between two variables X and Y

Definition 4. *Let X et Y two continuous random variables for which the dependence is described by the copula C. Spearman's coefficient of ρ C or indifferently of X and Y is given by:*

$$\rho_C = \rho_{X,Y} = 3Q(C,\Pi) = 12 \iint_{\mathbb{I}^2} C(u,v)\, du\, dv - 3$$

Using the explicit formula of the solution of (5), it is easy to prove

Proposition 4. *Spearmann $\rho_t = 12 \iint_{I^2} C(t,u,v) du dv - 3$ of the copula $C(t,.,.)$, is given by*

$$\rho_t = \frac{1}{4} + 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} K_{m,n}\left(C_0, \Pi\right) K_{m,n}\left(\mathbf{1}_{\mathbb{I}^2}, 0\right)$$

so

$$\rho_t = 48\sum_{m,n=1}^{\infty} \frac{1-(-1)^m}{m\pi}\frac{1-(-1)^n}{n\pi} e^{-\pi^2(m^2+n^2)t} K_{m,n}\left(C_0, \Pi\right) - 3$$

The asymmetry measure becomes recently an interesting topic of deep research owing to its relationship with easier explanation of ex-changeability of two variables. We refer, among others, to Siburg [23], [22] and [12]. Here we explain partially how the evolution preserve the symmetry and avoid it at the infinity. We recall that there are many ways to define asymmetry measures and orders (which are not in general total although the existence of Fréchet-Hoftding bounds for all copulas). The most practical measures, as re-used by Siburg in [23], are

Definition 5. *For a bivariate copulas C, one defines the $p-$measure of asymmetry, for all $p \geq 1$ as*

$$\mu_p(C) = \left(\iint_{I^2} |C(u,v) - C(v,u)|^p dudv \right)^{\frac{1}{p}}$$

and μ_∞ as

$$\mu_\infty(C) = \sup_{(u,v)\in I^2} |C(u,v) - C(v,u)|$$

Proposition 5. *According to the explicit formula given by Ishimura μ_∞ of $C(t,.,.)$ (denoted indifferently $C_t(.,.)$) may be estimated easily*

$$|C(t,u,v) - C(t,v,u)| \leq 4\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} |\sin(m\pi u)\sin(n\pi v)| K_{m,n}(C_0, C_0^T)$$

so

$$\mu(C_t) \leq 4\mu(C_0)\sum_{m,n=1}^{\infty} e^{-\pi^2(m^2+n^2)t} \leq 4\mu(C_0)\frac{e^{-2\pi^2 t}}{(1-e^{-\pi^2 t})^2} \tag{7}$$

From the estimation (7), one deduces immediately that $\lim_{t\to\infty}\mu(C_t) = 0$, which means that the behavior at infinity will be symmetric. This result is not surprising, since for smooth start copula data, the solution converges to the independence copula Π which is naturally symmetric.

At last, we give some analytic consequences and perspectives for future deeper analysis :

Topological consequence: The comprehensive copulas M and W are harmonic in the sense of distributions since the sum of their two mono-dimensional derivatives is null. It becomes then impossible to construct a process $C(t,.,.)$ of copulas that solves uniquely (5) and starting with M or W. Indeed, thanks to proposition (3) in (3), since such process should be automatically constant equal to the start datum, this yields to the contradiction $M = \Pi$ or $W = \Pi$. As a consequence, the approximation, of any given copula by a sequence of elements of $\mathcal{C}^2$ in the uniform topology sense fails. Roughly speaking, the set of regular copulas, say that belongs to $\mathcal{C}^2(\mathbb{I}^2)$, is not dense, for uniform convergence norm, in the set of all copulas. In particular, the approach of Ishimura is consistent but does not enable a regular approximation of general copulas.

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Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

The authors contributed to this work in the following ways:

- Elamrani O: Copulas and their simulation.
- El maazouz M: Harmonic analysis of copulas.
- Sani A: Semigroup theory and relationship with copulas.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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