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$\Lambda_{(\tau_1,\tau_2)}$-SETS AND RELATED TOPOLOGICAL SPACES

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Abstract. In this paper, we introduce the notions of $\Lambda_{(\tau_1,\tau_2)}$-sets and $\Lambda^{\star}_{(\tau_1,\tau_2)}$-sets. Furthermore, we investigate two topological spaces $(X, \Lambda_{(\tau_1,\tau_2)})$ and $(X, \Lambda^{\star}_{(\tau_1,\tau_2)})$ by utilizing $\Lambda_{(\tau_1,\tau_2)}$-sets and $\Lambda^{\star}_{(\tau_1,\tau_2)}$-sets, respectively.

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1. Introduction

The notions of closed sets and open sets are fundamental with respect to the investigation of general topology. Maki [16] called a subset A of a topological space (X, τ) a Λ-set if it is the intersection of open sets containing A. Arenas et al. [1] defined a subset A to be λ-closed if $A = L \cap F$, where L is a Λ-set and F is closed in (X, τ). Ganster et al. [13] introduced and studied the notion of pre-Λ-sets in topological spaces. Levine [15] introduced the concept of generalized closed sets. Dunham and Levine [12] investigated the further properties of generalized closed sets. Moreover, Levine defined a separation axiom called $T_{\frac{1}{2}}$ between T_0 and T_1. Dontchev and Ganster [10] introduced and investigated the notions of δ-generalized closed sets and $T_{\frac{3}{4}}$-spaces. As a modification of generalized closed sets, Palaniappan and Rao [18] introduced and studied the notion of regular generalized closed sets. As the further modification of regular generalized closed sets, Noiri and Popa [17] introduced and investigated the concept of regular generalized α-closed sets. Dungthaisong et al. [11] investigated the notion of generalized closed sets in bigeneralized topological spaces and studied some characterizations of pairwise μ-$T_{\frac{1}{2}}$ spaces. Viriyapong and Boonpok [20] introduced and investigated

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the notion of generalized (Λ, p)-closed sets. Furthermore, some properties of generalized (Λ, α)-closed sets, generalized $\delta p(\Lambda, s)$-closed sets, generalized (Λ, s)-closed sets and generalized (Λ, sp)-closed sets were presented in [2], [3], [4] and [5], respectively. Caldas et al. [8] introduced two new classes of sets called Λ_g-closed sets and Λ_g-open sets in topological spaces. It turns out that Λ_g-closed sets and Λ_g-open sets are weaker forms of closed sets and open sets, respectively and stronger forms of generalized closed sets and generalized open sets. Cammaroto and Noiri [9] introduced and studied three topological spaces (X, Λ_m), (X, Λ^*_{mc}) and $(X, \Lambda_{g\Lambda_m})$ by using Λ_m-sets, (Λ, m)-closed sets and generalized Λ_m-sets, respectively. In this paper, we introduce the concepts of $\Lambda_{(\tau_1,\tau_2)}$-closed sets and $\Lambda^\star_{(\tau_1,\tau_2)}$-closed sets. Moreover, some properties of $\Lambda_{(\tau_1,\tau_2)}$-closed sets and $\Lambda^\star_{(\tau_1,\tau_2)}$-closed sets are investigated.

2. Preliminaries

Throughout the present paper, spaces (X, τ_1, τ_2) and (Y, σ_1, σ_2) (or simply X and Y) always mean bitopological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of a bitopological space (X, τ_1, τ_2). The closure of A and the interior of A with respect to τ_i are denoted by τ_i-Cl(A) and τ_i-Int(A), respectively, for $i = 1, 2$. A subset A of a bitopological space (X, τ_1, τ_2) is called $\tau_1\tau_2$-*closed* [7] if $A = \tau_1$-Cl$(\tau_2$-Cl$(A))$. The complement of a $\tau_1\tau_2$-closed set is called $\tau_1\tau_2$-*open*. The intersection of all $\tau_1\tau_2$-closed sets of X containing A is called the $\tau_1\tau_2$-*closure* [7] of A and is denoted by $\tau_1\tau_2$-Cl(A). The union of all $\tau_1\tau_2$-open sets of X contained in A is called the $\tau_1\tau_2$-*interior* [7] of A and is denoted by $\tau_1\tau_2$-Int(A).

Lemma 1. [7] *Let A and B be subsets of a bitopological space (X, τ_1, τ_2). For the $\tau_1\tau_2$-closure, the following properties hold:*

(1) $A \subseteq \tau_1\tau_2$-*Cl*(A) *and* $\tau_1\tau_2$-*Cl*$(\tau_1\tau_2$-*Cl*$(A)) = \tau_1\tau_2$-*Cl*(A).
(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2$-*Cl*$(A) \subseteq \tau_1\tau_2$-*Cl*$(B)$.
(3) $\tau_1\tau_2$-*Cl*(A) *is* $\tau_1\tau_2$-*closed.*
(4) A *is* $\tau_1\tau_2$-*closed if and only if* $A = \tau_1\tau_2$-*Cl*(A).
(5) $\tau_1\tau_2$-*Cl*$(X - A) = X - \tau_1\tau_2$-*Int*$(A)$.

A subset A of a bitopological space (X, τ_1, τ_2) is said to be $(\tau_1, \tau_2)s$-*open* [6] (resp. $(\tau_1, \tau_2)p$-*open* [6], $(\tau_1, \tau_2)\beta$-*open* [6], $\alpha(\tau_1, \tau_2)$-*open* [19]) if $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$ (resp. $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$, $A \subseteq \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A)))$, $A \subseteq \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))))$. A subset A of a bitopological space (X, τ_1, τ_2) is called $(\tau_1, \tau_2)r$-*open* [20] (resp. $(\tau_1, \tau_2)r$-*closed*) if $A = \tau_1\tau_2$-Int$(\tau_1\tau_2$-Cl$(A))$ (resp. $A = \tau_1\tau_2$-Cl$(\tau_1\tau_2$-Int$(A))$).

Let A be a subset of a bitopological space (X, τ_1, τ_2). The set

$$\cap\{G \mid A \subseteq G \text{ and } G \text{ is } \tau_1\tau_2\text{-open}\}$$

is called the $\tau_1\tau_2$-*kernel* [7] of A and is denoted by $\tau_1\tau_2$-$ker(A)$.

Lemma 2. [7] *For subsets A, B of a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $A \subseteq \tau_1\tau_2\text{-}ker(A)$.

(2) *If* $A \subseteq B$, *then* $\tau_1\tau_2\text{-}ker(A) \subseteq \tau_1\tau_2\text{-}ker(B)$.

(3) *If* A *is* $\tau_1\tau_2$*-open, then* $\tau_1\tau_2\text{-}ker(A) = A$.

(4) $x \in \tau_1\tau_2\text{-}ker(A)$ *if and only if* $A \cap H \neq \emptyset$ *for every* $\tau_1\tau_2$*-closed set* H *containing* x.

3. $\Lambda_{(\tau_1,\tau_2)}$-SETS AND RELATED TOPOLOGICAL SPACES

In this paper, we introduce the concepts of $\Lambda_{(\tau_1,\tau_2)}$-closed sets and $\Lambda^{\star}_{(\tau_1,\tau_2)}$-sets. Moreover, we investigate two topological spaces $(X, \Lambda_{(\tau_1,\tau_2)})$ and $(X, \Lambda^{\star}_{(\tau_1,\tau_2)})$ by using $\Lambda_{(\tau_1,\tau_2)}$-sets and $\Lambda^{\star}_{(\tau_1,\tau_2)}$-sets, respectively.

Lemma 3. *Let* (X, τ_1, τ_2) *be a bitopological space and* $\{A_\gamma : \gamma \in \Gamma\}$ *be a family of subsets of* X*. Then, the following properties hold:*

(1) $\tau_1\tau_2\text{-}ker(\cap_{\gamma\in\Gamma}A_\gamma) \subseteq \cap_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma)$.

(2) $\tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma}A_\gamma) = \cup_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma)$.

Proof. (1) Suppose that $x \notin \cap_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma)$. Then, there exists $\gamma_0 \in \Gamma$ such that $x \notin \tau_1\tau_2\text{-}ker(A_{\gamma_0})$ and there exists a $\tau_1\tau_2$-open set U such that $x \notin U$ and $A_{\gamma_0} \subseteq U$. Since $\cap_{\gamma\in\Gamma}A_\gamma \subseteq A_{\gamma_0}$, $x \notin \tau_1\tau_2\text{-}ker(\cap_{\gamma\in\Gamma}A_\gamma)$ and hence $\tau_1\tau_2\text{-}ker(\cap_{\gamma\in\Gamma}A_\gamma) \subseteq \cap_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma)$.

(2) Since $A_\gamma \subseteq \cup_{\gamma\in\Gamma}A_\gamma$, by Lemma 2 (2), we have

$$\tau_1\tau_2\text{-}ker(A_\gamma) \subseteq \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma}A_\gamma)$$

and $\cup_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma) \subseteq \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma}A_\gamma)$. On the other hand, suppose that $x \notin \cup_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma)$ for each $\gamma \in \Gamma$ and hence there exists a $\tau_1\tau_2$-open set U_γ such that $A_\gamma \subseteq U_\gamma$ for each $\gamma \in \Gamma$. Therefore, $\cup_{\gamma\in\Gamma}A_\gamma \subseteq \cup_{\gamma\in\Gamma}U_\gamma$ and $\cup_{\gamma\in\Gamma}U_\gamma$ is a $\tau_1\tau_2$-open set not containing x. Thus, $x \notin \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma}A_\gamma)$ and hence

$$\cup_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(A_\gamma) \supseteq \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma}A_\gamma).$$

□

Definition 1. *A subset* A *of a bitopological space* (X, τ_1, τ_2) *is called a* $\Lambda_{(\tau_1,\tau_2)}$*-set if* $A = \tau_1\tau_2\text{-}ker(A)$.

The family of all $\Lambda_{(\tau_1,\tau_2)}$-sets of a bitopological space (X, τ_1, τ_2) is denoted by $\Lambda_{(\tau_1,\tau_2)}(X)$ (or simply $\Lambda_{(\tau_1,\tau_2)}$).

Lemma 4. *For subsets* A *and* $B_\gamma(\gamma \in \Gamma)$ *of a bitopological space* (X, τ_1, τ_2)*, the following properties hold:*

(1) $\tau_1\tau_2\text{-}ker(A)$ *is a* $\Lambda_{(\tau_1,\tau_2)}$*-set.*

(2) *If* A *is a* $\tau_1\tau_2$*-open set, then* A *is a* $\Lambda_{(\tau_1,\tau_2)}$*-set.*

(3) *If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*

(4) *If B_γ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} B_\gamma$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*

Proof. (1) and (2) are obvious.

(3) Let $B_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$. Then by Lemma 3 (2), we have

$$\cup_{\gamma\in\Gamma} B_\gamma = \cup_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(B_\gamma) = \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma} B_\gamma) \supseteq \cup_{\gamma\in\Gamma} B_\gamma.$$

Thus, $\cup_{\gamma\in\Gamma} B_\gamma = \tau_1\tau_2\text{-}ker(\cup_{\gamma\in\Gamma} B_\gamma)$ and hence $\cup_{\gamma\in\Gamma} B_\gamma \in \Lambda_{(\tau_1,\tau_2)}$.

(4) Let $B_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$. Then by Lemma 3 (1), we have

$$\cap_{\gamma\in\Gamma} B_\gamma = \cap_{\gamma\in\Gamma}\tau_1\tau_2\text{-}ker(B_\gamma) \supseteq \tau_1\tau_2\text{-}ker(\cap_{\gamma\in\Gamma} B_\gamma) \supseteq \cap_{\gamma\in\Gamma} B_\gamma.$$

Thus, $\cap_{\gamma\in\Gamma} B_\gamma = \tau_1\tau_2\text{-}ker(\cap_{\gamma\in\Gamma} B_\gamma)$ and so $\cap_{\gamma\in\Gamma} B_\gamma \in \Lambda_{(\tau_1,\tau_2)}$. □

Theorem 1. *For a bitopological space (X, τ_1, τ_2), the pair $(X, \Lambda_{(\tau_1,\tau_2)})$ is an Alexandroff space.*

Proof. (1) $\emptyset, X \in \Lambda_{(\tau_1,\tau_2)}$ since $\emptyset$ and X are $\tau_1\tau_2$-open sets.

(2) If $U_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} U_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ by Lemma 4 (3).

(3) If $U_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$, then $\cap_{\gamma\in\Gamma} U_\gamma \in \Lambda_{(\tau_1,\tau_2)}$ by Lemma 4 (4). □

Definition 2. [14] *A bitopological space (X, τ_1, τ_2) is said to be (τ_1, τ_2)-R_0 if for each $\tau_1\tau_2$-open set U and each $x \in U$, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$.*

Theorem 2. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-R_0 if and only if the topological space $(X, \Lambda_{(\tau_1,\tau_2)})$ is R_0.*

Proof. Let $V \in \Lambda_{(\tau_1,\tau_2)}$ and $x \in V$. Then, we have

$$x \in \tau_1\tau_2\text{-}ker(V) = \cap\{U \mid V \subseteq U, U \text{ is } \tau_1\tau_2\text{-open}\}$$

and $x \in U$ for every $\tau_1\tau_2$-open set U containing V. Since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ for every $\tau_1\tau_2$-open set U containing V. Thus,

$$\tau_1\tau_2\text{-Cl}(\{x\}) \subseteq \cap\{U \mid V \subseteq U, U \text{ is } \tau_1\tau_2\text{-open}\} = \tau_1\tau_2\text{-}ker(V) = V.$$

Since every $\tau_1\tau_2$-open set is a $\Lambda_{(\tau_1,\tau_2)}$-set, we have

$$\Lambda_{(\tau_1,\tau_2)}\text{-Cl}(\{x\}) \subseteq \tau_1\tau_2\text{-Cl}(\{x\}) \subseteq V.$$

This shows that $(X, \Lambda_{(\tau_1,\tau_2)})$ is R_0.

Conversely, suppose that $(X, \Lambda_{(\tau_1,\tau_2)})$ is R_0. Let V be a $\tau_1\tau_2$-open set and $x \in V$. Since every $\tau_1\tau_2$-open set is a $\Lambda_{(\tau_1,\tau_2)}$-set, we have $\Lambda_{(\tau_1,\tau_2)}$-Cl$(\{x\}) \subseteq V$. Since $X - \Lambda_{(\tau_1,\tau_2)}$-Cl$(\{x\}) \in \Lambda_{(\tau_1,\tau_2)}$,

$$X - \Lambda_{(\tau_1,\tau_2)}\text{-Cl}(\{x\}) = \cap\{U \mid X - \Lambda_{(\tau_1,\tau_2)}\text{-Cl}(\{x\}) \subseteq U, U \text{ is } \tau_1\tau_2\text{-open}\}.$$

Then, there exists a $\tau_1\tau_2$-open set U such that $X - \Lambda_{(\tau_1,\tau_2)}$-Cl$(\{x\}) \subseteq U$ and $x \notin U$. Thus, $x \in X - U \subseteq \Lambda_{(\tau_1,\tau_2)}$-Cl$(\{x\}) \subseteq V$. Since $X - U$ is $\tau_1\tau_2$-closed, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq X - U \subseteq V$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. □

Definition 3. *A bitopological space (X, τ_1, τ_2) is said to be:*

(1) (τ_1, τ_2)-T_0 *if for any pair of distinct points in X, there exists a $\tau_1\tau_2$-open set containing one of the points but not the other.*

(2) (τ_1, τ_2)-T_1 *if for any pair of distinct points x, y in X, there exist $\tau_1\tau_2$-open sets U and V such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.*

Theorem 3. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_0 if and only if the topological space $(X, \Lambda_{(\tau_1,\tau_2)})$ is T_0.*

Proof. This is obvious since every $\tau_1\tau_2$-open set is a $\Lambda_{(\tau_1,\tau_2)}$-set.

Conversely, let x and y be any pair of distinct points of X. Since (X, τ_1, τ_2) is (τ_1, τ_2)-T_0, there exists $V \in \Lambda_{(\tau_1,\tau_2)}$ such that either $x \in V$ and $y \notin V$ or $x \notin V$ and $y \in V$. In case $x \in V$ and $y \notin V$, there exists a $\tau_1\tau_2$-open set U such that $V \subseteq U$ and $y \notin U$. However, since $x \in V$, $x \in U$. In case $x \notin V$ and $y \in V$, similarly there exists a $\tau_1\tau_2$-open set U such that $x \notin U$ and $y \in U$. Thus, (X, τ_1, τ_2) is (τ_1, τ_2)-T_0. □

Lemma 5. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1*;*

(2) *for each $x \in X$, the singleton $\{x\}$ is $\tau_1\tau_2$-closed in X;*

(3) *for each $x \in X$, the singleton $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.*

Proof. (1) ⇒ (2): Let y be any point of X and $x \in X - \{y\}$. There exists a $\tau_1\tau_2$-open set V_x such that $x \in V_x$ and $y \notin V_x$. Thus, $X - \{y\} = \cup_{x \in X - \{y\}} V_x$. Therefore, the singleton $\{y\}$ is $\tau_1\tau_2$-closed in X.

(2) ⇒ (3): Let x be any point of X and $y \in X - \{x\}$. Then, $x \in X - \{y\}$ and $X - \{y\}$ is $\tau_1\tau_2$-open. By Lemma 2, $\tau_1\tau_2$-$ker(\{x\}) \subseteq X - \{y\}$. Therefore, $y \notin \tau_1\tau_2$-$ker(\{x\})$ and $\tau_1\tau_2$-$ker(\{x\}) \subseteq \{x\}$. This shows that $\tau_1\tau_2$-$ker(\{x\}) = \{x\}$. Thus, the singleton $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set.

(3) ⇒ (1): Suppose that the singleton $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set for each $x \in X$. Let x and y be any distinct points of X. Then, $y \notin \tau_1\tau_2$-$ker(\{x\})$ and there exists a $\tau_1\tau_2$-open set U_x such that $x \in U_x$ and $y \notin U_x$. Similarly, $x \notin \tau_1\tau_2$-$ker(\{y\})$ and there exists a $\tau_1\tau_2$-open set U_y such that $y \in U_y$ and $x \notin U_y$. This shows that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1. □

Theorem 4. *A bitopological space (X, τ_1, τ_2) is (τ_1, τ_2)-T_1 if and only if the topological space $(X, \Lambda_{(\tau_1,\tau_2)})$ is discrete.*

Proof. Suppose that (X, τ_1, τ_2) is (τ_1, τ_2)-T_1. Let $x \in X$. Then by Lemma 5, $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set and $\{x\}$ is open in $(X, \Lambda_{(\tau_1,\tau_2)})$. Thus, every subset of X is open in $(X, \Lambda_{(\tau_1,\tau_2)})$ and hence $(X, \Lambda_{(\tau_1,\tau_2)})$ is discrete.

Conversely, suppose that a topological space $(X, \Lambda_{(\tau_1,\tau_2)})$ is discrete. For any point $x \in X$, $\{x\}$ is open in $(X, \Lambda_{(\tau_1,\tau_2)})$ and hence $\{x\}$ is a $\Lambda_{(\tau_1,\tau_2)}$-set. By Lemma 5, (X, τ_1, τ_2) is (τ_1, τ_2)-T_1. □

Corollary 1. *For a bitopological space (X, τ_1, τ_2), the following properties are equivalent:*

(1) (X, τ_1, τ_2) *is* (τ_1, τ_2)-T_1*;*

(2) (X, τ_1, τ_2) *is* (τ_1, τ_2)-R_0 *and* (τ_1, τ_2)-T_0*;*

(3) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is* R_0 *and* T_0*;*

(4) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is* T_1*;*

(5) $(X, \Lambda_{(\tau_1,\tau_2)})$ *is discrete.*

Proof. (1) ⇒ (2): By Lemma 5, every (τ_1, τ_2)-T_1 space is (τ_1, τ_2)-R_0 and (τ_1, τ_2)-T_0.

(2) ⇒ (1): Since (X, τ_1, τ_2) is (τ_1, τ_2)-T_0, for any distinct points x, y of X, there exists a $\tau_1\tau_2$-open set U such that, say, $x \in U$ and $y \notin U$. Thus, $\tau_1\tau_2$-Cl$(\{x\}) \subseteq U$ since (X, τ_1, τ_2) is (τ_1, τ_2)-R_0. Then, $X - \tau_1\tau_2$-Cl$(\{x\})$ is a $\tau_1\tau_2$-open set such that $x \notin X - \tau_1\tau_2$-Cl$(\{x\})$ and $y \in X - U \subseteq X - \tau_1\tau_2$-Cl$(\{x\})$. Thus, (X, τ_1, τ_2) is (τ_1, τ_2)-T_1.

(2) ⇔ (3): This is an immediate consequence of Theorem 2 and 3. □

(3) ⇔ (4): This is obvious.

(5) ⇔ (1): This is an immediate consequence of Theorem 4.

Definition 4. *A subset A of a bitopological space (X, τ_1, τ_2) is said to be $\Lambda^\star_{(\tau_1,\tau_2)}$-set if $\tau_1\tau_2$-$ker(A) \subseteq F$ whenever $A \subseteq F$ and F is $\tau_1\tau_2$-closed.*

The family of all $\Lambda^\star_{(\tau_1,\tau_2)}$-sets of a bitopological space (X, τ_1, τ_2) is denoted by $\Lambda^\star_{(\tau_1,\tau_2)}(X)$ (or simply $\Lambda^\star_{(\tau_1,\tau_2)}$).

Lemma 6. *For a bitopological space (X, τ_1, τ_2), the following properties hold:*

(1) $\Lambda_{(\tau_1,\tau_2)} \subseteq \Lambda^\star_{(\tau_1,\tau_2)}$.

(2) *If $A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$.*

Proof. (1) Let $A \in \Lambda_{(\tau_1,\tau_2)}$. If $A \subseteq F$ and F is $\tau_1\tau_2$-closed, then $\tau_1\tau_2$-$ker(A) = A \subseteq F$. Thus, $A \in \Lambda^\star_{(\tau_1,\tau_2)}$ and hence $\Lambda_{(\tau_1,\tau_2)} \subseteq \Lambda^\star_{(\tau_1,\tau_2)}$.

(2) Let $\cup_{\gamma\in\Gamma} A_\gamma \subseteq F$ and F be any $\tau_1\tau_2$-closed set. Then, $A_\gamma \subseteq F$ and $\tau_1\tau_2$-$ker(A_\gamma) \subseteq F$ for each $\gamma \in \Gamma$ since $A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$. By Lemma 3, we have $\tau_1\tau_2$-$ker(\cup_{\gamma\in\Gamma} A_\gamma) = \cup_{\gamma\in\Gamma} A_\gamma \tau_1\tau_2$-$ker(A_\gamma) \subseteq F$ and hence $\cup_{\gamma\in\Gamma} A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$. □

Theorem 5. *For a bitopological space (X, τ_1, τ_2), the pair $(X, \Lambda^\star_{(\tau_1,\tau_2)})$ is an Alexandroff space.*

Proof. By Lemma 6, $\Lambda_{(\tau_1,\tau_2)} \subseteq \Lambda^\star_{(\tau_1,\tau_2)}$. Since $\emptyset, X \in \Lambda_{(\tau_1,\tau_2)}$, we have $\emptyset, X \in \Lambda^\star_{(\tau_1,\tau_2)}$. By Lemma 6, if $A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$ for each $\gamma \in \Gamma$, then $\cup_{\gamma\in\Gamma} A_\gamma \in \Lambda^\star_{(\tau_1,\tau_2)}$. Thus, by Theorem 1 $(X, \Lambda^\star_{(\tau_1,\tau_2)})$ is an Alexandroff space, where $A \in \Lambda^{\star}_{(\tau_1,\tau_2)}$ iff $A = \tau_1\tau_2\text{-}ker^{\star}(A)$ and

$$\tau_1\tau_2\text{-}ker^{\star}(A) = \cap\{U \mid A \subseteq U, U \in \Lambda^{\star}_{(\tau_1,\tau_2)}\}.$$

□

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

References

[1] F.G. Arenas, J. Dontchev, M. Ganster, On λ-sets and dual of generalized continuity, Quest. Answers Gen. Topol. 15 (1997), 3–13.

[2] C. Boonpok, M. Thongmoon, Some properties of generalized (Λ, α)-closed sets, Int. J. Anal. Appl. 21 (2023), 88. https://doi.org/10.28924/2291-8639-21-2023-88.

[3] C. Boonpok, N. Srisarakham, Properties of generalized $\delta p(\Lambda, s)$-closed sets, Eur. J. Pure Appl. Math. 16 (2023), 2581–2596. https://doi.org/10.29020/nybg.ejpam.v16i4.4736.

[4] C. Boonpok, C. Viriyapong, On some forms of closed sets and related topics, Eur. J. Pure Appl. Math. 16 (2023), 336–362. https://doi.org/10.29020/nybg.ejpam.v16i1.4582.

[5] C. Boonpok, C. Viriyapong, On generalized (Λ, sp)-closed sets, Eur. J. Pure Appl. Math. 15 (2022), 2127–2140. https://doi.org/10.29020/nybg.ejpam.v15i4.4302.

[6] C. Boonpok, $(\tau_1, \tau_2)\delta$-semicontinuous multifunctions, Heliyon, 6 (2020), e05367. https://doi.org/10.1016/j.heliyon.2020.e05367.

[7] C. Boonpok, C. Viriyapong, M. Thongmoon, On upper and lower (τ_1, τ_2)-precontinuous multifunctions, J. Math. Comp. Sci. 18 (2018), 282–293. https://doi.org/10.22436/jmcs.018.03.04.

[8] M. Caldas, S. Jafari, T. Noiri, On Λ-generalized closed sets in topological spaces, Acta Math. Hung. 118 (2008), 337–343. https://doi.org/10.1007/s10474-007-6224-1.

[9] F. Cammaroto, T. Noiri, On Λ_m-sets and related topological spaces, Acta Math. Hung. 109 (2005), 261–279. https://doi.org/10.1007/s10474-005-0245-4.

[10] J. Dontchev and M. Ganster, On δ-generalized closed sets and $T_{\frac{3}{4}}$-spaces, Mem. Fac. Sci. Kochi Univ. Ser. A Math. 17 (1996), 15–31.

[11] W. Dungthaisong, C. Boonpok, C. Viriyapong, Generalized closed sets in bigeneralized topological spaces, Int. J. Math. Anal. 5 (2011), 1175–1184.

[12] W. Dunham, N. Levine, Further results on generalized closed sets in topology, Kyungpook Math. J. 20 (1980), 169–175.

[13] M. Ganster, S. Jafari, T. Noiri, On pre-Λ-sets and pre-V-sets, Acta Math. Hung. 95 (2002), 337–343. https://doi.org/10.1023/a:1015605426358.

[14] B. Kong-ied, S. Sompong, C. Boonpok, On (τ_1, τ_2)-R_0 bitopological spaces, Asia Pac. J. Math. 11 (2024), 43. https://doi.org/10.28924/APJM/11-43.

[15] N. Levine, Generalized closed sets in topology, Rend. Circ. Mat. Palermo (2), 19 (1970), 89–96.

[16] H. Maki, Generalized Λ-sets and the associated closure operator, Special Issue in Commemoration of Prof. Kazusada IKEDA's Retirement, (1986), pp. 139–146. https://cir.nii.ac.jp/crid/1572543024674341120.

[17] T. Noiri, V. Popa, A note on modifications of rg-closed sets in topological spaces, Cubo. 15 (2013), 65–70. https://doi.org/10.4067/s0719-06462013000200006.

[18] N. Palaniappan, K.C. Rao, Regular generalized closed sets, Kyungpook Math. J. 33 (1993), 211–219.

[19] N. Viriyapong, S. Sompong, C. Boonpok, (τ_1, τ_2)-extremal disconnectedness in bitopological spaces, Int. J. Math. Comp. Sci. 19 (2024), 855–860.

[20] C. Viriyapong, C. Boonpok, $(\tau_1, \tau_2)\alpha$-continuity for multifunctions, J. Math. 2020 (2020), 6285763. https://doi.org/10.1155/2020/6285763.