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OPTIMAL INVESTMENT AND REINSURANCE POLICIES FOR LOSS-AVERSE INSURERS

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Abstract. This paper studies an optimal investment and reinsurance problem for loss-averse insurers. Specially, the insurers are allowed to purchase proportional reinsurance, acquire new business and invest in a financial market where the surplus of the insurers is approximated by a drifted Brownian motion and the financial market consists of one risk-free asset and one risky asset whose price process is modeled by a jump-diffusion process. The insurers need to manage risks from financial markets and insurance operations. The insurers' utility preferences are assumed to be loss-averse. Since this problem is not standard concave optimization problem, martingale method is applied to derive the explicit expressions of the optimal policies and the optimal wealth process. Moreover, Numerical examples are presented to show the economic behaviors of the optimal policies.

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1. Introduction

Optimal investment strategy for insurers has recently become an important subject. The insurers can participate in the financial market to avoid risk. Recently, many literatures have studied maximizing the utility of terminal value or minimizing the probability of ruin for the insurers. Browne [1] initiated the study of explicit solution for a firm to maximize the exponential utility of terminal wealth and minimize the probability of ruin with its surplus process given by the Lundberg risk model. For different claim sizes of insurers, the optimal strategy was given by the Bellman equation in Hipp and Plum [2] to minimize the ruin probability. Wang et al. [3] efficiently applied martingale methods to study the optimal portfolio selection for insurers under the mean-variance criterion as well as the expected constant absolute risk aversion (CARA) utility maximization. For more recent related papers see, for example, Yang and Zhang [4], Wang [5], Xu et al. [6], and Liu et al. [7].

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In addition to the risk of financial market, the insurers have to take into account the risk of insurance operations. The risk of insurance cannot be avoided by singly investing in the bonds and risky assets in the market. The business of reinsurance provides a way for the insurers to hedge this risk, and has also recently drawn much concern. The business of reinsurance comes up in different forms. Quota-share reinsurance and investment were originally investigated by Promislow and Young [8]. Proportional reinsurance was accessible in B$\ddot{a}$uerle [9] in which the author minimized the expected quadratic distance of the terminal value over a positive constant and successfully solved the related mean-variance problem. Under the constraint of no-shorting, Bai and Guo [10] studied the problem of optimal investment and reinsurance for an insurer under maximizing the expected exponential utility of terminal wealth as well as minimizing the probability of ruin. Zhang et al. [11] took into account the effect of transaction costs and obtained the explicit solution to maximize the expected utility of terminal wealth for an insurer. Recently, Zeng and Li [12] studied the optimal time-consistent investment and reinsurance problem for insurers under mean-variance criterion. Guan and Liang [13]considered the risk of interest rate and inflation for an insurer and obtained the explicit solution to maximize the constant relative risk aversion (CRRA) utility of terminal wealth.

Most works on the optimization problems for insurers care about maximizing the expectation of a smooth utility of terminal wealth. See for example, Browne [1], Wang et al. [3], Bai and Guo [10], Liang et al. [14], and Guan and Liang [13]. The decision makers are often assumed to be strictly risk averse. The optimal investment and reinsurance strategies to hedge the risk in the market often consist of a substantial allocation in risky assets and a large proportion of insurance business. However, some individuals are unwilling to take the risk from risky assets and insurance business. Besides, some individuals may be risk-seeking, invest more in risky asset and keep more insurance business. Therefore, the optimal terminal wealth in many literatures may led to huge risk for an insurer.

Since the existing works on optimization problems for insurers mainly care about the complexity in the market. We introduce here one different optimization criterion that is different from the smooth utility case. This criterion belongs to prospect theory. The breakthrough in Kahneman and Tversky [15] has been a cornerstone of the prospect theory, in which Kahneman and Tversky proposed reference point and distortion of probabilities in portfolio theory. These ideas have been proven to be of great use and can result in lowing risk for an investor. Because the prospect theory describes human behavior better, more and more literatures study the loss aversion utility and distortion of probability in portfolio selection. Berkelaar et al. [16] firstly employed the martingale method to derive the optimal investment strategies with two utility functions under loss aversion in a continuous case. Later, Gomes [17] considered the counterpart discrete model. Furthermore, Jarrow and Zhao [18] introduced a mean-variance framework under loss aversion. The above works only concerned the loss aversion in prospect theory. The distortion of probability in portfolio selection can refer to Bernard and Ghossoub [19], Jin and Zhou [20], He and and Zhou [21]and references therein.

However, there are relatively few studies in the literature on optimal investment and reinsurance problems for insurance companies under loss aversion. Guo [22] first investigated the optimal portfolio choice for an insurer under loss aversion, where a specific two-piece utility function is considered. Based on Guo [22], Chen and Yang [23] studied optimal reinsurance and investment strategies for an insurer in a stochastic market by considering the insurer's preference is represented by a two-piece utility function. Recently, Ma et al. [24] investigated optimal reinsurance and investment strategies with the assumption that the insurers can purchase proportional reinsurance contracts and invest their wealth in a financial market under an S-shaped utility. However, the financial market are both modeled as continuous time cases in Chen and Yang [23] and Ma et al. [24]. In this paper, we intend to investigate the optimal investment and reinsurance strategies for an insurer under loss aversion. Specifically, the surplus process of the insurer is assumed to follow a drifted Brownian motion. The financial market consists of one risk-free asset and one risky asset whose price process satisfies a jump-diffusion model. So we need to manage the risks of the risky asset and the insurance business. The goal is to maximize the expected utility of terminal wealth. The utility function under loss aversion we adopt is firstly studied in Kahneman and Tversky [15]. The utility function is convex under a reference point while concave above the point. This leads to a risk-seeking attitude towards losses. Since the optimization problem is not a concave maximization problem, the optimal terminal wealth is a discontinuous function and it seems that the stochastic programming method does not work here. We will apply the martingale method to derive the optimal investment and reinsurance strategies under loss aversion. Moreover, numerical examples in the end show that the loss-averse insurer may invest more or less in the risky asset, purchase more or less reinsurance, and acquire more or less new business based on the economic parameters. Specifically, when the reference level is high, the insurer judges the account by gains and acts as a risk averse investor. So, the loss-averse insurer becomes more concerned about volatilities that may cause the account of wealth to underperform the reference level, and thus, the lower wealth allocated in the risky asset and the less insurance business kept.

The organization of this paper is as follows. In section 2, the assumptions and model are described. Section 3 formulates the optimization problem we are going to consider under loss aversion. Section 4 solves the optimization problem and derives explicitly the corresponding optimal investment and proportional reinsurance strategies and the optimal wealth process by a martingale approach. Section 5 presents numerical examples to show the impact of the economic parameters on the optimal strategies. Finally, Section 6 concludes this paper.

2. Assumptions and Model

Let $(\Omega, \mathcal{F}, P)$ be a given complete probability space with a filtration $(\mathcal{F}_t)$, $t \in [0, T]$ satisfying the usual conditions, i.e. the filtration contains all P-null sets and is right continuous, where $T \in (0, +\infty)$ is a finite constant and represents the time horizon; $(\mathcal{F}_t)$ stands for the information available up to time t and any decision made at time t is based on this information. All stochastic processes in this paper are assumed to be well defined and adapted processes in this probability space.

2.1. **Financial market.**

We assume that the insurer can invest in a capital market where two assets are traded continuously on a finite horizon $[0, T]$. One is risk-free asset with price $P_0(t)$ given by

$$dP_0(t) = P_0(t)r(t)dt, \;\; P_0(0) = 1, \tag{1}$$

and one is risky asset with price $P_1(t)$ satisfying

$$dP_1(t) = P_1(t)\left[\mu(t)dt + \sigma_1(t)dW_1(t) + \gamma(t)dN(t)\right]\;, \;\; P_1(0) > 0, \tag{2}$$

where $r(t)$ is the risk-free interest rate; $\mu(t)$ is the appreciation rate; $\sigma_1(t) > 0$ and $\gamma(t) > 0$ are the volatilities; $W_1(t)$ is a 1-dimensional standard Brownian motion; $N(t)$ is a poisson process with intensity λ_1 on the filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$ which is assumed to be independent of $W_1(t)$. Hence, $M(t) = N(t) - \lambda_1 t$ is the compensated Poisson process defined on the filtered complete probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$. In general, we assume that $\mu(t) > r(t) \geq 0$.

2.2. **Surplus process.**

We consider an insurer whose surplus process is modeled by a diffusion approximation model. To understand the diffusion approximation model better, it is advantageous to start from the classical Cramér-Lundberg model. In the Cramér-Lundberg model, the claims arrive according to a homogeneous Poisson process $\{K(t)\}$ with intensity λ_2; Y_i is the size of the ith claim and $Y_i, i = 1, 2, 3...$ are assumed to be independent and identically distributed (i.i.d.) positive random variables with finite first-order moment $\mu_0 = EY$ and second-order moment $\sigma_0^2 = E(Y^2)$ and are assumed to be independent of $\{K(t)\}$. Then the surplus process of the insurer without reinsurance and investment follows

$$U(t) = x_0 + c_0 t - R(t) = x + c_0 t - \sum\nolimits_{i=1}^{K(t)} Y_i, \tag{3}$$

where $x_0 > 0$ is the initial reserve of an insurance company; c_0 is the premium rate which is assumed to be calculated according to the expected value principle, i.e., $c_0 = (1 + \theta)\lambda_2\mu_0$, where θ is the safety loading of the insurer. $R(t) = \sum_{i=1}^{K(t)} Y_i$ is a compound Poisson process defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t), P)$, which represents the cumulative amount of claims in time interval $[0, t]$.

By Grandll [25], the Cramér-Lundberg model can be approximated by the following diffusion model

$$dU(t) = \theta\lambda_2\mu_0 dt + \sigma_2 dW_2(t), \tag{4}$$

where $\theta\lambda_2\mu_0$ can be regarded as the premium return rate of the insurer; ${\sigma_2}^2 = \lambda_2{\sigma_0}^2$ measures the volatility of the insurer's surplus; $W_2(t)$ is a standard Brownian motion, which is independent of $W_1(t)$. It is worth pointing out that the diffusion approximation model (4) works well for the large insurance portfolios where an individual claim is relatively small compared to the size of surplus. The diffusion approximation model has been used in much existing literature, for example, Browne [1], Zeng and Li [12], Guan and Liang [13], and so on. It is assumed that $W_1(t)$, $W_2(t)$, and $N(t)$ are mutually independent.

In addition, the insurer is allowed to purchase proportional reinsurance and acquire new business (for example, acting as a reinsurer of other insurers, see B$\ddot{a}$uerle [9]) at each moment in order to control his or her insurance business risk. The proportional reinsurance or new business level is associated with the value of risk exposure $q(t) \in [0, +\infty)$ at any time $t \in [0, T]$. $q(t) \in [0, 1]$ corresponds to a proportional reinsurance cover; in this case the cedent should divert part of the premium to the reinsurer at the rate of $(1 - q(t))(1 + \eta)\lambda_2\mu_0$, where η is the safety loading of the reinsurer satisfying $\eta \geq \theta > 0$. In return, for each claim occurring at time t, the reinsurer pays $100(1 - q(t))\%$ of the claim, and the cedent pays the rest. $q(t) \in (1, +\infty)$ corresponds to acquiring new business (acting as a reinsurer for other insurers). When a reinsurance policy $\{q(t) : t \in [0, T]\}$ is adopted, the corresponding diffusion approximation dynamics for the surplus process becomes

$$dU(t) = [\theta\lambda\mu_0 - (1 - q(t))(1 + \eta)\lambda\mu_0]\, dt + \sigma_2 q(t) dW_2(t). \tag{5}$$

2.3. **Wealth process.**

Assume that the insurer can dynamically purchase proportional reinsurance, acquire new business and invest in the financial market over the time interval $[0, T]$ and there is no transaction cost in the financial market and the insurance market. A trading policy is denoted by a pair of stochastic processes $h = \{\pi(t), q(t)\}_{t\in[0,T]}$, where $q(t)$ and $\pi(t)$ are the value of the risk exposure and the dollar amount invested in the risky asset at time t, respectively. The dollar amount invested in the risk-free asset at time t is $X(t) - \pi(t)$, where $X(t)$ is the wealth process associated with strategy h. Then the evolution of $X(t)$ can be described as

$$\begin{cases} dX(t) = [X(t)r(t) + \pi(t)(\mu(t) - r(t)) + \theta\lambda_2\mu_0 - (1 - q(t))(1 + \eta)\lambda_2\mu_0]\, dt \\ \qquad\qquad + \sigma_1(t)\pi(t)dW_1(t) + \pi(t)\gamma(t)dN(t) + \sigma_2 q(t)dW_2(t) \\ X(0) = x_0. \end{cases} \tag{6}$$

Definition 2.1. (Admissible strategy) Let $\vartheta := [0, T] \times R$. For any fixed $t \in [0, T]$, a trading policy $h = \{\pi(t), q(t)\}_{t\in[0,T]}$ is said to be admissible if it satisfies that

(1) $\pi(t)$ and $q(t)$ are predictable mappings with respect to $\mathcal{F}_t$;

(2) for all $t \in [0,T]$, $q(t) \geq 0$ and $E\left[\int_t^T (\pi(s)^2 + q(s)^2)ds\right] < +\infty$;

(3) $(X(t), h)$ is the unique solution to the stochastic differential equation (6).

In addition, let $\prod(t,x)$ denote the set of all admissible strategies with respect to initial condition $(t,x) \in \vartheta$.

3. Formulation of the Problem

Most works on the optimization problems for insurers care about maximizing the expectation of a smooth utility of terminal wealth, in order to find the optimal strategies within $[0,T]$. However, in the real world, some individuals are unwilling to take the risk from the risky asset and the insurance business. They may be more interested in allocating funds to risk-free asset and keep less insurance business. Additionally, others may seek risk and invest more in risky assets and retain more insurance business. General optimization problems only characterize risk-averse investors and do not reflect the behavior of loss-averse people. In this section, we formulate one different optimization problem, which better manage the risks for the insurer.

This section formulates the optimization problem under loss aversion. Kahneman and Tversky [15] firstly established the theory of prospect theory. They stated that people always make decisions relative to some reference levels. The reference levels may be different for different people. The account of the wealth over (under) the reference is judged as gains (losses). People often act differently towards gains and losses. In fact, people are more sensitive to losses than gains. They also demonstrated their idea based on the following utility function:

$$U(X(T)) = \begin{cases} A(X(T)-\xi)^{\gamma_1}, & X(T) > \xi; \\ -B(\xi - X(T))^{\gamma_2}, & X(T) \leq \xi. \end{cases} \tag{7}$$

where A and B are positive constants, $0 < \gamma_1 \leq 1$, $0 < \gamma_2 \leq 1$. Statistics are showed in Kahneman and Tversky [15] to support the above utility function. The investor is risk-averse towards gains while risk-seeking towards losses. The reference point ξ is chosen in advance. For the insurer, the reference point ξ can be chosen to be connected with the premium rate and initial wealth. Figure 1 illustrates the properties of the loss aversion function (7). The utility function is convex when the wealth is less than ξ and concave when the wealth is bigger than ξ.

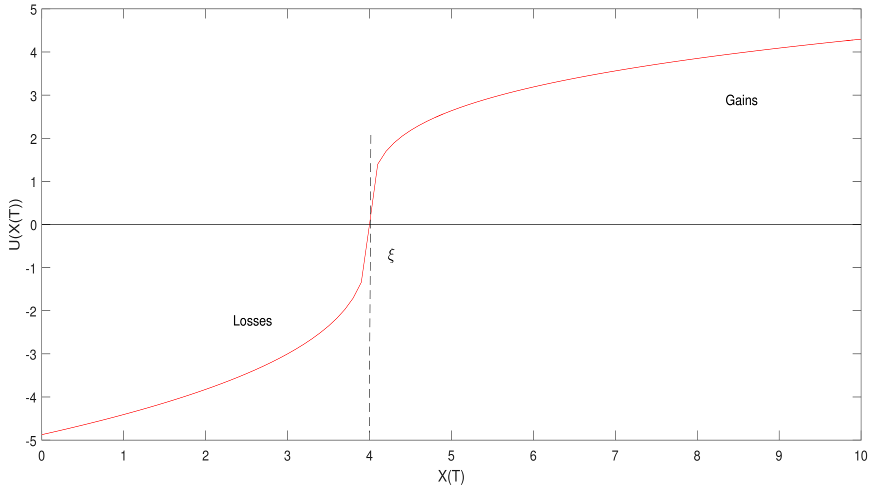

FIGURE 1. Representation of utility function (7) with parameters $A = 4$, $B = 3$, $\xi = 4$, $\gamma_1 = 0.3$, $\gamma_2 = 0.35$.

Following the utility maximization criterion, the problem of optimal investment and reinsurance strategies for an insurer can be formulated as follows:

$$\begin{cases} \max\limits_{h\in\Pi} \ E\{U[X(T)]\} \\ s.t.\, X(t) \text{ satisfies } (6) \\ X(t) \geq 0.\ \forall t \in [0,T]. \end{cases} \tag{8}$$

where $X(t) \geq 0$, $\forall t \in [0,T]$ reflects that the insurance company is not bankrupt throughout the investment period $[0,T]$.

4. Solution to the Optimization Problem

In this section, we use martingale method to solve problem (8). The previous section allows us to change dynamic maximization problem (8) with the mean constraint into a static problem. We are thus led to a constrained optimization problem which is solved by standard Lagrange multipliers methods.

Define

$$\begin{aligned} H(t) = \exp\Big\{ & -\int_0^t r(s)ds + \int_0^t \theta_1(s)dW_1(s) + \int_0^t \theta_2(s)dW_2(s) + \int_0^t \ln[1+\theta_3(s)]dN(s) \\ & -\frac{1}{2}\int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 + 2\lambda_1\theta_3(s)\right] ds \Big\}, \end{aligned} \tag{9}$$

where $\mu(t) - r(t) + \theta_1(t)\sigma_1(t) + [\theta_3(t)+1]\gamma(t)\lambda_1 = 0$ and $\theta_2(t) = -\frac{(1+\eta)\lambda_2\mu_0}{\sigma_2}$. Then, We have the following conclusion.

Proposition 4.1. *If $H(t)$ is defined by (9) for $t \in [0,T]$, then $H(t)X(t) + \int_0^t H(s)c_2 ds$ is a martingale under the probability measure P, where $c_2 = (1+\eta-\theta)\lambda_2\mu_0$.*

Proof. Differentiate $H(t)$, we have

$$dH(t) = H(t)\left[-r(t)dt + \theta_1(t)dW_1(t) + \theta_2(t)dW_2(t) + \theta_3(t)dM(t)\right]. \quad (10)$$

Applying $It\hat{o}'s$ formula, we obtain

$$\begin{aligned} d\left[H(t)X(t)\right] &= H(t)dX(t) + X(t)dH(t) + d\left[H(t), X(t)\right] \\ &= -H(t)c_2dt + H(t)\left[X(t)\theta_1(t) + \sigma_1(t)\pi(t)\right]dW_1(t) \\ &\quad + H(t)\left[X(t)\theta_2(t) + \sigma_2 q(t)\right]dW_2(t) \\ &\quad + H(t)\left[X(t)\theta_3(t) + \pi(t)\gamma(t)(\theta_3(t)+1)\right]dM(t), \end{aligned} \quad (11)$$

where $[H(t), X(t)]$ denotes the quadratic co-variation of $H(t)$ and $X(t)$. After integrating, we obtain

$$\begin{aligned} H(t)X(t) + \int_0^t H(s)c_2ds = x_0 &+ \int_0^t H(s)\left[X(s)\theta_1(s) + \sigma_1(s)\pi(s)\right]dW_1(s) \\ &+ \int_0^t H(s)\left[X(s)\theta_2(s) + \sigma_2 q(s)\right]dW_2(s). \\ &+ \int_0^t H(s)\left[X(s)\theta_3(s) + \pi(s)\gamma(s)(1+\theta_3(s))\right]dM(s). \end{aligned} \quad (12)$$

This shows that $H(t)X(t) + \int_0^t H(s)c_2ds$ can be represented as an $It\hat{o}$ integral with respect to the Brownian motions $W_1(t)$, $W_2(t)$ and the compensated Poisson process $M(t)$. Therefore $H(t)X(t) + \int_0^t H(s)c_2ds$ is a Martingale under P. □

Of course, a martingale must be super-martingale under P. The super-martingale property applied to (12) implies the following constraint:

$$E\left[H(T)X(T) + \int_0^T H(s)c_2ds\right] \leq x_0. \quad (13)$$

As in Karatzas et al. [26], we now show that the constraint (13) plays an important role in the optimization problem.

Theorem 4.1. *Let $\psi \geq 0$ be an $\mathcal{F}_t$-measurable random variable, then for a given initial wealth x_0 satisfying $E\left[H(T)\psi + \int_0^T H(s)c_2ds\right] = x_0$, there exists a policy $h = [\pi(t), q(t)]$, such that $h = [\pi(t), q(t)] \in \Pi$, $t \in [0, T]$, and $X^h(T) = \psi$.*

Proof. Define a martingale

$$M_1(t) = E\left[H(T)\psi + \int_0^T H(s)c_2ds|\mathcal{F}_t\right].$$

According to the Martingale representation theorem (e.g., Cont and Tankov, Proposition 9.4 [27]), there exist three predictable processes $\varphi_1 : \Omega \times [0, T] \mapsto R$, $\varphi_2 : \Omega \times [0, T] \mapsto R$ and $\varphi_3 : \Omega \times [0, T] \mapsto R$ satisfying

$$\int_0^T \varphi_i(s)^2 ds < \infty, \quad a.s., \quad i = 1, 2, 3.$$

such that

$$M_1(t) = E\left[H(T)\psi + \int_0^T H(s)c_2 ds\right] + \int_0^t \varphi_1(s)dW_1(s) + \int_0^t \varphi_2(s)dW_2(s) + \int_0^t \varphi_3(s)dM(s)$$
$$= x_0 + \int_0^t \varphi_1(s)dW_1(s) + \int_0^t \varphi_2(s)dW_2(s) + \int_0^t \varphi_3(s)dM(s). \tag{14}$$

Therefore, it is easy to see that

$$H(T)\psi + \int_0^T H(s)c_2 ds = x_0 + \int_0^T \varphi_1(s)dW_1(s) + \int_0^T \varphi_2(s)dW_2(s) + \int_0^T \varphi_3(s)dM(s). \tag{15}$$

Compare $dW_1(t)$-term and $dW_2(t)$-term respectively in (15) with those in (12) taking $t = T$, we have

$$\begin{cases} \pi(t) = \dfrac{1}{\sigma_1(t)H(t)}\left[\varphi_1(t) - H(t)X(t)\theta_1(t)\right], \\ q(t) = \dfrac{1}{\sigma_2 H(t)}\left[\varphi_2(t) - H(t)X(t)\theta_2(t)\right]. \end{cases} \tag{16}$$

Then we need to check whether the policy defined in (16) is admissible. To prove $\pi(t)$ is admissible, we only need to prove $\int_0^T |\pi(t)|dt < \infty$, a.s. Define some notations:

$$||f(t)||_\infty = \max_{0\le t\le T} |f(t)|, \quad ||f(t)||_2 = \left[\int_0^T |f(t)|^2 dt\right]^{\frac{1}{2}}.$$

According to (16), we have

$$\begin{aligned}\int_0^T |\pi(t)|dt &= \int_0^T \left|{\sigma_1}^{-1}(t)H(t)^{-1}\varphi_1(t) + X(t){\sigma_1}^{-1}(t)\theta_1(t)\right| dt \\ &\le \int_0^T \left| ||{\sigma_1}^{-1}(t)||_\infty ||H(t)^{-1}||_\infty \varphi_1(t) + ||X(t)||_\infty ||{\sigma_1}^{-1}(t)||_\infty \theta_1(t)\right| dt \\ &\le ||\sigma_1(t)^{-1}||_\infty ||H(t)^{-1}||_\infty ||\varphi_1(t)||_2 + ||\sigma_1(t)^{-1}||_\infty ||X(t)||_\infty ||\theta_1(t)||_2 \\ &< \infty, \text{a.s.}\end{aligned}$$

The last inequality follows from the uniformly bounded conditions.

Due to the non-negativity constraint on the admissible reinsurance strategy, we define two regions:

$$D_1 := \left\{(t, x) \in [0, T] \times R^+ | x < A_1(t)\right\},$$
$$D_2 := \left\{(t, x) \in [0, T] \times R^+ | x \ge A_1(t)\right\}.$$

where $A_1(t) = \frac{\varphi_2(t)}{H(t)\theta_2}$.

Firstly, we consider region D_1. It is obvious that $[\varphi_2(t) - H(t)X(t)\theta_2(t)] < 0$. Hence, we take the value $q(t) \equiv 0$, which satisfies the admissibility. In a word, in region D_1, we conjecture the form of trading strategy as follows:

$$\begin{cases} \pi(t) = \dfrac{1}{\sigma_1(t)H(t)}\left[\varphi_1(t) - H(t)X(t)\theta_1(t)\right], \\ q(t) = 0. \end{cases} \tag{17}$$

Then, we consider region D_2. In this region, it is easy to find that the policy defined in (16) is admissible. Hence, in region D_2, the conjecture of trading strategy is given by (16). □

According to Theorem 4.1, any $\mathcal{F}_t$-measurable random variable $\psi \geq 0$ with $E[H(T)\psi + \int_0^T H(s)c_2 ds] = x_0$ can be financed via trading an admissible policy h such that $X^h(T) = \psi$. So to determine the optimal policy h^* in the dynamic maximization problem (8), which depends on the time variable t, it is sufficient to maximize over all possible random variable ψ's . That is to say, the dynamic maximization problem (8) is equivalent to the following static optimization problem:

$$\begin{cases} \max\limits_{\psi \geq 0} E[U(\psi)] \\ s.t.\ E\left[H(T)\psi + \int_0^T H(s)c_2 ds\right] \leq x_0. \end{cases} \tag{18}$$

Theorem 4.2 characterizes the optimal solutions of the optimization problem (18).

Theorem 4.2. *The optimal terminal wealth of a loss-averse insurer with* $0 < \gamma_1 < 1$ *and* $0 < \gamma_2 < 1$ *is given by*

$$\psi^* = \begin{cases} \xi + \left\{x_0 - E\left[H(T)\xi + \int_0^T H(s)c_2 ds\right]\right\} \frac{H(T)^{\frac{1}{\gamma_1 - 1}}}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}}\right)}, & \xi \leq x_0 e^{\int_0^T r(t)dt} - c_2 \int_0^T e^{\int_t^T r(s)ds} dt; \\ 0, & \xi > x_0 e^{\int_0^T r(t)dt} - c_2 \int_0^T e^{\int_t^T r(s)ds} dt. \end{cases} \tag{19}$$

Proof. Denote $u_1(x) = A(x - \xi)^{\gamma_1}$, $u_2(x) = -B(\xi - x)^{\gamma_2}$. To solve the problem (18), firstly, we assume that

$$E\left[H(T)\xi + \int_0^T H(s)c_2 ds\right] \leq x_0.$$

If $\psi > \xi$, the Lagrangian function $L(\psi, y)$ of problem (18) can be written as

$$L(\psi, y) = E\left\{u_1(\psi) + y\left[x - H(T)\psi - \int_0^T H(s)c_2 ds\right]\right\}, \tag{20}$$

where y is the Lagrangian multiplier. Equating the derivatives of Lagrangian function (20) with respect to ψ and y respectively to zero, we obtain

$$\begin{cases} \dfrac{\partial L}{\partial \psi} = E\left[{u_1}'(\psi) - yH(T)\right] = 0, \\ \dfrac{\partial L}{\partial y} = x_0 - H(T)\psi - \int_0^T H(s)c_2 ds = 0. \end{cases} \tag{21}$$

Solving equation (21), we have

$$\psi_1^* = \xi + \left[\frac{A\gamma_1}{yH(T)}\right]^{\frac{1}{1-\gamma_1}}. \tag{22}$$

While, the Lagrangian multiplier y is determined by the constraint

$$E\left[H(T)\psi_1^* + \int_0^T H(s)c_2 ds\right] = E\left[H(T)\xi + (A\gamma_1)^{\frac{1}{1-\gamma_1}} y^{\frac{1}{\gamma_1-1}} H(T)^{\frac{\gamma_1}{\gamma_1-1}} + \int_0^T H(s)c_2 ds\right] = x_0,$$

which is satisfied by setting

$$y^{\frac{1}{\gamma_1-1}} = \frac{x_0 - E\left[H(T)\xi + \int_0^T H(s)c_2 ds\right]}{(A\gamma_1)^{\frac{1}{1-\gamma_1}} E\left(H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right)}.$$

Substitution of $y^{\frac{1}{\gamma_1-1}}$ in (22) gives us the optimal solution of (20) via the following formula:

$$\psi_1^* = \xi + \left\{x_0 - E\left[H(T)\xi + \int_0^T H(s)c_2 ds\right]\right\} \frac{H(T)^{\frac{1}{\gamma_1-1}}}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right).} \tag{23}$$

If $\psi \leq \xi$, the utility function $u_2(\psi)$ is continuous and convex in the closed interval $[0, \xi]$. Therefore the local optimal solution ψ_2^* is located at one of the boundaries $\psi_2^* = 0$ or $\psi_2^* = \xi$. Furthermore it is easy to check $\psi_2^* = 0$ and $\psi_2^* = \xi$ satisfy the constraint condition

$$E\left[H(T)\psi + \int_0^T H(s)c_2 ds\right] \leq x_0.$$

Since $U(\cdot)$ is not concave, we need to compare the local maxima ψ_1^* and ψ_2^* to determine the global maximum. Firstly we compare ψ_1^* to $\psi_2^* = \xi$:

$$\begin{aligned} U[\psi_1^*] - U[\xi] &= u_1(\psi_1^*) - u_2(\xi) \\ &= A[\psi_1^* - \xi]^{\gamma_1} \\ &= A\left[\frac{A\gamma_1}{yH(T)}\right]^{\frac{\gamma_1}{1-\gamma_1}} > 0. \end{aligned}$$

Hence $\psi_2^* = \xi$ is never the optimal level of terminal wealth. Similarly by comparing ψ_1^* to $\psi_2^* = 0$, we find

$$\begin{aligned} U[\psi_1^*] - U[\xi] &= u_1(\psi_1^*) - u_2(0) \\ &= A[\psi_1^* - \xi]^{\gamma_1} + B\xi^{\gamma_2} \\ &= A\left[\frac{A\gamma_1}{yH(T)}\right]^{\frac{\gamma_1}{1-\gamma_1}} + B\xi^{\gamma_2} > 0. \end{aligned}$$

So $\psi_2^* = 0$ is not the optimal level of terminal wealth either. We conclude that ψ_1^* is the optimal solution of the static problem (18), when

$$E\left[H(T)\xi + \int_0^T H(s)c_2 ds\right] \leq x_0.$$

Then, we assume that

$$E\left[H(T)\xi+\int_0^T H(s)c_2ds\right]>x_0.$$

If $\psi>\xi$, the Lagrangian function (20) has no optimal solution; If $\psi\le\xi$, similarly according to the continuity and convexity of the utility function $u_2(\psi)$, the local optimal solution ψ_2^* is located at one of the boundaries $\psi_2^*=0$ or $\psi_2^*=\xi$. But $\psi_2^*=\xi$ does not satisfy the constraint

$$E\left[H(T)\psi+\int_0^T H(s)c_2ds\right]\le x_0.$$

So we conclude that $\psi_2^*=0$ is the optimal solution of the static problem (18) when

$$E\left[H(T)\xi+\int_0^T H(s)c_2ds\right]>x_0.$$

It is easy to calculate that

$$E\left[H(T)\xi+\int_0^T H(s)c_2ds\right]=\xi e^{-\int_0^T r(s)ds}+c_2\int_0^T e^{-\int_0^t r(s)ds}dt.$$

Let ψ^* be the optimal solution of the problem (18). Then ψ^* can be written as

$$\psi^*=\begin{cases}\xi+\left\{x_0-E\left[H(T)\xi+\int_0^T H(s)c_2ds\right]\right\}\frac{H(T)^{\frac{1}{\gamma_1-1}}}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1-1}}\right)}, & \xi\le x_0e^{\int_0^T r(t)dt}-c_2\int_0^T e^{\int_t^T r(s)ds}dt;\\ 0, & \xi> x_0e^{\int_0^T r(t)dt}-c_2\int_0^T e^{\int_t^T r(s)ds}dt.\end{cases} \tag{24}$$

□

Note that the optimal terminal wealth is a discontinuous function. In good states ($\xi\le x_0e^{\int_0^T r(t)dt}-c_2\int_0^T e^{\int_t^T r(s)ds}dt$), the loss-averse agent behaves like the CRRA agent and obtains wealth above the aspiration level. In bad states ($\xi> x_0e^{\int_0^T r(t)dt}-c_2\int_0^T e^{\int_t^T r(s)ds}dt$), the insurer ends up with zero wealth. Since the insurer is mostly concerned with small changes in wealth relative to the reference level, the gambling behavior below the level causes the insurer to incur large losses in these bad states.

In the previous section, we characterize the optimal terminal wealth of a loss-averse insurer. In what follows, we derive closed-form expressions for the optimal policies when the price of risky asset follows a jump-diffusion model. When applying the martingale method the optimal strategies are derived not given in feedback form as with stochastic dynamic programming. Instead, the optimal strategies are derived as a function of the wealth process. Theorem 4.3 presents closed-form expressions of the optimal policy, the optimal wealth process and the optimal expected utility of terminal wealth.

Theorem 4.3. *Consider the optimal investment and reinsurance problem for an insurance company and the decision makers are assumed to be loss-averse. Then:*

(i) *The optimal trading policy* $h^* = [\pi^*(t), q^*(t)]$ *is given by*

$$\begin{cases} \pi^*(t) = \dfrac{\theta_1(t)}{(\gamma_1 - 1)\sigma_1(t)} \left[X^*(t) - \xi e^{-\int_t^T r(s)ds} - c_2 \displaystyle\int_t^T e^{-\int_t^s r(u)du} ds \right], \\ q^*(t) = \dfrac{\theta_2(t)}{(\gamma_1 - 1)\sigma_2} \left[X^*(t) - \xi e^{-\int_t^T r(s)ds} - c_2 \displaystyle\int_t^T e^{-\int_t^s r(u)du} ds \right]. \end{cases} \tag{25}$$

where $\pi^*(t)$ *and* $q^*(t)$ *denote the optimal investment strategy and the optimal reinsurance strategy respectively.*

(ii) *The corresponding optimal wealth process* $X^*(t), t \in [0, T]$ *is given by*

$$\begin{aligned} X^*(t) &= \xi e^{-\int_t^T r(s)ds} + c_2 \int_t^T e^{-\int_t^s r(u)du} ds + \left[x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt \right] \\ &\times \exp\left\{ \int_0^t r(s)ds + \frac{1}{\gamma_1 - 1} \int_0^t \theta_1(s)dW_1(s) + \theta_2(s)dW_2(s) + \ln[\theta_3(s) + 1]dN(s) \right. \\ &\left. + \frac{1 - 2\gamma_1}{2(\gamma_1 - 1)^2} \int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds + \int_0^t \left(1 + \theta_3(s) - (1 + \theta_3(s))^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \lambda_1 ds \right\}. \end{aligned} \tag{26}$$

(iii) *The insurer's optimal expected utility of terminal wealth is given by*

$$\begin{aligned} E[U(X^*(T))] &= A\left(x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt \right)^{\gamma_1} \\ &\times \exp\left\{ \gamma_1 \int_0^T r(s)ds + \frac{1}{2} \frac{\gamma_1}{1 - \gamma_1} \int_0^T \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds \right. \\ &\left. + (1 - \gamma_1) \int_0^T \left[(1 + \theta_3(s))^{\frac{\gamma_1}{\gamma_1 - 1}} - 1 \right] \lambda_1 ds + \int_0^T \gamma_1 \lambda_1 ds \right\}. \end{aligned} \tag{27}$$

Proof. We derive the optimal policy $h^* = \{\pi^*(t), q^*(t)\}_{t \in [0,T]}$ in the dynamic problem (8) with the corresponding optimal terminal wealth ψ_1^* satisfying

$$X^*(T) = X^{h^*}(T) = \psi_1^*.$$

Multiplying by $H(T)$ and then taking conditional expectation on both sides gives

$$E\left[H(T)X^*(T) + \int_0^T H(s)c_2 ds | \mathcal{F}_t \right] = E\left[H(T)\psi_1^* + \int_0^T H(s)c_2 ds | \mathcal{F}_t \right]. \tag{28}$$

According to Proportion 4.1, (28) can be rewritten as

$$\begin{aligned} H(t)X^*(t) + \int_0^t H(s)c_2 ds &= H(t)\xi e^{-\int_t^T r(s)ds} + \int_0^t H(s)c_2 ds + c_2 H(t) \int_t^T e^{-\int_t^s r(u)du} ds \\ &+ \left[x_0 - E\left(H(T)\xi + \int_0^T H(s)c_2 ds \right) \right] \frac{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}} | \mathcal{F}_t \right)}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}} \right)}. \end{aligned} \tag{29}$$

Then we obtain

$$\begin{aligned} H(t)X^*(t) &= H(t) \left[\xi e^{-\int_t^T r(s)ds} + c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right] \\ &+ \left[x_0 - E\left(H(T)\xi + \int_0^T H(s)c_2 ds \right) \right] \frac{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}} | \mathcal{F}_t \right)}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}} \right)}. \end{aligned} \tag{30}$$

Introduce an exponential martingale

$$
\begin{aligned}
Z(t) = \exp \Bigg\{ & \frac{\gamma_1}{\gamma_1 - 1} \int_0^t \theta_1(s) dW_1(s) + \theta_2(s) dW_2(s) + \ln[1 + \theta_3(s)] dN(s) \\
& - \frac{1}{2} \frac{(\gamma_1)^2}{(\gamma_1 - 1)^2} \int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds - \int_0^t \left([1 + \theta_3(s)]^{\frac{\gamma_1}{\gamma_1 - 1}} - 1 \right) \lambda_1 ds \Bigg\}. \qquad (31)
\end{aligned}
$$

According to $Z(t)$, $H(t)^{\frac{\gamma_1}{\gamma_1-1}}$ can be rewritten as

$$
\begin{aligned}
H(t)^{\frac{\gamma_1}{\gamma_1-1}} = & Z(t) \exp \Bigg\{ - \frac{\gamma_1}{\gamma_1 - 1} \int_0^t r(s) ds + \frac{1}{2} \frac{\gamma_1}{(\gamma_1 - 1)^2} \int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds \\
& + \int_0^t \left([1 + \theta_3(s)]^{\frac{\gamma_1}{\gamma_1 - 1}} - 1 \right) \lambda_1 ds - \frac{\gamma_1}{\gamma_1 - 1} \int_0^t \lambda_1 \theta_3(s) ds \Bigg\}. \qquad (32)
\end{aligned}
$$

Denote

$$
\begin{aligned}
f(t) = \exp \Bigg\{ & - \frac{\gamma_1}{\gamma_1 - 1} \int_0^t r(s) ds + \frac{1}{2} \frac{\gamma_1}{(\gamma_1 - 1)^2} \int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds \\
& + \int_0^t \left([1 + \theta_3(s)]^{\frac{\gamma_1}{\gamma_1 - 1}} - 1 \right) \lambda_1 ds - \frac{\gamma_1}{\gamma_1 - 1} \int_0^t \lambda_1 \theta_3(s) ds \Bigg\}, \qquad (33)
\end{aligned}
$$

then the fraction of (29) on the right-hand side can be rewritten as

$$
\frac{E\left(H(T)^{\frac{\gamma_1}{\gamma_1-1}} | \mathcal{F}_t \right)}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1-1}} \right)} = \frac{E\left[f(T) Z(T) | \mathcal{F}_t \right]}{E\left[f(T) Z(T) \right]} = \frac{f(T) E\left[Z(T) | \mathcal{F}_t \right]}{f(T) E\left[Z(T) \right]} = \frac{Z(t)}{Z(0)} = Z(t).
$$

The last equality holds because $Z(0) = 1$. Substituting back into (29), and since

$$
E\left[H(T)\xi + \int_0^T H(s) c_2 ds \right] = \xi e^{-\int_0^T r(s) ds} + c_2 \int_0^T e^{-\int_0^t r(s) ds} dt,
$$

we obtain

$$
\begin{aligned}
H(t) X^*(t) = H(t) & \left[\xi e^{-\int_t^T r(s) ds} + c_2 \int_t^T e^{-\int_t^s r(u) du} ds \right] \\
& + \left(x_0 - \xi e^{-\int_0^T r(s) ds} - c_2 \int_0^T e^{-\int_0^t r(s) ds} dt \right) Z(t). \qquad (34)
\end{aligned}
$$

Taking differential on both sides of (34), we get

$$
\begin{aligned}
d[H(t) X^*(t)] = & \left(\xi e^{-\int_t^T r(s) ds} + c_2 \int_t^T e^{-\int_t^s r(u) du} ds \right) dH(t) \\
& + H(t) d \left(\xi e^{-\int_t^T r(s) ds} + c_2 \int_t^T e^{-\int_t^s r(u) du} ds \right) \\
& + \left(x_0 - \xi e^{-\int_0^T r(s) ds} - c_2 \int_0^T e^{-\int_0^t r(s) ds} dt \right) dZ(t) \\
= & -c_2 H(t) dt + (\cdot) dM(t)
\end{aligned}
$$

$$+ \frac{1}{\gamma_1 - 1} H(t)\theta_1(t) \left(X^*(t)\gamma_1 - \xi e^{-\int_t^T r(s)ds} - c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right) dW_1(t)$$
$$+ \frac{1}{\gamma_1 - 1} H(t)\theta_2(t) \left(X^*(t)\gamma_1 - \xi e^{-\int_t^T r(s)ds} - c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right) dW_2(t), \tag{35}$$

where

$$\begin{aligned}(\cdot) =& H(t) \left\{ \theta_3(t) \left[\xi e^{-\int_t^T r(s)ds} + c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right] \right. \\ &+ \left. \left(X^*(t) - \xi e^{-\int_t^T r(s)ds} - c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right) \left[(1 + \theta_3(t))^{\frac{\gamma_1}{\gamma_1 - 1}} - 1 \right] \right\}.\end{aligned} \tag{36}$$

Since $H(t)X^*(t)$ also satisfies (11), we have

$$\begin{aligned} d[H(t)X^*(t)] &= H(t)dX^*(t) + X^*(t)dH(t) + d[H(t), X^*(t)] \\ &= -H(t)c_2 dt + H(t)[X^*(t)\theta_1(t) + \sigma_1(t)\pi(t)] dW_1(t) \\ &\quad + H(t)[X^*(t)\theta_2(t) + \sigma_2 q(t)] dW_2(t) \\ &\quad + H(t)[X^*(t)\theta_3(t) + \pi(t)\gamma(t)(\theta_3(t) + 1)] dM(t). \end{aligned} \tag{37}$$

Comparing $dW_1(t)$-term and $dW_2(t)$-term of equation (35) with those of equation (37), the optimal policy is given by

$$\begin{cases} \pi^*(t) = \dfrac{\theta_1(t)}{(\gamma_1 - 1)\sigma_1(t)} \left[X^*(t) - \xi e^{-\int_t^T r(s)ds} - c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right], \\ q^*(t) = \dfrac{\theta_2(t)}{(\gamma_1 - 1)\sigma_2} \left[X^*(t) - \xi e^{-\int_t^T r(s)ds} - c_2 \int_t^T e^{-\int_t^s r(u)du} ds \right]. \end{cases} \tag{38}$$

Finally, it is easy to prove that the policy in Equation (38) is admissible. So, it is the optimal policy of the optimization problem (8). From (34), we easily derive the optimal wealth process:

$$\begin{aligned} X^*(t) &= \xi e^{-\int_t^T r(s)ds} + c_2 \int_t^T e^{-\int_t^s r(u)du} ds + \left[x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt \right] \frac{Z(t)}{H(t)} \\ &= \xi e^{-\int_t^T r(s)ds} + c_2 \int_t^T e^{-\int_t^s r(u)du} ds + \left[x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt \right] \\ &\quad \times \exp \left\{ \int_0^t r(s)ds + \frac{1}{\gamma_1 - 1} \int_0^t \theta_1(s)dW_1(s) + \theta_2(s)dW_2(s) + \ln[\theta_3(s) + 1]dN(s) \right. \\ &\quad \left. + \frac{1 - 2\gamma_1}{2(\gamma_1 - 1)^2} \int_0^t \left[\theta_1(s)^2 + \theta_2(s)^2 \right] ds + \int_0^t \left(1 + \theta_3(s) - (1 + \theta_3(s))^{\frac{\gamma_1}{\gamma_1 - 1}} \right) \lambda_1 ds \right\}. \end{aligned} \tag{39}$$

Substituting ψ_1^* into the value function in the maximization problem (18), we can derive the optimal expected utility:

$$\begin{aligned} E[U(X^*(T))] &= E[U(\psi_1^*)] \\ &= E[u_1(\psi_1^*)] \end{aligned}$$

$$= A\left(x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt\right)^{\gamma_1} E\left(\frac{H(T)^{\frac{1}{\gamma_1 - 1}}}{E\left(H(T)^{\frac{\gamma_1}{\gamma_1 - 1}}\right)}\right)^{\gamma_1}$$

$$= A\left(x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt\right)^{\gamma_1} E\left(\frac{H(T)^{\frac{1}{\gamma_1 - 1}}}{f(T)E[Z(T)]}\right)^{\gamma_1}$$

$$= A\left(x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt\right)^{\gamma_1} f(T)^{1-\gamma_1}.$$

The last equality holds because $E[Z(T)] = 1$. Substituting the expression of $f(t)$ into the above formula, the insurer's optimal expected utility of terminal wealth is given by

$$\begin{aligned} E[U(X^*(T))] = {} & A\left(x_0 - \xi e^{-\int_0^T r(s)ds} - c_2 \int_0^T e^{-\int_0^t r(s)ds} dt\right)^{\gamma_1} f(T)^{1-\gamma_1} \\ & \times \exp\left\{\gamma_1 \int_0^T r(s)ds + \frac{1}{2}\frac{\gamma_1}{1-\gamma_1}\int_0^T \left[\theta_1(s)^2 + \theta_2(s)^2\right] ds \right. \\ & \left. + (1-\gamma_1)\int_0^T \left[(1+\theta_3(s))^{\frac{\gamma_1}{\gamma_1 - 1}} - 1\right]\lambda_1 ds + \int_0^T \gamma_1 \lambda_1 ds \right\}. \end{aligned} \quad (40)$$

□

Note that (1) the optimal policy depends on the wealth process $X^*(t)$; (2) the parameters of the capital market and the insurance market have an impact on the optimal policy; (3) the reference point of the insurer ξ has an impact on the optimal policy.

5. Numerical Examples

In this section, we present numerical examples to explore the economic behavior of the optimal investment and reinsurance strategies. Since the optimal strategies are stochastic, we apply the Monte Carlo Methods (MCM) to show the impacts of economic parameters on the optimal strategies. For convenience, but without loss of generality, we only analyze the results of the original model with $r(t) \equiv r$, $\mu(t) \equiv \mu$, $\sigma_1 \equiv \sigma$, and $\gamma(t) \equiv \gamma$ for all $t \in [0, T]$. Throughout the numerical analysis, unless otherwise stated, the basic parameters are given by: $\mu = 0.2$, $r = 0.05$, $\sigma_0 = 1$, $\sigma_1 = 2$, $\eta = 1.5$, $\mu_0 = 0.1$, $x_0 = 10$, $\lambda_1 = 0.3$, $\lambda_2 = 0.2$, $\gamma = 1.5$, $\theta = 1$, $T = 10$, $\xi = 5$, $\gamma_1 = 0.2$, $\gamma_2 = 0.3$, $A = 1$, $B = 2.25$. Without loss of generality, we only picture the cases at $t = 1$.

5.1. **Impact of parameters on the optimal investment policy.**

This subsection works on analyzing how the parameters of the insurance market, the coefficient of the insurer's risk aversion and the reference level influence the optimal investment policy.

Figure 2 shows that the optimal dollar amount invested in the risky asset increases with respect to the expected instantaneous rate of return of the risky asset μ, while decreasing with respect to the volatility of risky asset σ_1, namely, as the appreciation rate μ increases or as the volatility of risky asset σ_1 decreases, the insurer will invest more money in the risky asset.

Figure 3 displays that the optimal dollar amount invested in the risky asset is decreasing with respect to the volatility of risky asset γ, while increasing with respect to the jump intensity of the jump of risky asset's price λ_1, that is to say, the insurer will invest more money in the risky asset as the the jump intensity of the jump of risky asset's price λ_1 increases or as the volatility of risky asset γ decreases.

Figure 4 illustrates that the optimal dollar amount invested in the risky asset is increasing with respect to the coefficient of risk aversion γ_1, which indicates that the more risk-averse an insurance company is, the more the insurance company invests in risky asset. However, the optimal investment policy is decreasing with respect to the reference level ξ. When the reference level is increased, the insurer tends to adopt a lower allocation in risky asset, since the loss-averse insurer with higher reference level becomes more concerned about the volatilities of the risky asset that may cause the account of wealth to underperform the reference level.

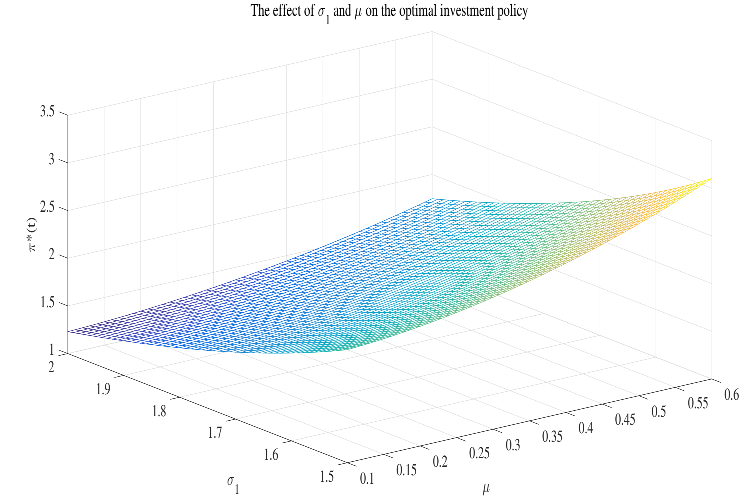

Figure 2. The effect of σ_1 and μ on the optimal investment policy.

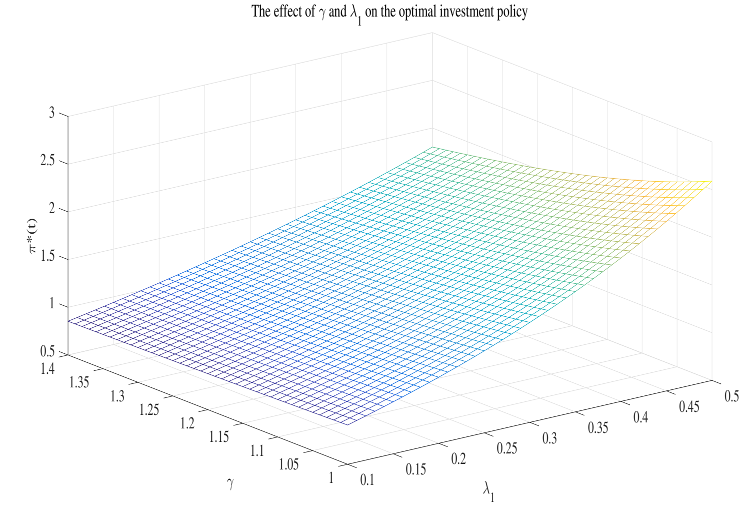

Figure 3. The effect of γ and λ_1 on the optimal investment policy.

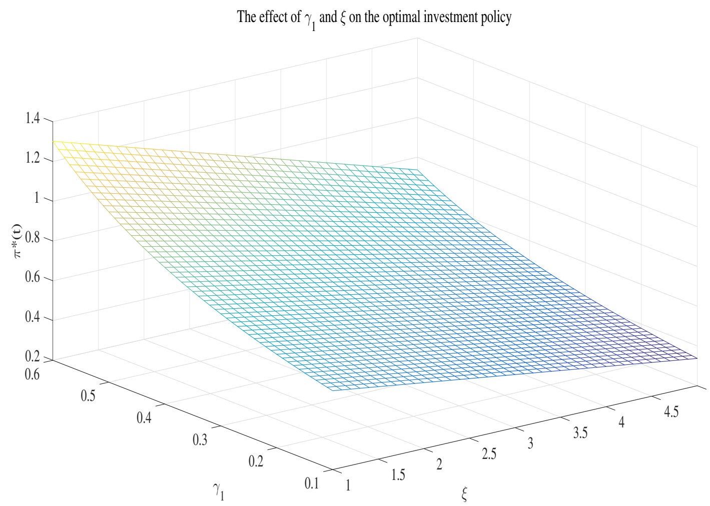

FIGURE 4. The effect of γ_1 and ξ on the optimal investment policy.

5.2. **Impact of parameters on the optimal reinsurance policy.**

In this section, we analyze how the parameters of the insurance market, the coefficient of the insurer's risk aversion and the reference level influence the optimal reinsurance policy.

Figure 5 demonstrates that the optimal reinsurance proportion increases with respect to both the expectation of the size of each claim μ_0 and the relative safety loading of the reinsurer η, which reveals that when the expectation of the size of each claim μ_0 or the relative safety loading of the reinsurer η increases, the insurer will purchase less reinsurance or acquire more new business.

Figure 6 reveals that the optimal reinsurance proportion is decreasing with the risk component of the insurance business σ_0, while increasing with the intensity of the claims λ_2, that is to says, as the the intensity of the claims λ_2 increases or the risk component of the insurance business σ_0 decreases, the insurer will purchase less reinsurance or acquire more new business.

Figure 7 shows that the optimal reinsurance proportion increases with respect to the coefficient of risk aversion γ_1, which reveals that when the insurer is less risk-averse, the insurer will purchase less reinsurance or acquire more new business. However, the optimal reinsurance proportion is decreasing with respect to the reference level ξ. When the reference level is increased, the insurer tends to purchase less reinsurance or acquire more new business, since the loss-averse insurer with higher reference level becomes more concerned about the volatilities of the insurance that may cause the account of wealth to underperform the reference level.

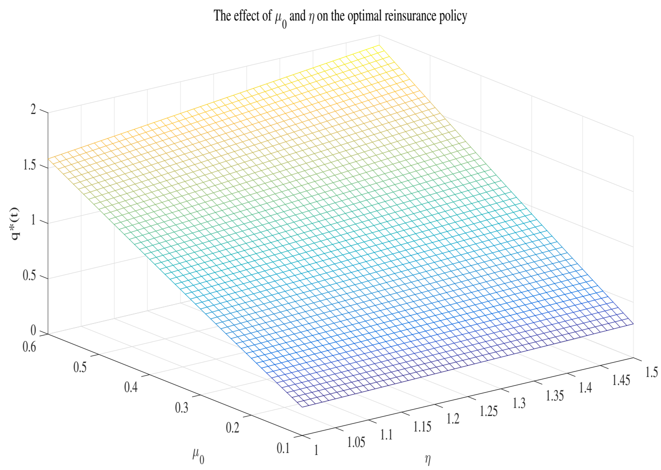

FIGURE 5. The effect of μ_0 and η on the optimal reinsurance policy.

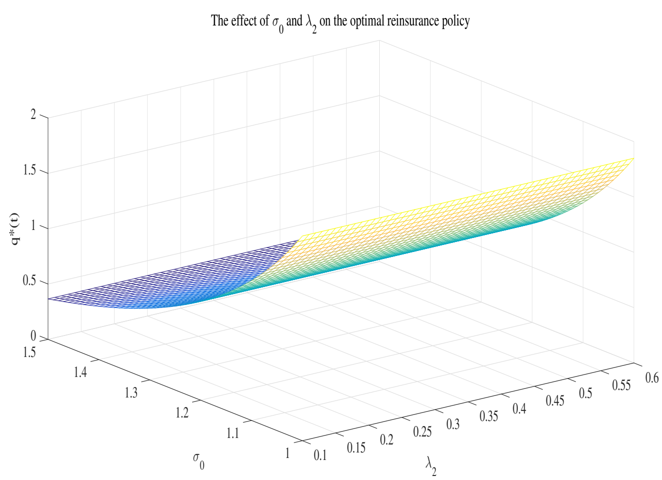

FIGURE 6. The effect of σ_0 and λ_2 on the optimal reinsurance policy.

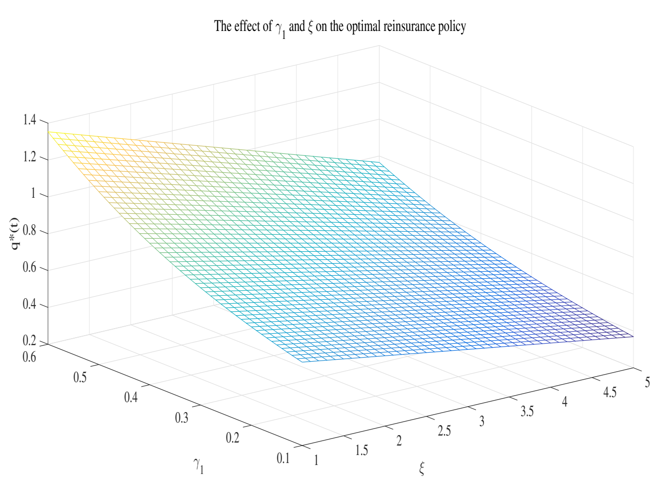

FIGURE 7. The effect of γ_1 and ξ on the optimal reinsurance policy.

6. Conclusions

In this paper, we study the optimal investment and reinsurance problem for an insurer under loss aversion. The insurer is allowed to invest in a financial market and purchase proportional reinsurance. The surplus process of the insurer is assumed to follow a diffusion approximation model and the financial market consists of one risk-free asset and one risky asset. So, the goal is to find the best asset allocation and reinsurance proportion. In general, stochastic programming methods and the martingale method can be applied in the work to maximize the expectation of a smooth utility function of terminal wealth. However, in our work, only martingale method is suitable since the optimization problem is not strictly concave.

With the help of martingale approach, we change the dynamic maximization problem into a static optimization problem. The closed-form expressions for the optimal policies, the optimal wealth process and the optimal terminal wealth are derived respectively. The optimal terminal wealth is a discontinuous function. In good states, the loss-averse agent behaves like the CRRA agent and obtains wealth above the aspiration level. In bad states, the insurer ends up with zero wealth. Numerical analysis in the end shows the impact of economic parameters on the optimal policy.

This paper considers the optimal investment and reinsurance problem for an insurer where the utility preference of the insurer is assumed to be loss-averse. However, this article does not consider the situation of incomplete financial market which affects the resource allocation and risk management of financial institutions, as happened in the 2008 financial crisis. Incomplete financial markets require us to clearly describe the asset trading process. How this will affect the insurance investment problems are interesting questions. Besides, in recent years, behavioral economics has attracted a great deal of attention to the hypothesis of non-self-interest of economic individuals. Will this affect the asset allocation and risk management of insurance companies in the same way as loss aversion? These are all research directions worth exploring.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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