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LATTICE VALUED FUZZY SETS IN HILBERT ALGEBRAS

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Abstract. In this paper, we introduce and study the concept of lattice valued fuzzy subgroups/ideals in Hilbert algebras. Also, we study the characteristic LFSs, t-level subsets, and the Cartesian product of LFSs in Hilbert algebras.

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1. Introduction

The concept of fuzzy sets was proposed by Zadeh [12]. The theory of fuzzy sets has several applications in real-life situations, and many scholars have researched fuzzy set theory. After the introduction of the concept of fuzzy sets, several research studies were conducted on the generalizations of fuzzy sets. The idea of intuitionistic fuzzy sets suggested by Atanassov [1] is one of the extensions of fuzzy sets with better applicability. Applications of intuitionistic fuzzy sets appear in various fields, including medical diagnosis, optimization problems, and multicriteria decision making [8–10]. The concept of Hilbert algebra was introduced in early 50-ties by L. Henkin and T. Skolem for some investigations of implication in intuicionistic and other non-classical logics. In 60-ties, these algebras were studied especially by A. Horn and A. Diego from algebraic point of view. A. Diego proved (cf. [6] that Hilbert algebras form a variety which is locally finite. Hilbert algebras were treated by D. Busneag (cf. [2], [3]) and Y. B. Jun (cf. [11]) and some of their filters forming deductive systems were recognized. In this paper, we introduce and study the concept of lattice valued fuzzy subgroups / ideals of Hilbert algebras.

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2. Preliminaries

Definition 2.1. [6] A Hilbert algebra is a triplet $H = (H, *, 1)$, where H is a nonempty set, $*$ is a binary operation and 1 is fixed element of H such that the following axioms hold for each $x, y, z \in \mathcal{H}$.

(1) $x * (y * x) = 1$,
(2) $(x * (y * z)) * ((x * y) * (x * z)) = 1$,
(3) $x * y = 1$ and $y * x = 1$ imply $x = y$.

Lemma 2.2. [7] *Let $H = (H, *, 1)$ be a Hilbert algebra and $x, y, z \in \mathcal{H}$. Then*

(1) $x * x = 1$,
(2) $1 * x = x$,
(3) $x * 1 = 1$,
(4) $x * (y * z) = y * (x * z)$.

It is easily checked that in a Hilbert algebra H the relation $\leq y$ defined by $x \leq y \;\Leftrightarrow\; x * y = 1$ is a partial order on H with 1 as the largest element.

Definition 2.3. [4] A nonempty subset I of a Hilbert algebra $H = (H, *, 1)$ is called an ideal of H if

(1) $1 \in I$,
(2) $x * y \in I$ for all $x \in \mathcal{H}$, $y \in I$,
(3) $(y_1 * (y_2 * x)) * x \in I$ for all $x \in \mathcal{H}$, $y_1, y_2 \in I$.

Lemma 2.4. [5] *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice. Then the following properties hold:*

(1) $(\forall u, v \in \mathscr{L})((u \vee v)' = u' \wedge v')$,
(2) $(\forall u, v \in \mathscr{L})((u \wedge v)' = u' \vee v')$,
(3) $(\forall u, v \in \mathscr{L})(u \leq v \Leftrightarrow u' \geq v')$,
(4) $(\forall u, v \in \mathscr{L})(u = v \Leftrightarrow u' = v')$,
(5) $(\forall u, v \in \mathscr{L})(u < v \Leftrightarrow u' > v')$.

3. Lattice Valued Fuzzy Hilbert Algebras

Definition 3.1. Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a lattice. Then an LFS $\mathscr{L}$ in $\mathcal{H}$ is called an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ if it satisfies

$$(\forall x, y \in \mathcal{H})\left(\mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) \right). \tag{1}$$

Proposition 3.2. *Every $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ satisfies $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(x)$ for all $x \in \mathcal{H}$.*

Proof. For any $x \in \mathcal{H}$, $\mathscr{L}_\mu(1) = \mathscr{L}_\mu(x * x) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(x) = \mathscr{L}_\mu(x)$. □

Definition 3.3. Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a lattice. Then an LFS $\mathscr{L}$ in $\mathcal{H}$ is called an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ if it satisfies

$$(\forall x \in \mathcal{H}) \left(\ \mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(x) \ \right), \tag{2}$$

$$(\forall\, x, y \in \mathcal{H}) \left(\ \mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(y) \ \right), \tag{3}$$

$$(\forall\, x, y_1, y_2 \in \mathcal{H}) \left(\ \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) \ \right). \tag{4}$$

Proposition 3.4. *If an LFS $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$, then*

$$(\forall x, y \in \mathcal{H}) \left(\ \mathscr{L}_\mu((y * x) * x) \geq \mathscr{L}_\mu(y) \ \right). \tag{5}$$

Proof. Putting $y_1 = y$ and $y_2 = 1$ in (4), we have $\mathscr{L}_\mu((y * x) * x) \geq \mathscr{L}_\mu(y) \wedge \mathscr{L}_\mu(1) = \mathscr{L}_\mu(y)$. □

Lemma 3.5. *If an LFS $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$, then we have the following*

$$(\forall\, x, y \in \mathcal{H}) \left(\ x \leq y \ \Rightarrow \mathscr{L}_\mu(x) \leq \mathscr{L}_\mu(y) \ \right). \tag{6}$$

Proof. Let $x, y \in \mathcal{H}$ be such that $x \leq y$. Then $x * y = 1$ and so

$$\begin{aligned} \mathscr{L}_\mu(y) &= \mathscr{L}_\mu(1 * y) \\ &= \mathscr{L}_\mu(((x * y) * (x * y)) * y) \\ &\geq \mathscr{L}_\mu(x * y) \wedge \mathscr{L}_\mu(x) \\ &\geq \mathscr{L}_\mu(1) \wedge \mathscr{L}_\mu(x) \\ &= \mathscr{L}_\mu(x). \end{aligned}$$

□

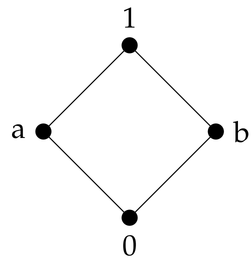
Theorem 3.6. *Every $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$.*

Proof. Let $\mathscr{L}$ be an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Since $y \leq x * y$ for all $x, y \in \mathcal{H}$. Then from Lemma 3.5 that $\mathscr{L}_\mu(y) \geq \mathscr{L}_\mu(x * y)$. It follows from (3) that $\mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(y) \geq \mathscr{L}_\mu(x * y) \wedge \mathscr{L}_\mu(x) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y)$. Hence $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. □

Example 3.7. Let $A = \{1, a, b, c\}$ be a Hilbert algebra with a fixed element 1 and a binary operation $\cdot$ defined by the following Cayley table, as the following table.

$\cdot$	1	a	b	c
1	1	a	b	c
a	1	1	b	b
b	1	a	1	a
c	1	1	1	1

Consider a lattice $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$, where $\mathcal{L} = \{1, a, b, 0\}$ as drawn in the following



We define an LFS $\mathscr{L}$ as follows:

$$\mathscr{L}_\mu = \begin{pmatrix} 1 & a & b & c \\ 1 & 0 & 0 & a \end{pmatrix}.$$

Hence $\mathcal{L}$ is $\mathcal{L}$-fuzzy subalgebra but not $\mathcal{L}$-fuzzy ideal of A.

Now we shall determine $\mathcal{L}$ is a complete lattice $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, 0_{\mathcal{L}}, 1_{\mathcal{L}})$.

Let A be a subset of $\mathcal{H}$. Then the characteristic function χ_A of $\mathcal{H}$ is a function of $\mathcal{H}$ into $\{0_{\mathcal{L}}, 1_{\mathcal{L}}\}$ defined as follows:

$$\chi_A(x) = \begin{cases} 1_{\mathcal{L}} & \text{if } x \in A \\ 0_{\mathcal{L}} & \text{if } x \notin A. \end{cases}$$

By the definition of characteristic function, χ_A is a function of $\mathcal{H}$ into $\{0_{\mathcal{L}}, 1_{\mathcal{L}}\} \subset \mathcal{L}$. We denote the LFS $\mathscr{L}_A$ in $\mathcal{H}$ is described by its membership function χ_A, is called the characteristic LFS of A in $\mathcal{H}$.

Lemma 3.8. *Let the constant* 1 *of* $\mathcal{H}$ *is in* A. *Then* $\chi_A(1) \geq \chi_A(a)$ *for all* $a \in \mathcal{H}$.

Proof. Assume that $1 \in A$. Then for all $a \in \mathcal{H}$, $\chi_A(1) = 1_{\mathcal{L}} \geq \chi_A(a)$. □

Lemma 3.9. *Let* A *be a nonempty subset of a Hilbert algebra* $\mathcal{H}$. *If* $\chi_A(1) \geq \chi_A(a)$ *for all* $a \in \mathcal{H}$, *then the constant* 1 *of* $\mathcal{H}$ *is in* A.

Proof. Assume that $\chi_A(1) \geq \chi_A(a)$ for all $a \in \mathcal{H}$. Since A is a nonempty subset of $\mathcal{H}$, we have an element u in A, that is, $\chi_A(u) = 1_{\mathcal{L}}$. Thus $1_{\mathcal{L}} \geq \chi_A(1) \geq \chi_A(u) = 1_{\mathcal{L}}$. So $\chi_A(1) = 1_{\mathcal{L}}$, that is, $1 \in A$. □

Theorem 3.10. *A nonempty subset* A *of* $\mathcal{H}$ *is a subalgebra of* $\mathcal{H}$ *if and only if the characteristic LFS* $\mathscr{L}_A$ *is an* $\mathcal{L}$*-fuzzy subalgebra of* $\mathcal{H}$.

Proof. Assume that A is a subalgebra of $\mathcal{H}$. Let $a, b \in \mathcal{H}$. Case 1: $a, b \in A$. Then $\chi_A(a) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since A is a subalgebra of $\mathcal{H}$, $a * b \in A$ and so $\chi_A(a * b) = 1_{\mathcal{L}}$. Therefore, $\chi_A(a * b) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(a) \wedge \chi_A(b)$.
Case 2: $a \notin A$ or $b \notin A$. Then $\chi_A(a) = 0_{\mathcal{L}}$ or $\chi_A(b) = 0_{\mathcal{L}}$, so $\chi_A(a) \wedge \chi_A(b) = 0_{\mathcal{L}}$. Therefore, $\chi_A(a * b) \geq 0_{\mathcal{L}} = \chi_A(a) \wedge \chi_A(b)$. Hence, $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Conversely, assume that $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Let $a, b \in A$. Then $\chi_A(a) = 1_{\mathcal{L}} = \chi_A(b)$, so $\chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. Since $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$, we have $1_{\mathcal{L}} \geq \chi_A(a * b) \geq \chi_A(a) \wedge \chi_A(b) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(a * b) = 1_{\mathcal{L}}$, that is, $a * b \in A$. Hence, A is a subalgebra of $\mathcal{H}$. □

Theorem 3.11. *A nonempty subset A of $\mathcal{H}$ is an ideal of $\mathcal{H}$ if and only if the characteristic LFS $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$.*

Proof. Assume that A is an ideal of $\mathcal{H}$. Since $1 \in A$, it follows from Lemma 3.9 that $\chi_A(1) \geq \chi_A(x)$ for all $x \in \mathcal{H}$. Let $x, y \in \mathcal{H}$. Case 1: $y \in A$. Then $\chi_A(y) = 1_{\mathcal{L}}$. Since A is an ideal of $\mathcal{H}$, $x * y \in A$ and so $\chi_A(x * y) = 1_{\mathcal{L}}$. Therefore, $\chi_A(x * y) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(y)$. Case 2: $y \notin A$. Then $\chi_A(y) = 0_{\mathcal{L}}$. Therefore, $\chi_A(x * y) \geq 0_{\mathcal{L}} = \chi_A(y)$. Let $x, y_1, y_2 \in \mathcal{H}$. Case 1: $y_1, y_2 \in A$. Then $\chi_A(y_1) = 1_{\mathcal{L}} = \chi_A(y_2)$, so $\chi_A(y_1) \wedge \chi_A(y_2) = 1_{\mathcal{L}}$. Since A is an ideal of $\mathcal{H}$, $(y_1 * (y_2 * x)) * x \in A$ and so $\chi_A((y_1 * (y_2 * x)) * x) = 1_{\mathcal{L}}$. Therefore, $\chi_A((y_1 * (y_2 * x)) * x) = 1_{\mathcal{L}} \geq 1_{\mathcal{L}} = \chi_A(y_1) \wedge \chi_A(y_2)$. Case 2: $y_1 \notin A$ or $y_2 \notin A$. Then $\chi_A(y_1) = 0_{\mathcal{L}}$ or $\chi_A(y_2) = 0_{\mathcal{L}}$, so $\chi_A(y_1) \wedge \chi_A(y_2) = 0_{\mathcal{L}}$. Therefore, $\chi_A((y_1 * (y_2 * x)) * x) \geq 0_{\mathcal{L}} = \chi_A(y_1) \wedge \chi_A(y_2)$. Hence, $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Conversely, assume that $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Since $\chi_A(1) \geq \chi_A(x)$ for all $x \in \mathcal{H}$, by Lemma 3.9 that $1 \in A$. Let $x, y \in \mathcal{H}$ such that $y \in A$. Then $\chi_A(y) = 1_{\mathcal{L}}$. Since $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$, $1_{\mathcal{L}} \geq \chi_A(x * y) \geq \chi_A(y) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A(x * y) = 1_{\mathcal{L}}$, that is, $x * y \in A$. Let $x, y_1, y_2 \in \mathcal{H}$ such that $y_1, y_2 \in A$. Then $\chi_A(y_1) = 1_{\mathcal{L}} = \chi_A(y_2)$, so $\chi_A(y_1) \wedge \chi_A(y_2) = 1_{\mathcal{L}}$. Since $\mathscr{L}_A$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$, $1_{\mathcal{L}} \geq \chi_A((y_1 * (y_2 * x)) * x) \geq \chi_A(y_1) \wedge \chi_A(y_2) = 1_{\mathcal{L}}$. By anti-symmetry, we have $\chi_A((y_1 * (y_2 * x)) * x) = 1_{\mathcal{L}}$, that is, $(y_1 * (y_2 * x)) * x \in A$. Hence A is an ideal of $\mathcal{H}$. □

Definition 3.12. Let $\mathcal{L}$ be an LFS in $\mathcal{H}$ with the membership function $\mathscr{L}_\mu$. For any $t \in \mathcal{L}$, the sets

$$U(\mathscr{L}_\mu, t) = \{x \in \mathcal{H} : \mathscr{L}_\mu(x) \geq t\}$$
$$U^+(\mathscr{L}_\mu, t) = \{x \in \mathcal{H} : \mathscr{L}_\mu(x) > t\}$$
$$L(\mathscr{L}_\mu, t) = \{x \in \mathcal{H} : \mathscr{L}_\mu(x) \leq t\}$$
$$L^-(\mathscr{L}_\mu, t) = \{x \in \mathcal{H} : \mathscr{L}_\mu(x) < t\}$$

are referred to as an upper t-level subset, an upper t-strong level subset, a lower t-level subset and a lower t-strong level subset of $\mathcal{L}$, respectively.

Theorem 3.13. *An LFS $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ if and only if $U(\mathscr{L}_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $U(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} x, y \in U(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(x) \geq t, \mathscr{L}_\mu(y) \geq t \\ &\Rightarrow \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) \geq t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) \geq t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq t \\ &\Rightarrow x * y \in U(\mathscr{L}_\mu, t). \end{aligned}$$

Hence, $U(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $U(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ if it is nonempty. Let $x, y \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(x) \geq t$ and $\mathscr{L}_\mu(y) \geq t$. Thus

$x, y \in U(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ and so $x * y \in U(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) \geq t = \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y)$. Hence $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. □

Lemma 3.14. *Let $\mathscr{L}$ be an LFS in $\mathcal{H}$. Then $\mathscr{L}$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$ if and only if $U(\mathscr{L}_\mu, t)$, if it is nonempty, contains $1 \in \mathcal{H}$ for every $t \in \mathscr{L}$.*

Proof. Let $t \in \mathscr{L}$ be such that $U(\mathscr{L}_\mu, t) \neq \emptyset$. Let $a \in \mathcal{H}$. Then

$$\begin{aligned} a \in U(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(a) \geq t \\ &\Rightarrow \mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a) \geq t \\ &\Rightarrow 1 \in U(\mathscr{L}_\mu, t). \end{aligned}$$

Conversely, assume for all $t \in \mathscr{L}$, $U(\mathscr{L}_\mu, t)$ contains $1 \in \mathcal{H}$ if it is nonempty. Choose $t = \mathscr{L}_\mu(a) \in \mathcal{L}$. Then $\mathscr{L}_\mu(a) \geq t$. Thus $a \in U(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, $1 \in U(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(1) \geq t = \mathscr{L}_\mu(a)$. □

Theorem 3.15. *An LFS $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ if and only if $U(\mathscr{L}_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $U(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} y \in U(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y) \geq t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(y) \geq t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq t \\ &\Rightarrow x * y \in U(\mathscr{L}_\mu, t). \end{aligned}$$

Let $x, y_1, y_2 \in \mathcal{H}$. Then

$$\begin{aligned} y_1, y_2 \in U(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y_1) \geq t, \mathscr{L}_\mu(y_2) \geq t \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) \geq t \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \geq t \\ &\Rightarrow (y_1 * (y_2 * x)) * x \in U(\mathscr{L}_\mu, t). \end{aligned}$$

By Lemma 3.14, we have $1 \in U(\mathscr{L}_\mu, t)$. Hence $U(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $U(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ if it is nonempty. Let $x, y \in \mathcal{H}$. By Lemma 3.14. we have $\mathscr{L}_\mu$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y) \geq t$. Thus $y \in U(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $x * y \in U(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) \geq t = \mathscr{L}_\mu(y)$. Let $x, y_1, y_2 \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y_1) \geq t$ and $\mathscr{L}_\mu(y_2) \geq t$. Thus $y_1, y_2 \in U(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $y_1, y_2 \in U(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(y_1 * (y_2 * x)) * x) \geq t = \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2)$. Hence $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. □

Theorem 3.16. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ if and only if $U^+(\mathscr{L}_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $U^+(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then $\mathscr{L}_\mu(x)$ and $\mathscr{L}_\mu(y)$ are compatible. Suppose that $\mathscr{L}_\mu(x) \geq \mathscr{L}_\mu(y)$, that is, $\mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) = \mathscr{L}_\mu(y)$. Then

$$\begin{aligned} x, y \in U^+(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(x) > t, \mathscr{L}_\mu(y) > t \\ &\Rightarrow \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) = \mathscr{L}_\mu(y) > t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) > t \\ &\Rightarrow x * y \in U^+(\mathscr{L}_\mu, t). \end{aligned}$$

Hence, $U^+(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $U^+(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ if it is nonempty. Suppose there exist $x, y \in \mathcal{H}$ such that $\mathscr{L}_\mu(x * y) \ngeq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y)$. It means that $\mathscr{L}_\mu(x * y) < \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y)$. Choose $t = \mathscr{L}_\mu(x * y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) > t$ and so $\mathscr{L}_\mu(x) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) > t$ and $\mathscr{L}_\mu(y) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y) > t$. Thus $x, y \in U^+(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ and so $x * y \in U^+(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) > t = \mathscr{L}_\mu(x * y)$, a contradiction. Hence $\mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(x) \wedge \mathscr{L}_\mu(y)$ for all $x, y \in \mathcal{H}$. Hence $\mathscr{L}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. □

Lemma 3.17. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathscr{L}$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$ if and only if $U^+(\mathscr{L}_\mu, t)$, if it is nonempty, contains $1 \in \mathcal{H}$ for every $t \in \mathscr{L}$.*

Proof. Let $t \in \mathscr{L}$ be such that $U^+(\mathscr{L}_\mu, t) \neq \emptyset$. Let $a \in \mathcal{H}$. Then

$$\begin{aligned} a \in U^+(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(a) > t \\ &\Rightarrow \mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a) > t \\ &\Rightarrow 1 \in U^+(\mathscr{L}_\mu, t). \end{aligned}$$

Conversely, assume for all $t \in \mathscr{L}$, $U^+(\mathscr{L}_\mu, t)$ contains $1 \in \mathcal{H}$ if it is nonempty. Suppose there exists $x \in \mathcal{H}$ such that $\mathscr{L}_\mu(1) \ngeq \mathscr{L}_\mu(x)$. It means that $\mathscr{L}_\mu(1) < \mathscr{L}_\mu(x)$. Choose $t = \mathscr{L}_\mu(1) \in \mathscr{L}$. Then $\mathscr{L}_\mu(x) > t$. Thus $x \in U^+(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $1 \in U^+(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(1) > t = \mathscr{L}_\mu(1)$, a contradiction. Hence $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(x)$ for all $x \in \mathcal{H}$. □

Theorem 3.18. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ if and only if $U^+(\mathscr{L}_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $U^+(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} y \in U^+(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y) > t \\ &\Rightarrow \mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(y) > t \\ &\Rightarrow x * y \in U^+(\mathscr{L}_\mu, t). \end{aligned}$$

Let $x, y_1, y_2 \in \mathcal{H}$. Then $\mathscr{L}_\mu(y_1)$ and $\mathscr{L}_\mu(y_2)$ are compatible. Suppose that $\mathscr{L}_\mu(y_1) \geq \mathscr{L}_\mu(y_2)$, that is, $\mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) = \mathscr{L}_\mu(y_2)$. Then

$$\begin{aligned} y_1, y_2 \in U^+(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y_1) > t, \mathscr{L}_\mu(y_2) > t \\ &\Rightarrow \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) = \mathscr{L}_\mu(y_2) > t \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) > t \\ &\Rightarrow (y_1 * (y_2 * x)) * x \in U^+(\mathscr{L}_\mu, t). \end{aligned}$$

By Lemma 3.14, we have $1 \in U^+(\mathscr{L}_\mu, t)$. Hence, $U^+(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $U^+(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ if it is nonempty. Suppose there exist $x, y \in \mathcal{H}$ such that $\mathscr{L}_\mu(x * y) \ngeq \mathscr{L}_\mu(y)$. It means that $\mathscr{L}_\mu(x * y) < \mathscr{L}_\mu(y)$. By Lemma 3.14, we have $\mathscr{L}$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(x * y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y) > t$ and so $\mathscr{L}_\mu(x * y) \geq \mathscr{L}_\mu(y) > t$. Thus $y \in U^+(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $x * y \in U^+(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x*y) > t = \mathscr{L}_\mu(x*y)$, a contradiction. Hence $\mathscr{L}_\mu(x*y) \geq \mathscr{L}_\mu(y)$ for all $x, y \in \mathcal{H}$. Suppose there exist $x, y_1, y_2 \in \mathcal{H}$ such that $\mathscr{L}_\mu((y_1*(y_2*x))*x) \ngeq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2)$. It means that $\mathscr{L}_\mu((y_1*(y_2*x))*x) < \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2)$. By Lemma 3.14, we have $\mathscr{L}$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) > t$ and so $\mathscr{L}_\mu(y_1) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) > t$ and $\mathscr{L}_\mu(y_2) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) > t$. Thus $y_1, y_2 \in U^+(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $U^+(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $(y_1*(y_2*x))*x \in U^+(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu((y_1*(y_2*x))*x) > t = \mathscr{L}_\mu((y_1*(y_2*x))*x)$, a contradiction. Hence $\mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2)$ for all $x, y_1, y_2 \in \mathcal{H}$. Hence $\mathscr{L}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. □

Definition 3.19. Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice. Let $\mathscr{L}$ be LFS in $\mathcal{H}$. The LFS $\mathscr{L}'$ defined by $(\forall a \in \mathcal{H})(\mathscr{L}'_\mu(a) = (\mathscr{L}_\mu(a))' = \mathscr{L}_\mu(a)')$ is called the complement of $\mathscr{L}$ in $\mathcal{H}$.

Theorem 3.20. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice. Then $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ if and only if $L(\mathscr{L}_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $L(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} x, y \in L(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(x) \leq t, \mathscr{L}_\mu(y) \leq t \\ &\Rightarrow \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) \leq t \\ &\Rightarrow ((\mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y))' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' \geq \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y) \leq t \\ &\Rightarrow x * y \in L(\mathscr{L}_\mu, t). \end{aligned}$$

Hence, $L(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $L(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ if it is nonempty. Let $x, y \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(x) \leq t$ and $\mathscr{L}_\mu(y) \leq t$. Thus $x, y \in U(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ and so $x * y \in L(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) \leq t = \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y)$. By Lemma 2.4 (1), we have $\mathscr{L}_\mu(x * y)' \geq t = \mathscr{L}_\mu(x)' \wedge \mathscr{L}'_\mu(y)$. Hence $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. □

Lemma 3.21. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice. and $\mathscr{L}$ be an LFS in $\mathcal{H}$. Then $\mathscr{L}'$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$ if and only if $L(\mathscr{L}_\mu, t)$, if it is nonempty, contains $1 \in \mathcal{H}$ for every $t \in \mathscr{L}$.*

Proof. Let $t \in \mathscr{L}$ be such that $L(\mathscr{L}_\mu, t) \neq \emptyset$. Let $a \in \mathcal{H}$. Then

$$\begin{aligned} a \in L(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(a) \leq t \\ &\Rightarrow \mathscr{L}_\mu(a)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1)' \geq \mathscr{L}_\mu(a)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1) \leq t \\ &\Rightarrow 1 \in L(\mathscr{L}_\mu, t). \end{aligned}$$

Conversely, assume for all $t \in \mathscr{L}$, $L(\mathscr{L}_\mu, t)$ contains $1 \in \mathcal{H}$ if it is nonempty. Choose $t = \mathscr{L}_\mu(a) \in \mathcal{L}$. Then $\mathscr{L}_\mu(a) \leq t$. Thus $a \in L(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, $1 \in L(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(1) \leq t = \mathscr{L}_\mu(a)$. By Lemma 2.4 (3), we have $\mathscr{L}_\mu(1)' \geq \mathscr{L}_\mu(a)'$. □

Theorem 3.22. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice. Then $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ if and only if $L(\mathscr{L}_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $L(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} y \in L(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y) \leq t \\ &\Rightarrow \mathscr{L}_\mu(y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' \geq \mathscr{L}_\mu(y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(x * y) \leq t \\ &\Rightarrow x * y \in L(\mathscr{L}_\mu, t). \end{aligned}$$

Let $x, y_1, y_2 \in \mathcal{H}$. Then

$$\begin{aligned} y_1, y_2 \in L(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y_1) \leq t, \mathscr{L}_\mu(y_2) \leq t \\ &\Rightarrow \mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2) \leq t \\ &\Rightarrow (\mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2))' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(y_1)' \wedge \mathscr{L}_\mu(y_2)' = (\mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2))' \geq t' \end{aligned}$$

$$\begin{aligned}
&\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x)' \geq \mathscr{L}_\mu(y_1)' \wedge \mathscr{L}_\mu(y_2)' \geq t' \\
&\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x)' \geq t' \\
&\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) \leq t \\
&\Rightarrow (y_1 * (y_2 * x)) * x \in L(\mathscr{L}_\mu, t).
\end{aligned}$$

By Lemma 3.21, we have $1 \in L(\mathscr{L}_\mu, t)$. Hence, $L(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $L(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ if it is nonempty. Let $x, y \in \mathcal{H}$. By Lemma 3.21, we have $\mathscr{L}_\mu$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y) \leq t$. Thus $y \in L(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $x * y \in L(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) \leq t = \mathscr{L}_\mu(y)$. Let $x, y_1, y_2 \in \mathcal{H}$. Choose $t = \mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2) \in \mathcal{L}$. Then $\mathscr{L}_\mu(y_1) \leq t$ and $\mathscr{L}_\mu(y_2) \leq t$. Thus $y_1, y_2 \in L(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L(\mathscr{L}_\mu, t)$ is an ideal of $\mathcal{H}$ and so $y_1, y_2 \in L(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(y_1 * (y_2 * x)) * x) \leq t = \mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2)$. By Lemma 2.4 (1), we have $\mathscr{L}_\mu(y_1 * (y_2 * x)) * x)' \geq \mathscr{L}_\mu(y_1)' \wedge \mathscr{L}_\mu(y_2)'$. Hence $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. □

Theorem 3.23. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice with $\leq$ a linearly ordered set. Then $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$ if and only if $L^-(\mathscr{L}_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $L^-(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in L^-(\mathscr{L}_\mu, t)$. Then $\mathscr{L}_\mu(x)$ and $\mathscr{L}_\mu(y)$ are compatible. Suppose that $\mathscr{L}_\mu(x) \leq \mathscr{L}_\mu(y)$, that is, $\mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) = \mathscr{L}_\mu(y)$. Then

$$\begin{aligned}
x, y \in L^-(\mathscr{L}_\mu, t) \;&\Rightarrow\; \mathscr{L}_\mu(x) < t, \mathscr{L}_\mu(y) < t \\
&\Rightarrow\; \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) = \mathscr{L}_\mu(y) < t \\
&\Rightarrow\; (\mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y))' = \mathscr{L}_\mu(y)' > t' \\
&\Rightarrow\; \mathscr{L}_\mu(x * y)' \geq \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)' > t' \\
&\Rightarrow\; \mathscr{L}_\mu(x * y)' > t' \\
&\Rightarrow\; \mathscr{L}_\mu(x * y) < t \\
&\Rightarrow\; x * y \in L^-(\mathscr{L}_\mu, t).
\end{aligned}$$

Hence, $L^-(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $L^-(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ if it is nonempty. Suppose there exist $x, y \in \mathcal{H}$ such that $\mathscr{L}_\mu(x * y)' \ngeq \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)'$. It means that $\mathscr{L}_\mu(x * y)' < \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)'$. By Lemma 2.4 (1), we have $\mathscr{L}_\mu(x * y)' < \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)' = (\mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y))'$. By Lemma 2.4 (5), we have $\mathscr{L}_\mu(x * y) > \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y)$. Choose $t = \mathscr{L}_\mu(x * y) \in \mathcal{L}$. Then $\mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) < t$ and so $\mathscr{L}_\mu(x) \leq \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) < t$ and $\mathscr{L}_\mu(y) \leq \mathscr{L}_\mu(x) \vee \mathscr{L}_\mu(y) < t$. Thus $x, y \in L^-(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, we get $L^-(\mathscr{L}_\mu, t)$ is a subalgebra of $\mathcal{H}$ and so $x * y \in L^-(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(x * y) < t = \mathscr{L}_\mu(x * y)$, a contradiction. Hence $\mathscr{L}_\mu(x * y)' \geq \mathscr{L}_\mu(x)' \wedge \mathscr{L}_\mu(y)'$ for all $x, y \in \mathcal{H}$. Hence $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}$. □

Lemma 3.24. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ a linearly ordered set and $\mathscr{L}'$ be an LFS in $\mathcal{H}$. Then $\mathscr{L}'$ satisfies the condition $\mathscr{L}_\mu(1) \geq \mathscr{L}_\mu(a)$ for all $a \in \mathcal{H}$ if and only if $L^-(\mathscr{L}_\mu, t)$, if it is nonempty, contains $1 \in \mathcal{H}$ for every $t \in \mathscr{L}$.*

Proof. Let $t \in \mathscr{L}$ be such that $L(\mathscr{L}_\mu, t) \neq \emptyset$. Let $a \in \mathcal{H}$. Then

$$\begin{aligned} a \in L(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(a) \leq t \\ &\Rightarrow \mathscr{L}_\mu(a)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1)' \geq \mathscr{L}_\mu(a)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1)' \geq t' \\ &\Rightarrow \mathscr{L}_\mu(1) \leq t \\ &\Rightarrow 1 \in L(\mathscr{L}_\mu, t). \end{aligned}$$

Conversely, assume for all $t \in \mathscr{L}$, $L(\mathscr{L}_\mu, t)$ contains $1 \in \mathcal{H}$ if it is nonempty. Suppose there exists $a \in \mathcal{H}$ such that $\mathscr{L}_\mu(1)' \ngeq \mathscr{L}_\mu(a)'$. It means that $\mathscr{L}_\mu(1)' < \mathscr{L}_\mu(a)'$. By Lemma 2.4 (5), we have $\mathscr{L}_\mu(1) > \mathscr{L}_\mu(a)$. Choose $t = \mathscr{L}_\mu(1) \in \mathcal{L}$. Then $\mathscr{L}_\mu(a) < t$. Thus $a \in L^-(\mathscr{L}_\mu, t) \neq \emptyset$. As the hypothesis, $1 \in L^-(\mathscr{L}_\mu, t)$. Thus $\mathscr{L}_\mu(1) < t = \mathscr{L}_\mu(1)$, a contradiction. Hence $\mathscr{L}_\mu(1)' \geq \mathscr{L}_\mu(a)'$ for all $a \in \mathcal{H}$. □

Theorem 3.25. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ be a Boolean lattice with $\leq$ a linearly ordered set. Then $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$ if and only if $L(\mathscr{L}_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}$ for every $t \in \mathcal{L}$.*

Proof. Assume $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. Let $t \in \mathcal{L}$ be such that $L^-(\mathscr{L}_\mu, t) \neq \emptyset$. Let $x, y \in \mathcal{H}$. Then

$$\begin{aligned} y \in L^-(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y) < t \\ &\Rightarrow \mathscr{L}_\mu(y)' > t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' \geq \mathscr{L}_\mu(y)' > t' \\ &\Rightarrow \mathscr{L}_\mu(x * y)' > t' \\ &\Rightarrow \mathscr{L}_\mu(x * y) < t \\ &\Rightarrow x * y \in L^-(\mathscr{L}_\mu, t). \end{aligned}$$

Let $x, y_1, y_2 \in \mathcal{H}$. Then $\mathscr{L}_\mu(y_1)$ and $\mathscr{L}_\mu(y_2)$ are compatible. Suppose that $\mathscr{L}_\mu(y_1) \geq \mathscr{L}_\mu(y_2)$, that is, $\mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2) = \mathscr{L}_\mu(y_2)$. Then

$$\begin{aligned} y_1, y_2 \in L^-(\mathscr{L}_\mu, t) &\Rightarrow \mathscr{L}_\mu(y_1) < t, \mathscr{L}_\mu(y_2) < t \\ &\Rightarrow \mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2) < t \\ &\Rightarrow (\mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2))' > t' \\ &\Rightarrow \mathscr{L}_\mu(y_1)' \wedge \mathscr{L}_\mu(y_2)' = (\mathscr{L}_\mu(y_1) \vee \mathscr{L}_\mu(y_2))' > t' \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x)' \geq \mathscr{L}_\mu(y_1)' \wedge \mathscr{L}_\mu(y_2)' > t' \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x)' > t' \\ &\Rightarrow \mathscr{L}_\mu((y_1 * (y_2 * x)) * x) < t \\ &\Rightarrow (y_1 * (y_2 * x)) * x \in L^-(\mathscr{L}_\mu, t). \end{aligned}$$

By Lemma 3.24, we have $1 \in L^{-}(\mathscr{L}_{\mu}, t)$. Hence $L^{-}(\mathscr{L}_{\mu}, t)$ is an ideal of $\mathcal{H}$. Conversely, assume for all $t \in \mathcal{L}$, $L^{-}(\mathscr{L}_{\mu}, t)$ is an ideal of $\mathcal{H}$ if it is nonempty. Suppose there exist $x, y \in \mathcal{H}$ such that $\mathscr{L}_{\mu}(x * y)' \ngeq \mathscr{L}_{\mu}(y)'$. It means that $\mathscr{L}_{\mu}(x * y)' < \mathscr{L}_{\mu}(y)'$. By Lemma 2.4 (5), we have $\mathscr{L}_{\mu}(x * y) > \mathscr{L}_{\mu}(y)$. Choose $t = \mathscr{L}_{\mu}(x * y) \in \mathcal{L}$. Then $\mathscr{L}_{\mu}(y) < t$ and so $\mathscr{L}_{\mu}(x * y) \leq \mathscr{L}_{\mu}(x * y) \vee \mathscr{L}_{\mu}(y) < t$. Thus $y \in L^{-}(\mathscr{L}_{\mu}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mathscr{L}_{\mu}, t)$ is an ideal of $\mathcal{H}$ and so $x * y \in L^{-}(\mathscr{L}_{\mu}, t)$. Thus $\mathscr{L}_{\mu}(x * y) < t = \mathscr{L}_{\mu}(x * y)$, a contradiction. Hence $\mathscr{L}_{\mu}(x * y)' \geq \mathscr{L}_{\mu}(y)'$ for all $x, y \in \mathcal{H}$. Suppose there exist $x, y_1, y_2 \in \mathcal{H}$ such that $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x)' \ngeq \mathscr{L}_{\mu}(y_1)' \wedge \mathscr{L}_{\mu}(y_2)'$. It means that $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x)' < \mathscr{L}_{\mu}(y_1)' \wedge \mathscr{L}_{\mu}(y_2)'$. By Lemma 2.4 (1), we have $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x)' < \mathscr{L}_{\mu}(y_1)' \wedge \mathscr{L}_{\mu}(y_2)' = (\mathscr{L}_{\mu}(y_1) \vee \mathscr{L}_{\mu}(y_2))'$. By Lemma 2.4 (5), we have $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x) > \mathscr{L}_{\mu}(y_1) \vee \mathscr{L}_{\mu}(y_2)$. Choose $t = \mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x) \in \mathcal{L}$. Then $\mathscr{L}_{\mu}(y_1) \vee \mathscr{L}_{\mu}(y_2) < t$ and so $\mathscr{L}_{\mu}(y_1) \leq \mathscr{L}_{\mu}(y_1) \vee \mathscr{L}_{\mu}(y_2) < t$ and $\mathscr{L}_{\mu}(y_2) \leq \mathscr{L}_{\mu}(y_1) \vee \mathscr{L}_{\mu}(y_1) < t$. Thus $y_1, y_2 \in L^{-}(\mathscr{L}_{\mu}, t) \neq \emptyset$. As the hypothesis, we get $L^{-}(\mathscr{L}_{\mu}, t)$ is an ideal of $\mathcal{H}$ and so $y_1 * (y_2 * x)) * x \in L^{-}(\mathscr{L}_{\mu}, t)$. Thus $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x) < t = \mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x)$, a contradiction. Hence $\mathscr{L}_{\mu}(y_1 * (y_2 * x)) * x)' \geq \mathscr{L}_{\mu}(y_1)' \wedge \mathscr{L}_{\mu}(y_2)'$ for all $x, y \in \mathcal{H}$. Hence $\mathscr{L}'$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}$. □

4. Cartesian Product of LFSs

Definition 4.1. Let $\mathscr{L}$ and $\mathscr{M}$ be LFSs in nonempty sets $\mathcal{U}_1$ and $\mathcal{U}_2$, respectively. The Cartesian product of $\mathscr{L}$ and $\mathscr{M}$ is $\mathscr{L} \times \mathscr{M} : \mathcal{U}_1 \times \mathcal{U}_2 \to L$ described by its membership function $(\mathscr{L} \times \mathscr{M})_{\mu}$ such that $(\forall a \in \mathcal{U}_1, b \in \mathcal{U}_2)((\mathscr{L} \times \mathscr{M})_{\mu}(a, b) = \mathscr{L}_{\mu}(a) \wedge \mathscr{M}_{\mu}(b))$. It is clearly that $(\mathscr{L} \times \mathscr{M})$ is an LFS in $\mathcal{U}_1 \times \mathcal{U}_2$.

Remark 4.2. Let $\mathcal{H}_1 = (\mathcal{H}_1, *, 1_1)$ and $\mathcal{H}_2 = (\mathcal{H}_2, \circ, 1_2)$ be Hilbert algebras. We can easily prove that $\mathcal{H}_1 \times \mathcal{H}_2$ is a Hilbert algebra defined by $(\forall a, b \in \mathcal{H}_1, u, v \in \mathcal{H}_2)((a, u) \circledast (b, v) = (a * b, u \circ v))$.

Theorem 4.3. *Let $\mathscr{L}$ and $\mathscr{M}$ be $\mathcal{L}$-fuzzy subalgebras of Hilbert algebras $\mathcal{H}_1 = (\mathcal{H}_1, *, 1_1)$ and $\mathcal{H}_2 = (\mathcal{H}_2, \circ, 1_2)$, respectively. Then $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy subalgebra of a Hilbert algebra $\mathcal{H}_1 \times \mathcal{H}_2$.*

Proof. Let $a, b \in \mathcal{H}_1$, $u, v \in \mathcal{H}_2$. Then

$$\begin{aligned}
(\mathscr{L} \times \mathscr{M})_{\mu}((a, u) \circledast (b, v)) &= (\mathscr{L} \times \mathscr{M})_{\mu}(a * b, u \circ v) \\
&= \mathscr{L}_{\mu}(a * b) \wedge \mathscr{M}_{\mu}(u \circ v) \\
&\geq (\mathscr{L}_{\mu}(a) \wedge \mathscr{L}_{\mu}(b)) \wedge (\mathscr{M}_{\mu}(u) \wedge \mathscr{M}_{\mu}(v)) \\
&= (\mathscr{L}_{\mu}(a) \wedge \mathscr{M}_{\mu}(u)) \wedge (\mathscr{L}_{\mu}(b) \wedge \mathscr{M}_{\mu}(v)) \\
&= (\mathscr{L} \times \mathscr{M})_{\mu}(a, u) \wedge (\mathscr{L} \times \mathscr{M})_{\mu}(b, v).
\end{aligned}$$

Hence, $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}_1 \times \mathcal{H}_2$. □

Theorem 4.4. *Let $\mathscr{L}$ and $\mathscr{M}$ be $\mathcal{L}$-fuzzy ideals of Hilbert algebras $\mathcal{H}_1 = (\mathcal{H}_1, *, 1_1)$ and $\mathcal{H}_2 = (\mathcal{H}_2, \circ, 1_2)$, respectively. Then $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy ideal of a Hilbert algebra $\mathcal{H}_1 \times \mathcal{H}_2$.*

Proof. Let $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then

$$\begin{aligned}(\mathscr{L} \times \mathscr{M})_\mu(1,1) &= \mathscr{L}_\mu(1) \wedge \mathscr{M}_\mu(1)\\ &\geq \mathscr{L}_\mu(x) \wedge \mathscr{M}_\mu(y)\\ &= (\mathscr{L} \times \mathscr{M})_\mu(x,y).\end{aligned}$$

Let $(x_1, x_2), (y_1, y_2) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then

$$\begin{aligned}(\mathscr{L} \times \mathscr{M})_\mu((x_1,x_2) * (y_1,y_2)) &= (\mathscr{L} \times \mathscr{M})_\mu((x_1 * y_1),(x_2 * y_2))\\ &= \mathscr{L}_\mu((x_1 * y_1)) \wedge \mathscr{M}_\mu((x_2 * y_2))\\ &\geq \mathscr{L}_\mu(y_1) \wedge \mathscr{L}_\mu(y_2)\\ &= (\mathscr{L} \times \mathscr{M})_\mu(y_1,y_2),\end{aligned}$$

Let $(x_1, y_1), (x_2, y_2), (x_3, y_3) \in \mathcal{H}_1 \times \mathcal{H}_2$. Then
$(\mathscr{L} \times \mathscr{M})_\mu(((x_2, y_2) * ((x_3, y_3) * (x_1, y_1))) * (x_1, y_1))$

$$\begin{aligned}&= (\mathscr{L} \times \mathscr{M})_\mu((x_2 * (x_3 * x_1)) * x_1, (y_2 * (y_3 * y_1)) * y_1)\\ &= \mathscr{L}_\mu((x_2 * (x_3 * x_1)) * x_1) \wedge \mathscr{M}_\mu((y_2 * (y_3 * y_1)) * y_1)\}\\ &\geq (\mathscr{L}_\mu(x_2) \wedge \mathscr{M}_\mu(x_3)) \wedge \mathscr{L}_\mu(y_2) \wedge \mathscr{M}_\mu(y_3)\\ &= (\mathscr{L}_\mu(x_2) \wedge \mathscr{M}_\mu(y_2)) \wedge (\mathscr{L}_\mu(x_3) \wedge \mathscr{M}_\mu(y_3)\\ &= (\mathscr{L} \times \mathscr{M})_\mu(x_2,y_2) \wedge (\mathscr{L} \times \mathscr{M})_\mu(x_3,y_3).\end{aligned}$$

Hence $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}_1 \times \mathcal{H}_2$. □

The following theorem is a straightforward result of Theorems 3.13, 3.15 and **??**.

Theorem 4.5. (1) *An LFS $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}_1 \times \mathcal{H}_2$ if and only if $U((\mathscr{L} \times \mathscr{M})_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}_1 \times \mathcal{H}_2$ for every $t \in \mathcal{L}$.*

(2) *An LFS $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ if and only if $U((\mathscr{L} \times \mathscr{M})_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ for every $t \in \mathcal{L}$.*

The following theorem is a straightforward result of Theorems 3.16 and 3.18.

Theorem 4.6. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee)$ be a linearly ordered set. Then the following statements are true.*

(1) *An LFS $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy subalgebra of $\mathcal{H}_1 \times \mathcal{H}_2$ if and only if $U^+((\mathscr{L} \times \mathscr{M})_\mu, t)$ is, if it is nonempty, a subalgebra of $\mathcal{H}_1 \times \mathcal{H}_2$ for every $t \in \mathcal{L}$.*

(2) *An LFS $\mathscr{L} \times \mathscr{M}$ is an $\mathcal{L}$-fuzzy ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ if and only if $U^+((\mathscr{L} \times \mathscr{M})_\mu, t)$ is, if it is nonempty, an ideal of $\mathcal{H}_1 \times \mathcal{H}_2$ for every $t \in \mathcal{L}$.*

The following theorem is a straightforward result of Theorems 3.20 and 3.22.

Theorem 4.7. *Let $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee,', 0_\mathcal{L}, 1_\mathcal{L})$ be a Boolean lattice. Then the following statements are true.*

(1) *An LFS* $(\mathscr{L} \times \mathscr{M})'$ *is an* $\mathcal{L}$*-fuzzy subalgebra of* $\mathcal{H}_1 \times \mathcal{H}_2$ *if and only if* $L((\mathscr{L} \times \mathscr{M})_\mu, t)$ *is, if it is nonempty, a subalgebra of* $\mathcal{H}_1 \times \mathcal{H}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $(\mathscr{L} \times \mathscr{M})'$ *is an* $\mathcal{L}$*-fuzzy ideal of* $\mathcal{H}_1 \times \mathcal{H}_2$ *if and only if* $L((\mathscr{L} \times \mathscr{M})_\mu, t)$ *is, if it is nonempty, an ideal of* $\mathcal{H}_1 \times \mathcal{H}_2$ *for every* $t \in \mathcal{L}$.

The following theorem is a straightforward result of Theorems 3.23 and 3.25.

Theorem 4.8. *Let* $\mathcal{L} = (\mathcal{L}, \leq, \wedge, \vee, ', 0_{\mathcal{L}}, 1_{\mathcal{L}})$ *be a Boolean lattice with* $\leq$ *is a linear order. Then the following statements are true.*

(1) *An LFS* $(\mathscr{L} \times \mathscr{M})'$ *is an* $\mathcal{L}$*-fuzzy subalgebra of* $\mathcal{H}_1 \times \mathcal{H}_2$ *if and only if* $L^-((\mathscr{L} \times \mathscr{M})_\mu, t)$ *is, if it is nonempty, a subalgebra of* $\mathcal{H}_1 \times \mathcal{H}_2$ *for every* $t \in \mathcal{L}$.

(2) *An LFS* $(\mathscr{L} \times \mathscr{M})'$ *is an* $\mathcal{L}$*-fuzzy ideal of* $\mathcal{H}_1 \times \mathcal{H}_2$ *if and only if* $L^-((\mathscr{L} \times \mathscr{M})_\mu, t)$ *is, if it is nonempty, an ideal of* $\mathcal{H}_1 \times \mathcal{H}_2$ *for every* $t \in \mathcal{L}$.

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Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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