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ON SEQUENCES OF ZEROS OF POLYNOMIALS INVOLVING PISOT AND SALEM NUMBERS

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Abstract. In this paper, we investigate zeros of a family of polynomials that involve Pisot numbers and Salem Numbers. We establish that the sequences of those zeros are monotone.

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1. Introduction

A real algebraic integer θ is called a *Pisot number* (or a Pisot-Vijayaraghavan number or a P.V. number for short) if $\theta > 1$ and all of its conjugates lie inside the unit circle. The set of all Pisot numbers is usually denoted by $\mathcal{S}$. In 1955, Dufresnoy and Pisot [4] introduced a powerful method for investigating the property of $\mathcal{S}$, especially its derived set. This method can produce a family of minimal polynomails of Pisot numbers. In 1945, Salem [5] introduced a new class of algebraic integers that are now known as *Salem numbers*. A Salem number is a real algebraic integer $\tau > 1$ whose conjugates have modulus at most 1 and at least one of them has a modulus 1. The set of Salem numbers is normally denoted by $\mathcal{T}$. Boyd showed in [2] that indeed each Salem number can be derives via the minimal polynomial of a Pisot number. Furthermore, it was established in [1, Theorem 6.4.3, p. 113–115] that if R is the minimal polynomial of a Salem number, then there exist minimal polynomials P_1 and P_2 of Pisot numbers such that $(z^2+1)R(z) = zP_1(z) + P_1^*(z)$ and $(z-1)R(z) = zP_2(z) - P_2^*(z)$, where $P^*(z) = z^{\deg P}P(z^{-1})$. In this article, we will investigate some property of sequences of Salem numbers that can be derived by using the method invented by Dufresnoy and Pisot and the relation of the minimal polynomials of Salem numbers and Pisot numbers.

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2. Preliminary Results and Notations

Here let P be the minimal polynomial of a Pisot number θ and f denote the power series $f(z) := \frac{A(z)}{P^*(z)} = \sum_{n=0}^{\infty} u_n z^n$, where $A(z) = 1$ if $P^*(z) = P(z)$, otherwise $A(z) = \epsilon P(z)$, where $\epsilon = \text{sign } P(0)$. In [4, Théorème 1, p. 81] Dufresnoy and Pisot showed that the sequence of coefficients $\{u_n\}$ satisfies the recursive system of inequalities:

$$1 \leq u_0 ,$$

$$w_1 = u_0^2 - 1 \leq u_1 ,$$

$$w_m(u_0, \ldots, u_{m-1}) \leq u_m \leq w_m^*(u_0, \ldots, u_{m-1}) \text{ for } m \geq 2,$$

where $w_2^* = \infty$ if $u_0 = 1$. The values of w_m and w_m^* can be determined from the preceding $u_0, u_1 \ldots, u_{m-1}$ as follows. When the first m coefficients of the series f are given, let $D_m(z) = -z^m + d_1 z^{m-1} + \ldots + d_m$ be such that the Maclaurin series of $\dfrac{D_m(z)}{E_m(z)}$, where $E_m(z) = -z^m D_m(z^{-1})$, has the form

$$\frac{D_m(z)}{E_m(z)} = u_0 + u_1 z + \ldots + u_{m-1} z^{m-1} + \ldots . \tag{1}$$

Then w_m is the coefficient of z^m in the series (1). To determine w_m^*, let $\widetilde{D}_m(z) = z^m + d_1^* z^{m-1} + \ldots + d_m^*$ be such that

$$\frac{\widetilde{D}_m(z)}{\widetilde{E}_m(z)} = u_0 + u_1 z + \ldots + u_{m-1} z^{m-1} + \ldots ., \tag{2}$$

where $\widetilde{E}_m(z) = z^m \widetilde{D}_m(z^{-1})$. Then w_m^* is the coefficient of z^m in the series (2). We can easily obtain the first few polynomials D_m and $\widetilde{D}_m$:

$$D_1(z) = u_0 - z, \qquad \widetilde{D}_1(z) = z + u_0,$$

$$D_2(z) = u_0 + \frac{u_1}{1+u_0} z - z^2, \qquad \widetilde{D}_2(z) = u_0 + \frac{u_1}{1-u_0} z + z^2, \text{ if } u_0 \neq 1,$$

$$w_1 = u_0^2 - 1, \qquad w_1^* = 1 - u_0^2,$$

$$w_2 = \frac{u_1^2}{u_0+1} + u_0^2 - 1, \qquad w_2^* = 1 - u_0^2 + \frac{u_1^2}{u_0 - 1}.$$

Note that if $u_0 = 1$, $\widetilde{D}_2$ is not defined. In this case, we let $w_2^* = \infty$. The following are some facts about w_m and w_m^*:

(1) $w_m = u_m$ or $w_m^* = u_m$ only if $f = D_m/E_m$ or $f = \widetilde{D}_m/\widetilde{E}_m$, respectively. If this is the case, then $P = -D_m$ or $P = \widetilde{D}_m$, respectively, and $m = \deg P$. In either case we say that ***the rank of** f* is m.

(2) If $w_k < u_k < w_k^*$ for all $m \geq 2$, we say that ***the rank of** f **is infinite***. In this case the Pisot number θ is a limit point of $\mathcal{S}$.

Dufresnoy and Pisot [4, Théorème 1 & 2, p. 81–82] showed that if $2 \le n \le s$, where s is the rank of f, each D_n and $\widetilde{D}_n$ has a unique zero $\theta_n > 1$ and $\theta_n^+ > 1$, respectively, outside the unit circle, all the other zeros lying inside the unit circle. Hence if the coefficients of D_n and $\widetilde{D}_n$ are integers, then both are minimal polynomials of some Pisot numbers θ_n^- and θ_n^+, respectively. Furthermore, $\theta_n^- \le \theta \le \theta_n^+$ and the sequences $\{\theta_n\}_{n=1}^s$ and $\{\theta_n^+\}_{n=3}^s$ are monotone convergent to θ (see [4, p. 84] or [1, Theorem 7.1.5, p. 130]).

In this paper, we focus on the case $P \ne P^*$ and $|P(0)| \ge 2$. Hence we obtain $u_0 = |P(0)| \ge 2$ and $\deg P > 2$. In [4, Lemme 4 & 5, p.83] and [1, Theorem 7.1.2, p. 125], it was proved that the polynomials D_n and $\widetilde{D}_n$ can be derived by using the recurrence relations

$$D_{m+1}(z) = (1+z)D_m(z) - \frac{u_m - w_m}{u_{m-1} - w_{m-1}} z D_{m-1}(z), \tag{3}$$

$$\widetilde{D}_{m+1}(z) = (1+z)\widetilde{D}_m(z) - \frac{w_m^* - u_m}{w_{m-1}^* - u_{m-1}} z \widetilde{D}_{m-1}(z). \tag{4}$$

If f is of finite rank s, that is, either $P = -D_s$ or $P = \widetilde{D}_s$, then for all $k > s$ we get $w_k = u_k = w_k^*$,

$$D_k(z) := (1+z)^{k-s} D_s(z), \quad \widetilde{D}_k(z) := (1 - z^{k-s}) D_s(z),$$

when $P = -D_s$. For the case $P = \widetilde{D}_s$, we have

$$D_k(z) := (1 - z^{k-s})\widetilde{D}_s(z) \quad \text{and } \widetilde{D}_k(z) := (1+z)^{k-s}\widetilde{D}_s(z).$$

For $n \in \mathbb{N}$ let

$$Q_n^-(z) = z^n P(z) - P^*(z) \quad \text{and} \quad Q_n^+(z) = z^n P(z) + P^*(z).$$

These polynomials were introduced by Salem in [6] and were investigated further by Boyd [2]. It was proved that Q_n^+ is reciprocal and always has a zero $\tau_n^+ > 1$ and the remaining zeros are inside the closed unit disk. So τ_n^+ is a Salem number. For Q_n^-, it will have a zero τ_n^- outside the unit circle if, and only if, $n > \deg P - 2\frac{P'(1)}{P(1)}$. Similarly (see [3]), the polynomial $z^n D_m(z) - D_m^*(z)$ has at most one zero, say $\gamma_{n,m} > 1$ in $|z| > 1$. If no such zero exists, we let $\gamma_{n,m} = 1$. Also the polynomial $z^n \widetilde{D}_m(z) + \widetilde{D}_m^*(z)$ always has exactly a zero, say $\widetilde{\gamma}_{n,m} > 1$ in $|z| > 1$. From [1, Theorem 6.4.3, p. 113–115], we know that every minimal polynomial of a Salem number τ can be written as $\frac{zP(z) - P^*(z)}{z-1}$ for some minimal polynomial P of a Pisot number. Therefore, we can restrict n to be 1. For simplicity, let

$$Q(z) := zP(z) - P^*(z), \quad Q_m(z) := zD_m(z) - D_m^*(z),$$

$$\widetilde{Q}_m(z) := z\widetilde{D}_m(z) + \widetilde{D}_m^*(z), \quad \gamma_m := \gamma_{1,m}, \quad \text{and} \quad \widetilde{\gamma}_m := \widetilde{\gamma}_{1,m}.$$

As mentioned above, the polynomial Q will have a Salem number τ as one of its zeros only if $\deg P - 2\frac{P(1)}{P'(1)} < 1$. If no such zero exists, we let $\tau = 1$. Similarly, γ_m is greater than 1 only if $\deg D_m - 2\frac{D_m(1)}{D_m'(1)} < 1$. Therefore, it is worth studying the behavior of the sequence $K_m := m - 2\frac{D_m(1)}{D_m'(1)}$. In the next section we will show that the sequences $\{\gamma_m\}$ and $\{\widetilde{\gamma}_m\}$ are monotone. We also prove in Section 4 that the sequence $\{K_m\}$ is decreasing.

3. $\{\gamma_m\}$ and $\{\widetilde{\gamma}_m\}$ Are Monotone

Let τ, γ_m and $\tilde{\gamma}_m$ be the zeros of Q, Q_m and $\widetilde{Q}_m$, respectively, as defined in Section 2. We also suppose that $u_0 = |P(0)| \geq 2$ and s is the rank of f.

Theorem 1. *The sequence $\{\gamma_k\}_{k\geq 1}$ is non-decreasing.*

Proof. Since $\deg D_1 - 2\frac{D_1(1)}{D_1'(1)} = \frac{u_0+1}{u_0-1} > 1$, we have $\gamma_1 = 1$. It follows from the definition that $\gamma_2 \geq \gamma_1$. Next suppose that $\gamma_k \geq \gamma_{k-1}$ for some $k \geq 2$. For $2 \leq k \leq s-1$, by using the identity (3) we have

$$
\begin{aligned}
Q_{k+1}(z) =& zD_{k+1}(z) - D^*_{k+1}(z) \\
=& z\{(1+z)D_k(z) - a_k z D_{k-1}(z)\} - \{(1+z)D^*_k(z) - a_k z D^*_{k-1}(z)\} \\
=& (1+z)\{zD_k(z) - D^*_k(z)\} - a_k z\{zD_{k-1}(z) - D^*_{k-1}(z)\} \\
=& (1+z)Q_k(z) - a_k z Q_{k-1}(z),
\end{aligned} \tag{5}
$$

where $a_k = \frac{u_k - w_k}{u_{k-1} - w_{k-1}}$ which is positive. Since $\gamma_k \geq \gamma_{k-1}$ and the leading ocefficient of Q_{k-1} is negative, we obtain that $Q_{k-1}(\gamma_k) \leq 0$. And since γ_k is a zero of Q_k, it follows from (5) that $Q_{k+1}(\gamma_k) = -a_k\gamma_k Q_{k-1}(\gamma_k) \geq 0$. This implies $\gamma_{k+1} \geq \gamma_k$ for all $2 \leq k \leq s-1$. This proves the theorem for the case that s is infinity.

Now suppose that s is finite and $k > s$. So we have either $P = -D_s$ or $P = \widetilde{D}_s$. If $P = \widetilde{D}_s$, then $D_k(z) = (1+z)^{k-s}D_s(z)$ and

$$
Q_k(z) = z(1+z)^{k-s}D_s(z) - (1+z)^{k-s}D^*_s(z) = (1+z)^{k-s}Q_s(z).
$$

In this case we have $\gamma_k = \gamma_s = \tau$ for all $k > s$. If $P = \widetilde{D}_s$, then $D_k(z) = (1 - z^{k-s})\widetilde{D}_s(z)$ and

$$
\begin{aligned}
Q_k(z) =& z(1 - z^{k-s})\widetilde{D}_s(z) - (z^{k-s} - 1)\widetilde{D}^*_s(z) \\
=& (1 - z^{k-s})(z\widetilde{D}_s(z) + \widetilde{D}^*_s(z)) = (1 - z^{k-s})\widetilde{Q}_s(z).
\end{aligned}
$$

Hence, in this case $\gamma_k = \tilde{\gamma}_s = \tau$ for all $k > s$. The proof is now complete. □

Theorem 2. *The sequence $\{\tilde{\gamma}_k\}_{k\geq 2}$ is non-increasing.*

Proof. First note that since $\widetilde{Q}_1(z) = z\widetilde{D}_1(z) + \widetilde{D}^*_1(z) = z^2 + 2u_0 z + 1$, $\widetilde{Q}_1(z) > 0$ for all $z \geq 1$. So $\tilde{\gamma}_1$ does not exist. From the definition, we have

$$
\widetilde{Q}_2(z) = (1+z)\left(z^2 + \frac{u_0^2 - u_1 - 2u_0 + 1}{u_0 - 1}z + 1\right)
$$

which has $\tilde{\gamma}_2 = \frac{u_1+2u_0-u_0^2-1+\sqrt{(u_1+2u_0-u_0^2-1)^2-4(u_0-1)^2}}{2(u_0-1)} > 1$ as one of its zeros. For $2 \le k \le s-1$, by using the identity (4) we get

$$\begin{aligned}\widetilde{Q}_{k+1}(z) =& z\{(1+z)\widetilde{D}_k(z) - b_k z\widetilde{D}_{k-1}(z)\} + \{(1+z)\widetilde{D}_k^*(z) - b_k z\widetilde{D}_{k-1}^*(z)\} \\ =&(1+z)\widetilde{Q}_k(z) - b_k z\widetilde{Q}_{k-1}(z),\end{aligned} \tag{6}$$

where $b_k = \frac{w_k^*-u_k}{w_{k-1}^*-u_{k-1}}$. Note that $b_2 < 0$ and $b_k > 0$ for all $3 \le k \le s-1$. Since $\widetilde{Q}_1(\tilde{\gamma}_2) > 0$ and $b_2 < 0$, $\widetilde{Q}_3(\tilde{\gamma}_2) = -b_2\tilde{\gamma}_2\widetilde{Q}_1(\tilde{\gamma}_2) > 0$. It implies $\tilde{\gamma}_2 > \tilde{\gamma}_3$.

Now suppose that $\tilde{\gamma}_{k-1} > \tilde{\gamma}_k$ for some k, $3 \le k \le s-1$. So $\widetilde{Q}_{k-1}(\tilde{\gamma}_k) < 0$. From (6), we have $\widetilde{Q}_{k+1}(\tilde{\gamma}_k) = -b_k\tilde{\gamma}_k\widetilde{Q}_{k-1}(\tilde{\gamma}_k) > 0$ because $b_k > 0$ and $\widetilde{Q}_{k-1}(\tilde{\gamma}_k) < 0$. Hence we obtain $\tilde{\gamma}_k > \tilde{\gamma}_{k+1}$. This proves the theorem for the case that s is infinity.

Suppose that s is finite and $k > s$. So we have either $P = -D_s$ or $P = \widetilde{D}_s$. If $P = -D_s$, then $\widetilde{D}_k(z) = (1-z^{k-s})D_s(z)$ and

$$\begin{aligned}\widetilde{Q}_k(z) =& z(1-z^{k-s})D_s(z) + (z^{k-s}-1)D_s^*(z) \\ =&(1-z^{k-s})(zD_s(z) - D_s^*(z)) = (1-z^{k-s})Q_s(z).\end{aligned}$$

Thus we get $\tilde{\gamma}_k = \tau = \gamma_s$. If $P = \widetilde{D}_s$, then $\widetilde{D}_k(z) = (1+z)^{k-s}\widetilde{D}_s(z)$ and

$$\widetilde{Q}_k(z) = z(1+z)^{k-s}\widetilde{D}_s(z) + (1+z)^{k-s}\widetilde{D}_s^*(z) = (1+z)^{k-s}\widetilde{Q}_s(z).$$

This gives us $\tilde{\gamma}_k = \tau = \tilde{\gamma}_s$ for all $k > s$. The proof is now complete. □

Since the construction of D_{k+1} depends on choice of u_k, we next show that the values of γ_{k+1} and $\tilde{\gamma}_{k+1}$ decrease with respect to u_k.

Theorem 3. *For $2 \le k < s$, the values of γ_{k+1} and $\tilde{\gamma}_{k+1}$ decrease if u_k decreases.*

Proof. Recall that, from (5), we have

$$Q_{k+1}(z) = (1+z)Q_k(z) - a_k zQ_{k-1}(z),$$

where $a_k = \frac{u_k-w_k}{u_{k-1}-w_{k-1}}$. If $\gamma_k \le z \le \gamma_{k+1}$, then $Q_{k-1}(z) \le 0$, $Q_k(z) \le 0$ and $Q_{k+1}(z) \ge 0$ because $\gamma_{k-1} \le \gamma_k \le \gamma_{k+1}$. So, when z is fixed, the value of $(1+z)Q_k(z) - a_k zQ_{k-1}(z)$ decreases if a_k decreases (because $a_k > 0$). Hence the minimum value of γ_{k+1} is attained when a_k is the minimal, that is, when u_k is the minimal.

Similarly, from (6) we have

$$\widetilde{Q}_{k+1}(z) = (1+z)\widetilde{Q}_k(z) - b_k z\widetilde{Q}_{k-1}(z),$$

where $b_k = \frac{w_k^*-u_k}{w_{k-1}^*-u_{k-1}}$. We get $\widetilde{Q}_{k-1}(z) \le 0$, $\widetilde{Q}_k(z) \le 0$, and $\widetilde{Q}_{k+1}(z) \ge 0$ if $\tilde{\gamma}_{k+1} \le z \le \tilde{\gamma}_k$. Since $b_k > 0$ for $k \ge 3$, the value of $(1+z)\widetilde{Q}_k(z) - b_k z\widetilde{Q}_{k-1}(z)$ increases if b_k increases and z is fixed. If $\widetilde{Q}_{k+1}(z)$

increases, it implies that the value of $\tilde{\gamma}_{k+1}$ decreases (because $\widetilde{Q}_{k+1}(x) > 0\,, \forall x > \tilde{\gamma}_{k+1}$). Hence, the minimum value of $\tilde{\gamma}_{k+1}$ is attained when b_k is the maximal. Since $b_k = \frac{w_k^* - u_k}{w_{k-1}^* - u_{k-1}}$, b_k is the maximal if u_k is the minimal. This completes the proof. ☐

4. $\{K_m\}$ Is Decreasing

Recall we define

$$K_m := \begin{cases} \deg D_m - 2\frac{D_m'(1)}{D_m(1)} & \text{if } D_m(1) \neq 0, \\ -\infty & \text{if } D_m(1) = 0. \end{cases}$$

Theorem 4. *Suppose that $u_0 = |P(0)| > 1$. The sequence $\{K_m\}_{m\geq 1}$ is non-increasing. If s is finite and $P(z) = -D_s(z)$, then $K_m = K_s$ for all $m \geq s$. If s is finite and $P(z) = \widetilde{D}_s(z)$, then $K_m = -\infty$ for all $m > s$.*

Proof. Since $D_1(z) = u_0 - z$ and $D_2(z) = u_0 + \frac{u_1}{1+u_0}z - z^2$, we get

$$K_1 = 1 + \frac{2}{u_0 - 1} = \frac{u_0 + 1}{u_0 - 1}$$

$$K_2 = 2 - 2\frac{\frac{u_1}{1+u_0} - 2}{u_0 + \frac{u_1}{1+u_0} - 1} = 2\frac{(u_0+1)^2}{u_0^2 + u_1 - 1}.$$

Since $u_1 \in \mathbb{N}$ and $u_1 > w_1 = u_0^2 - 1$, we obtain

$$K_2 \leq 2\frac{(u_0+1)^2}{2u_0^2 - 1}.$$

We have $\frac{2(u_0+1)^2}{2u_0^2-1} \leq \frac{u_0+1}{u_0-1}$ if, and only if, $\frac{2(u_0^2-1)}{2u_0^2-1} \leq 1$ which is true for all $u_0 \geq 2$. Hence, $K_1 \geq K_2$.

Suppose that $K_n \leq K_{n-1}$ for some $2 \leq n \leq s-1$. We want to show that $K_{n+1} \leq K_n$. By the identity (3), we get

$$D_{n+1}(1) = 2D_n(1) - a_n D_{n-1}(1), \tag{7}$$

$$D_{n+1}'(1) = D_n(1) + 2D_n'(1) - a_n(D_{n-1}(1) + D_{n-1}'(1)), \tag{8}$$

where $a_n = \frac{u_n - w_n}{u_{n-1} - w_{n-1}} > 0$. For convenience, we write $D_k(1)$ and $D_k'(1)$ as D_k and D_k', respectively. We have $K_{n+1} \leq K_n$ if, and only if,

$$n + 1 - 2\frac{D_{n+1}'}{D_{n+1}} \leq n - 2\frac{D_n'}{D_n}$$

which is equivalent to

$$D_n(D_{n+1} - 2D_{n+1}') \leq -2D_n' D_{n+1}. \tag{9}$$

Note that $D_k(1) > 0$ for all $k \leq s$. From (7) and (8), we have

$$\begin{aligned} D_{n+1} - 2D_{n+1}' &= 2D_n - a_n D_{n-1} - 2\left[D_n + 2D_n' - a_n(D_{n-1} + D_{n-1}')\right] \\ &= a_n D_{n-1} - 4D_n' + 2a_n D_{n-1}'. \end{aligned}$$

So (9) becomes

$$D_n(a_nD_{n-1} - 4D'_n + 2a_nD'_{n-1}) \leq -2D'_n(2D_n - a_nD_{n-1})$$

which is equivalent to

$$\frac{D_{n-1} + 2D'_{n-1}}{D_{n-1}} \leq 2\frac{D'_n}{D_n} \tag{10}$$

because a_n, D_n and D_{n-1} are positive. Note that $K_n \leq K_{n-1}$ if, and only if, the inequality (10) holds. This shows that $K_{n+1} \leq K_n$, as desired. Therefore, by induction, we obtain that $K_n \leq K_{n-1}$ for all $2 \leq n \leq s$.

Now suppose that s is finite. If $P(z) = \widetilde{D}_s(z)$, then $D_n(1) = 0$ for all $n \geq s+1$. Hence $K_n = -\infty$ in this case. If $P(z) = -D_s(z)$, then $D_n(z) = (1+z)^{n-s}D_n(z)$ for all $n \geq s+1$. So for $n \geq s+1$

$$\begin{aligned} K_n =& n - 2\frac{(n-s)2^{n-s-1}D_s(1) + 2^{n-s}D'_s(1)}{2^{n-s}D_s(1)} \\ =& n - (n-s) - 2\frac{D'_s(1)}{D_s(1)} = K_s. \end{aligned}$$

The proof now is complete. □

Since the construction of D_m depends on choice of u_{m-1}, we can consider K_m as a function with variable u_{m-1}. The next theorem shows that K_m is decreasing with respect to u_{m-1}.

Theorem 5. *For $3 \leq m$, K_m is a decreasing function with respect to u_{m-1}.*

Proof. Here we write $D_n(1) = D_n$ for all n. By the definition and the identities (7) and (8), we kave

$$\begin{aligned} K_{m+1} =& m + 1 - 2\frac{D'_{m+1}}{D_{m+1}} \\ =& m + 1 - 2\frac{D_m + 2D'_m - a_m(D_{m-1} + D'_{m-1})}{2D_m - a_mD_{m-1}}, \end{aligned}$$

where $a_m = \frac{u_m - w_m}{u_{m-1} - w_{m-1}}$. We will show that K_{m+1} is a decreasing function with respect to u_m, that is, we want to show that

$$\frac{\partial K_{m+1}}{\partial u_m} = -2\frac{\partial}{\partial a_m}\left[\frac{D_m + 2D'_m - a_m(D_{m-1} + D'_{m-1})}{2D_m - a_mD_{m-1}}\right] \cdot \frac{\partial a_m}{\partial u_m} \leq 0.$$

Since $\frac{\partial a_m}{\partial u_m} = \frac{1}{u_{m-1} - w_{m-1}} > 0$, it suffices to show that

$$\frac{\partial}{\partial a_m}\left[\frac{D_m + 2D'_m - a_m(D_{m-1} + D'_{m-1})}{2D_m - a_mD_{m-1}}\right] \geq 0.$$

Since

$$\frac{D_m + 2D'_m - a_m(D_{m-1} + D'_{m-1})}{2D_m - a_mD_{m-1}} = 1 + \frac{-D_m + 2D'_m - a_mD'_{m-1}}{2D_m - a_mD_{m-1}},$$

we need to show that

$$\frac{\partial}{\partial a_m}\left[\frac{2D'_m - D_m - a_m D'_{m-1}}{2D_m - a_m D_{m-1}}\right] = \frac{-2D_m D'_{m-1} + D_{m-1}(2D'_m - D_m)}{(2D_m - a_m D_{m-1})^2} \geq 0. \tag{11}$$

Since, by Theorem 4, $\{K_m\}$ is decreasing, we can write

$$K_m = m - 2\frac{D'_m}{D_m} = 1 + \delta_m, \tag{12}$$

$$K_{m-1} = (m-1) - 2\frac{D'_{m-1}}{D_{m-1}} = 1 + \delta_{m-1}, \tag{13}$$

for some $\delta_{m-1} \geq \delta_m$. From (12) and (13), we get

$$2D'_m - D_m = (m - 2 - \delta_m)D_m \quad \text{and} \quad D'_{m-1} = \frac{m - 2 - \delta_{m-1}}{2}D_{m-1}.$$

Thus, we obtain

$$-2D_m D'_{m-1} + D_{m-1}(2D'_m - D_m) = D_m D_{m-1}(\delta_{m-1} - \delta_m) \geq 0$$

because $\delta_{m-1} \geq \delta_m$ and $D_m, D_{m-1} > 0$. Therefore the inequality (11) holds. This proves $\frac{\partial K_{m+1}}{\partial u_m} \geq 0$, as desired. □

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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