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DIFFERENT OPERATORS FOR THE POLYNOMIALS $S_n(\delta, \zeta, \lambda|q)$

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Abstract. In 2010, Saad and Sukhi defined the polynomials $S_n(\delta, \zeta, \lambda|q)$. They simply derived its generating function by utilizing the operator $L(b\theta_{xy})$. In this study, we provide Rogers' formula using the q-exponential operator $T(bD_q)$, Mehler's formulas using the operator $L(b\theta_{xy})$, and a linearization formula for the polynomials $S_n(\delta, \zeta, \lambda|q)$. In addition, we employ the Cauchy companion operator $E(a, b; \theta)$ to recover the generating function and provide the Rogers formula, Mehler's formula, and an extended Rogers formula for the polynomials $S_n(\delta, \zeta, \lambda|q)$.

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1. Introduction

We assume that $0 < |q| < 1$. The q-shifted factorial is given for every $a \in \mathbb{C}$ as [10, 13]:

$$(\beta; q)_0 = 1, \qquad (\beta; q)_r = \prod_{k=0}^{r-1}(1 - \beta q^k), \qquad (\beta; q)_\infty = \prod_{k=0}^{\infty}(1 - \beta q^k).$$

$$(\beta; q)_r = (\beta; q)_\infty / (\beta q^r; q)_\infty.$$

$$(\beta; q)_{r+k} = (\beta; q)_k \, (\beta q^k; q)_r.$$

and [1]

$$(\beta q^{-r}; q)_n = (-1)^r \beta^r q^{\binom{r}{2} - r^2} \, (q/\beta; q)_r. \tag{1.1}$$

The multiple q-shifted factorials is [13]

$$(\beta_1, \beta_2, \cdots, \beta_m; q)_r = (\beta_1; q)_r (\beta_2; q)_r \cdots (\beta_m; q)_r.$$

$$(\beta_1, \beta_2, \cdots, \beta_m; q)_\infty = (\beta_1; q)_\infty (\beta_2; q)_\infty \cdots (\beta_m; q)_\infty \,.$$

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The q-binomial coefficient is [2]:

$$\begin{bmatrix} r \\ t \end{bmatrix} = \frac{(q;q)_r}{(q;q)_t (q;q)_{r-t}}.$$

The q-hypergeometric series is [13]:

$${}_i\phi_j \left(\begin{matrix} \beta_1, \ldots, \beta_i \\ \alpha_1, \ldots, \alpha_j \end{matrix} ; q, x \right) = \sum_{r=0}^{\infty} \frac{(\beta_1, \ldots, \beta_i; q)_r}{(q, \alpha_1, \ldots, \alpha_j; q)_r} \left[(-1)^r q^{\binom{r}{2}} \right]^{1+j-i} x^r.$$

The Cauchy identity is [13,16]:

$$\sum_{t=0}^{\infty} \frac{(\beta;q)_k}{(q;q)_t} \mu^t = \frac{(\beta\mu;q)_\infty}{(\mu;q)_\infty}, \quad |\mu| < 1.$$

The Euler's identities are [13]:

$$\sum_{r=0}^{\infty} \frac{\mu^r}{(q;q)_r} = \frac{1}{(\mu;q)_\infty}, \quad |\mu| < 1. \tag{1.2}$$

$$\sum_{r=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} \mu^r}{(q;q)_r} = (\mu;q)_\infty. \tag{1.3}$$

Cauchy polynomials are given by [17,18,20]:

$$P_n(w,t) = (w-t)(w-qt)\cdots(w-q^{n-1}t) = (t/w;q)_n\, w^n,$$

which has the generating function [4,5,7]:

$$\sum_{n=0}^{\infty} P_n(w,t) \frac{\nu^n}{(q;q)_n} = \frac{(t\nu;q)_\infty}{(w\nu;q)_\infty}, \quad |w\nu| < 1, \tag{1.4}$$

The q-differential operator, or q-derivative, is defined by [8,15]:

$$D_q \{f(a)\} = \frac{f(a) - f(aq)}{a}.$$

The q-exponential operator was defined as follows [8,22] as

$$T(\lambda D_q) = \sum_{n=0}^{\infty} \frac{(\lambda D_q)^n}{(q;q)_n}. \tag{1.5}$$

Chen et al. [11] presented the following identity:

$$T(\lambda D_q) \left\{ \frac{(z\alpha, x\alpha; q)_\infty}{(y\alpha;q)_\infty} \right\} = \frac{(x\alpha, z\alpha;q)_\infty}{(y\alpha;q)_\infty} \sum_{r=0}^{\infty} \frac{(-1)^r q^{\binom{r}{2}} (y\alpha;q)_r (z\lambda)^r}{(q;q)_r (x\alpha, z\alpha;q)_r} {}_2\phi_1 \left(\begin{matrix} x/y, 0 \\ x\alpha q^r \end{matrix} ; q, y\lambda \right). \tag{1.6}$$

where $\max\{|y\alpha|, |y\lambda|\} < 1$.

A special type of Rogers-Szegö polynomials is [12]:

$$r_i(g,f) = \sum_{k=0}^{i} \begin{bmatrix} i \\ k \end{bmatrix} g^k f^{i-k},$$

such that

$$T(fD_q)\{g^n\} = r_n(g,f). \tag{1.7}$$

The operator θ is defined by [9,14,19]:

$$\theta\{f(a)\} = \frac{f(aq^{-1}) - f(a)}{aq^{-1}}.$$

Chen [6] defined the following Cauchy companion operator:

$$E(d,e;\theta) = \sum_{n=0}^{\infty} \frac{(d;q)_n}{(q;q)_n}(e\theta)^n. \tag{1.8}$$

Some operator identities was given [6]:

Proposition 1.1. [6]. *We have*

$$E(d,e;\theta)\{w^n\} = \sum_{t=0}^{n} \begin{bmatrix} n \\ t \end{bmatrix} (d;q)_t(-eq)^t w^{n-t} q^{\binom{t}{2}} q^{-nt}. \tag{1.9}$$

$$E(d,e;\theta)\{(wt;q)_\infty\} = \frac{(det,wt;q)_\infty}{(et;q)_\infty}, \quad |e\,t| < 1. \tag{1.10}$$

$$E(d,e;\theta)\left\{\frac{(wt;q)_\infty}{(wv;q)_\infty}\right\} = \frac{(wt;q)_\infty}{(wv;q)_\infty}\, {}_2\phi_1\left(\begin{matrix} d, t/v \\ q/wv \end{matrix}; q, eq/w\right), \quad \max\{|wv|, |eq/w|\} < 1. \tag{1.11}$$

$$E(d,e;\theta)\left\{\frac{(ws,wt;q)_\infty}{(wv;q)_\infty}\right\} = \frac{(deq/w,ws,wt;q)_\infty}{(eq/w,wv;q)_\infty}\, {}_3\phi_2\left(\begin{matrix} d, q/ws, q/wt \\ deq/w, q/wv \end{matrix}; q, est/v\right), \tag{1.12}$$

where max $\max\{|wv|, |eq/w|, |est/v|\} < 1.$

The homogeneous q-difference operator is [21]:

$$\theta_{xy}\{f(x,y)\} = \frac{f(q^{-1}x,y) - f(x,qy)}{q^{-1}x - y}.$$

The homogeneous q-shift operator is [21]:

$$L(b\theta_{xy}) = \sum_{r=0}^{\infty} \frac{q^{\binom{r}{2}}(b\theta_{xy})^r}{(q;q)_r}.$$

Proposition 1.2. [21]. *We have*

$$L(b\theta_{\alpha\beta})\{P_t(\beta,\alpha)\} = \sum_{k=0}^{t} \begin{bmatrix} t \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} b^k P_{t-k}(\beta,\alpha). \tag{1.13}$$

$$L(b\theta_{\alpha\beta})\left\{\frac{(\alpha t;q)_\infty}{(\beta t;q)_\infty}\right\} = \frac{(\alpha t, bt;q)_\infty}{(\beta t;q)_\infty}, \quad |\beta t| < 1. \tag{1.14}$$

Saad and Sukhi [21] defined the polynomials

$$S_n(\delta,\zeta,\lambda|q) = \sum_{k=0}^{j} \begin{bmatrix} j \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}}\, \lambda^k\, P_{j-k}(\zeta,\delta), \tag{1.15}$$

with the generating function

$$\sum_{r=0}^{\infty} S_r(\delta,\zeta,\lambda|q)\frac{t^r}{(q;q)_r}=\frac{(\delta t,\lambda t;q)_\infty}{(\zeta t;q)_\infty},\quad |\zeta t|<1. \tag{1.16}$$

They represented $S_n(\delta,\zeta,\lambda|q)$ by the homogeneous q-shift operator as:

$$L(\lambda\theta_{\delta\zeta})\left\{P_n(\zeta,\delta)\right\}=S_n(\delta,\zeta,\lambda|q). \tag{1.17}$$

Abdlhusein [3] provided the transformation

$${}_1\phi_1\left(\begin{matrix} wt \\ dt\end{matrix};q,ds\right)=\frac{(wt,ds;q)_\infty}{(dt;q)_\infty}\,{}_2\phi_1\left(\begin{matrix} d/w,0 \\ ds\end{matrix};q,wt\right), \tag{1.18}$$

where $\max\{|dt|,|wt|\}<1$.

The following describes the paper's structure: The generating function for $S_n(\delta,\zeta,\lambda|q)$ is recovered in section 2 by employing the operator $E(\delta/\zeta,\zeta;\theta)$. In section 3, we present two Rogers formulations utilizing the operators $E(\delta/\zeta,\zeta;\theta)$ and $T(tD_q)$, respectively. Our formula for linearizing $E(\delta/\zeta,\zeta;\theta)$ is derived from the first Rogers formula. Additionally, by utilizing the operator $E(\delta/\zeta,\zeta;\theta)$, we provide an extension for the Rogers formula for the polynomials $S_n(\delta,\zeta,\lambda|q)$. Using the operator $E(\delta/\zeta,\zeta;\theta)$, we present two Mehlers formulas in section 4 for $S_n(\delta,\zeta,\lambda|q)$.

2. The Generating Function for the Polynomials $S_n(\delta,\zeta,\lambda;q)$

The generating function for the polynomials $S_n(\delta,\zeta,\lambda;q)$ is recovered in this section.

The generating function for the polynomials $S_n(\delta,\zeta,\lambda;q)$ is recovered in this section by employing the Cauchy companion operator.

Suppose that $E(\delta/\zeta,\zeta;\theta)$ acts on λ. We give the following representation:

Theorem 2.1. *We have*

$$E(\delta/\zeta,\zeta;\theta)\{(-1)^n q^{\binom{n}{2}}\lambda^n\}=S_n(\delta,\zeta,\lambda;q). \tag{2.1}$$

Proof.

$$\begin{aligned}
&E(\delta/\zeta,\zeta;\theta)\{(-1)^n q^{\binom{n}{2}}\lambda^n\}\\
&=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(-1)^n q^{\binom{n}{2}}(\delta/\zeta;q)_k(-\zeta q)^k\lambda^{n-k}q^{\binom{k}{2}-nk}\quad \text{(by using (1.9))}\\
&=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(-1)^{n+k}P_k(\zeta,\delta)\lambda^{n-k}q^{\binom{n}{2}+\binom{k}{2}+k-nk}\\
&=\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}(-1)^{k}P_{n-k}(\zeta,\delta)\lambda^{k}q^{\binom{n}{2}+\binom{n-k}{2}+(n-k)-n(n-k)}
\end{aligned}$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k q^{\binom{k}{2}} \lambda^k P_{n-k}(\zeta,\delta)$$
$$= S_n(\delta,\zeta,\lambda;q) \quad \text{(by using (1.15))}.$$

□

The generating function for $S_n(\delta,\zeta,\lambda;q)$ is recovered depending on the operator representation (2.1) with identity (1.10).

Theorem 2.2 (The generating function for $S_n(\delta,\zeta,\lambda;q)$)**.** *We have*

$$\sum_{n=0}^{\infty} S_n(\delta,\zeta,\lambda;q)\frac{\nu^n}{(q;q)_n} = \frac{(\delta\nu,\lambda\nu;q)_\infty}{(\zeta\nu;q)_\infty}, \quad |\zeta\nu|<1.$$

Proof.

$$\begin{aligned}
&\sum_{r=0}^{\infty} S_r(\delta,\zeta,\lambda;q)\frac{\nu^r}{(q;q)_r} \\
&\quad= \sum_{r=0}^{\infty} E(\delta/\zeta,\zeta;\theta)\{(-1)^r\, q^{\binom{r}{2}}\, \lambda^r\}\frac{\nu^r}{(q;q)_r} \quad \text{(by using (2.1))} \\
&\quad= E(\delta/\zeta,\zeta;\theta)\left\{\sum_{r=0}^{\infty}\frac{(-1)^r\, q^{\binom{r}{2}}\,(\lambda\nu)^r}{(q;q)_r}\right\} \\
&\quad= E(\delta/\zeta,\zeta;\theta)\,\{(\lambda\nu;q)_\infty\} \quad \text{(by using (1.10))} \\
&\quad= \frac{(\delta\nu,\lambda\nu;q)_\infty}{(\zeta\nu;q)_\infty}.
\end{aligned}$$

□

3. Two Rogers Formulas for the Polynomials $S_n(\delta,\zeta,\lambda|q)$

In this section, two Rogers formulas are provided using the operators $E(\delta/\zeta,\zeta;\theta)$ and $T(tD_q)$, respectively.

The first Rogers formula for $S_n(\delta,\zeta,\lambda|q)$ will be given by employing the operator $T(\lambda D_q)$.

Theorem 3.1 (First Rogers formula for $S_n(\delta,\zeta,\lambda|q)$)**.** *We have*

$$\begin{aligned}
&\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty} S_{r+\ell}(\delta,\zeta,\lambda|q)\,\frac{t^r}{(q;q)_r}\frac{v^\ell}{(q;q)_\ell} \\
&\quad= \frac{(\delta v,\lambda v;q)_\infty}{(\zeta v;q)_\infty}\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\zeta v;q)_\rho(\lambda t)^\rho}{(q;q)_\rho(\delta v,\lambda v;q)_\rho}\,{}_2\phi_1\left(\begin{matrix}\delta/\zeta,0\\ \delta v q^\rho\end{matrix};q,\zeta t\right),
\end{aligned} \tag{3.1}$$

where $\max\{|\zeta v|,|\zeta t|\}<1$.

Proof. Let $\ell \to \ell - r$ in the L.H.S. of equation (3.1), we have

$$\begin{aligned}
&\sum_{r=0}^{\infty}\sum_{\ell=r}^{\infty} S_\ell(\delta,\zeta,\lambda|q)\,\frac{t^r}{(q;q)_r}\frac{v^{\ell-r}}{(q;q)_{\ell-r}}\\
&\quad=\sum_{\ell=0}^{\infty}\sum_{r=0}^{\ell} S_\ell(\delta,\zeta,\lambda|q)\frac{t^r}{(q;q)_r}\frac{v^{\ell-r}}{(q;q)_{\ell-r}}\\
&\quad=\sum_{\ell=0}^{\infty} S_\ell(\delta,\zeta,\lambda|q)\frac{1}{(q;q)_\ell}\sum_{r=0}^{\ell}\begin{bmatrix}\ell\\ r\end{bmatrix} v^{\ell-r}t^r\\
&\quad=\sum_{\ell=0}^{\infty} S_\ell(\delta,\zeta,\lambda|q)\frac{1}{(q;q)_\ell}\, r_\ell(v,t)\\
&\quad=\sum_{\ell=0}^{\infty} S_\ell(\delta,\zeta,\lambda|q))\frac{1}{(q;q)_\ell}\, T(tD_q)\{v^\ell\}\quad \text{(by using (1.7))}\\
&\quad=T(tD_q)\left\{\sum_{\ell=0}^{\infty} S_\ell(\delta,\zeta,\lambda|q)\frac{v^\ell}{(q;q)_\ell}\right\}\\
&\quad=T(tD_q)\left\{\frac{(\delta v,\lambda v;q)_\infty}{(\zeta v;q)_\infty}\right\}\quad \text{(by using (1.16))}\\
&\quad=\frac{(\delta v,\lambda v;q)_\infty}{(\zeta v;q)_\infty}\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\zeta v;q)_\rho(\lambda t)^\rho}{(q;q)_\rho(\delta v,\lambda v;q)_\rho}\,{}_2\phi_1\left(\begin{matrix}\delta/\zeta,0\\ \delta v q^\rho\end{matrix};q,\zeta t\right)\quad \text{(by using (1.6))}.
\end{aligned}$$

□

The linearization formula for $S_n(\delta,\zeta,\lambda|q)$ can be derived from the first Rogers formula.

Theorem 3.2 (Linearization formula for $S_n(\delta,\zeta,\lambda|q)$)**.** *We have*

$$S_{\ell+r}(\delta,\zeta,\lambda|q) = \sum_{\rho=0}^{\ell}\begin{bmatrix}\ell\\ \rho\end{bmatrix}(-1)^\rho q^{\binom{\rho}{2}+\rho r}\,\lambda^\rho\,P_{\ell-\rho}(\zeta,\delta)\,S_r(\delta q^{\ell-\rho},\zeta,\lambda|q). \tag{3.2}$$

Proof. Write equation (3.1) as

$$\begin{aligned}
&\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty} S_{\ell+r}(\delta,\zeta,\lambda|q)\,\frac{t^\ell}{(q;q)_\ell}\frac{v^r}{(q;q)_r}\\
&\quad=\frac{(\delta v,\lambda v;q)_\infty}{(\zeta \ell;q)_\infty}\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\zeta v;q)_\rho(\lambda t)^\rho}{(q;q)_\rho(\delta v,\lambda v;q)_\rho}\sum_{\ell=0}^{\infty}\frac{(\delta/\zeta;q)_\ell}{(\delta v q^\rho;q)_\ell(q;q)_\ell}(\zeta t)^\ell\\
&\quad=\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\lambda t)^\rho}{(q;q)_\rho}\sum_{\ell=0}^{\infty}\frac{(\delta/\zeta;q)_\ell\,(\zeta t)^\ell}{(q;q)_\ell}\sum_{r=0}^{\infty} S_r(\delta q^\ell,\zeta,\lambda|q)\frac{(vq^\rho)^r}{(q;q)_r}\quad \text{(by using (1.16))}.
\end{aligned}$$

Letting $\ell \to \ell - \rho$, we get

$$\sum_{r=0}^{\infty}\sum_{\ell=0}^{\infty} S_{\ell+r}(\delta,\zeta,\lambda|q)\,\frac{t^\ell}{(q;q)_\ell}\frac{v^r}{(q;q)_r}$$

$$= \sum_{\rho=0}^{\infty} \frac{(-1)^{\rho} q^{\binom{\rho}{2}} \lambda^{\rho}}{(q;q)_{\rho}} \sum_{\ell=\rho}^{\infty} \frac{(\delta/\zeta;q)_{\ell-\rho}\, \zeta^{\ell-\rho}}{(q;q)_{\ell-\rho}} \sum_{r=0}^{\infty} S_r(\delta q^{\ell-\rho}, \zeta, \lambda; q) \frac{q^{\rho r}}{(q;q)_r}\, t^{\ell}\, v^{r}$$

$$= \sum_{\ell=0}^{\infty} \sum_{\rho=0}^{\ell} \frac{(-1)^{\rho} q^{\binom{\rho}{2}} \lambda^{\rho}}{(q;q)_{\rho}} \frac{P_{\ell-\rho}(\zeta,\delta)}{(q;q)_{\ell-\rho}} \sum_{r=0}^{\infty} S_r(\delta q^{\ell-\rho}, \zeta, \lambda; q) \frac{q^{\rho r}}{(q;q)_r}\, t^{\ell}\, v^{r}.$$

Equating the coefficients of $t^{\ell}\, v^{r}$, we get the desired identity. □

By combining the identity (1.11) with the representation (2.1), it is possible to get the second Rogers-type formula for the polynomials $S_n(\delta,\zeta,\lambda|q)$.

Theorem 3.3 (Second Rogers formula for $S_n(\delta,\zeta,\lambda|q)$)**.** *We have*

$$\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} S_{t+r}(\delta,\zeta,\lambda|q)(-1)^t q^{-\binom{t}{2}-tr} \frac{\mu^t}{(q;q)_t} \frac{\nu^r}{(q;q)_r} = \frac{(\lambda\nu;q)_{\infty}}{(\lambda\mu;q)_{\infty}}\, {}_2\phi_1\left(\begin{matrix} \delta/\zeta, \nu/\mu \\ q/\lambda\mu \end{matrix} ; q, \zeta q/\lambda \right), \tag{3.3}$$

where $\max\{|\lambda\mu|, |\zeta q/\lambda|\} < 1$.

Proof.

$$\sum_{t=0}^{\infty}\sum_{r=0}^{\infty} S_{t+r}(\delta,\zeta,\lambda|q)(-1)^t q^{-\binom{t}{2}-tr} \frac{\mu^t}{(q;q)_t} \frac{\nu^r}{(q;q)_r}$$

$$= E(\delta/\zeta,\zeta;\theta) \left\{ \sum_{t=0}^{\infty} \frac{(\lambda\mu)^t}{(q;q)_t} \sum_{r=0}^{\infty} \frac{(-1)^r\, q^{\binom{r}{2}}\, (\lambda\nu)^r}{(q;q)_r} \right\} \quad \text{(by using (2.1))}$$

$$= E(\delta/\zeta,\zeta;\theta) \left\{ \frac{(\lambda\nu;q)_{\infty}}{(\lambda\mu;q)_{\infty}} \right\}$$

$$= \frac{(\lambda\nu;q)_{\infty}}{(\lambda\mu;q)_{\infty}}\, {}_2\phi_1\left(\begin{matrix} \delta/\zeta, \nu/\mu \\ q/\lambda\mu \end{matrix} ; q, \zeta q/\lambda \right). \quad \text{(by using (1.11))}$$

□

Using the representation (1.12) and equation (2.1), the extended Rogers formula of the polynomials $S_n(\delta,\zeta,\lambda|q)$ was discovered.

Theorem 3.4 (Extended Rogers formula for $S_n(\delta,\zeta,\lambda|q)$)**.** *We have*

$$\sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} S_{t+r+s}(\delta,\zeta,\lambda|q)(-1)^s q^{-\binom{s}{2}-s(t+r)-tr} \frac{\alpha^t}{(q;q)_t} \frac{\beta^r}{(q;q)_r} \frac{\gamma^s}{(q;q)_s}$$

$$= \frac{(\delta q/\lambda, \lambda\beta, \lambda\alpha;q)_{\infty}}{(\zeta q/\lambda, \lambda\gamma;q)_{\infty}}\, {}_3\phi_2\left(\begin{matrix} \delta/\zeta, q/\lambda\beta, q/\lambda\alpha \\ \delta q/\lambda, q/\lambda\gamma \end{matrix} ; q, \zeta\alpha\beta/\gamma \right), \tag{3.4}$$

where max $\max\{|\zeta q/\lambda|, |\lambda\gamma|, |\zeta\alpha\beta/\gamma|\} < 1$.

Proof.

$$\sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} S_{t+r+s}(\delta,\zeta,\lambda;q)(-1)^s q^{-\binom{s}{2}-s(t+r)-tr} \frac{\alpha^t}{(q;q)_t} \frac{\beta^r}{(q;q)_r} \frac{\gamma^s}{(q;q)_s}$$

$$= \sum_{t=0}^{\infty}\sum_{r=0}^{\infty}\sum_{s=0}^{\infty} E(\delta/\zeta,\zeta;\theta)\{(-b)^{t+r+s}q^{\binom{t+r+s}{2}}\}(-1)^s q^{-\binom{s}{2}-s(t+r)-tr}\frac{\alpha^t}{(q;q)_t}\frac{\beta^r}{(q;q)_r}\frac{\gamma^s}{(q;q)_s}$$

(by using (2.1))

$$= E(\delta/\zeta,\zeta;\theta)\left\{\sum_{t=0}^{\infty}\frac{(-1)^t q^{\binom{t}{2}}(\lambda\alpha)^t}{(q;q)_t}\sum_{r=0}^{\infty}\frac{(-1)^r q^{\binom{r}{2}}(\lambda\beta)^r}{(q;q)_r}\sum_{s=0}^{\infty}\frac{(\lambda\gamma)^s}{(q;q)_s}\right\}$$

$$= E(\delta/\zeta,\zeta;\theta)\left\{\frac{(\lambda\beta,\lambda\alpha;q)_\infty}{(\lambda\gamma;q)_\infty}\right\} \quad \text{(by using (1.2)and(1.3))}$$

$$= \frac{(\delta q/\lambda,\lambda\beta,\lambda\alpha;q)_\infty}{(\zeta q/\lambda,\lambda\gamma;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\delta/\zeta, q/\lambda\beta, q/\lambda\alpha\\ \delta q/\lambda, q/\lambda\gamma\end{matrix};q,\zeta\alpha\beta/\gamma\right). \quad \text{(by using (1.12))}$$

□

4. Two Mehler's Formula for the Polynomials $S_n(\delta,\zeta,\lambda|q)$

Two Mehler's formulas are given in this section utilizing the operators $L(b\lambda\theta_{\delta\zeta})$ and $E(\delta/\zeta,\zeta;\theta)$, respectively.

Theorem 4.1. *For $|\zeta s|<1$, we have*

$$L(b\lambda\theta_{\delta\zeta})\left\{\frac{P_j(\zeta,\delta)}{(\delta s;q)_j}\frac{(\delta s;q)_\infty}{(\zeta s;q)_\infty}\right\}$$
$$= \frac{(\lambda s,\delta s q^r;q)_\infty}{(\zeta s;q)_\infty}\sum_{k=0}^{r}\begin{bmatrix} r\\ k\end{bmatrix}(-1)^k q^{\binom{k}{2}}\frac{(\zeta s;q)_k(\delta/\zeta;q)_{r-k}}{(\lambda s;q)_k}\zeta^{r-k}\lambda^k. \tag{4.1}$$

Proof. Rewrite the L.H.S. of equation (3.1) as

$$\sum_{j=0}^{\infty}\sum_{\rho=0}^{\infty}S_{j+\rho}(\delta,\zeta,\lambda|q)\frac{r^j}{(q;q)_j}\frac{s^\rho}{(q;q)_\rho}$$
$$= \sum_{j=0}^{\infty}\sum_{\rho=0}^{\infty}L(\lambda\theta_{\delta\zeta})\{P_{j+\rho}(\zeta,\delta)\}\frac{r^j}{(q;q)_j}\frac{s^\rho}{(q;q)_\rho} \quad \text{(by using (1.17))}$$
$$= L(z\theta_{\delta\zeta})\left\{\sum_{j=0}^{\infty}P_j(\zeta,\delta)\frac{r^j}{(q;q)_j}\sum_{\rho=0}^{\infty}P_\rho(\zeta,\delta q^j)\frac{s^\rho}{(q;q)_\rho}\right\}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{j=0}^{\infty}P_j(\zeta,\delta)\frac{r^j}{(q;q)_j}\frac{(\delta s q^j;q)_\infty}{(\zeta s;q)_\infty}\right\} \quad \text{(by using (1.4))}$$
$$= \sum_{j=0}^{\infty}L(\lambda\theta_{\delta\zeta})\left\{\frac{P_j(\zeta,\delta)}{(\delta s;q)_j}\frac{(\delta s;q)_\infty}{(\zeta s;q)_\infty}\right\}\frac{r^j}{(q;q)_j}.$$

Letting $j\to j-\rho$ in the R.H.S. of equation (3.1), we get

$$\frac{(\delta s,\lambda s;q)_\infty}{(\zeta s;q)_\infty}\sum_{\rho=0}^{\infty}\sum_{j=\rho}^{\infty}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\zeta s;q)_\rho}{(q;q)_\rho(\delta s,zs;q)_\rho}\frac{(\delta/\zeta;q)_{j-\rho}}{(\delta s q^\rho;q)_{j-\rho}(q;q)_{j-\rho}}\zeta^{j-\rho}\lambda^\rho r^j$$

$$= \frac{(\lambda s;q)_\infty}{(\zeta s;q)_\infty}\sum_{j=0}^{\infty}(\delta s q^j;q)_\infty \sum_{\rho=0}^{j}\frac{(-1)^\rho q^{\binom{\rho}{2}}(\zeta s;q)_\rho}{(q;q)_\rho(\lambda s;q)_\rho}\frac{(\delta/\zeta;q)_{j-\rho}}{(q;q)_{j-\rho}}\zeta^{j-\rho}\,\lambda^\rho\, r^j.$$

The proof is finished after equating the coefficients of r^j. □

The operator $L(\lambda\theta_{\delta\zeta})$ will be used to provide the first Mehler's formula for $S_n(\delta,\zeta,\lambda|q)$.

Theorem 4.2 (First Mehler's formula for $S_n(\delta,\zeta,\lambda|q)$)**.** *We have*

$$\sum_{\rho=0}^{\infty} S_\rho(\delta,\zeta,\lambda|q)\, S_\rho(\alpha,v,\beta;q)\frac{(-1)^\rho\, q^{-\binom{\rho}{2}}\, e^\rho}{(q;q)_\rho}$$
$$= \frac{(\lambda\beta e,\delta\beta e,\lambda\alpha e;q)_\infty}{(\zeta\beta e,\lambda v e;q)_\infty}\,{}_3\phi_2\left(\begin{matrix}\delta/\zeta,\alpha/v,q/\lambda\beta e\\ q/\zeta\beta e, q/\lambda v e\end{matrix};q,q\right), \tag{4.2}$$

provided that $\delta/\zeta = q^{-r}$, $\alpha/v = q^{-r}$ *where* $\max\{|\zeta\beta e q^{-n}|,|\lambda v e q^{-n}|\}<1$ *of nonnegative integer* r.

Proof.

$$\sum_{\rho=0}^{\infty} S_\rho(\delta,\zeta,\lambda|q)\, S_\rho(\alpha,v,\beta;q)\frac{(-1)^\rho\, q^{-\binom{\rho}{2}}\, e^\rho}{(q;q)_\rho}$$
$$= \sum_{\rho=0}^{\infty} L(\lambda\theta_{\delta\zeta})\,\{P_\rho(\zeta,\delta)\}\; S_\rho(\alpha,v,\beta;q)\frac{(-1)^\rho q^{-\binom{\rho}{2}}\, e^\rho}{(q;q)_\rho}\quad \text{(by using (1.17))}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{\rho=0}^{\infty} P_\rho(\zeta,\gamma)(-1)^\rho q^{-\binom{\rho}{2}}\sum_{k=0}^{\rho}\begin{bmatrix}\rho\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}P_{\rho-k}(v,\alpha)\beta^k\frac{e^\rho}{(q;q)_\rho}\right\}\quad \text{(by using (1.9))}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{k=0}^{\infty}\sum_{\rho=k}^{\infty} P_\rho(\zeta,\delta)(-1)^\rho q^{-\binom{\rho}{2}}(-1)^k q^{\binom{k}{2}}P_{\rho-k}(v,\alpha)\beta^k\frac{e^\rho}{(q;q)_k(q;q)_{\rho-k}}\right\}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{\rho,k=0}^{\infty} P_{\rho+k}(\zeta,\delta)(-1)^{\rho+k} q^{-\binom{\rho+k}{2}}(-1)^k q^{\binom{k}{2}}P_{\rho}(v,\alpha)\beta^k\frac{e^{\rho+k}}{(q;q)_k(q;q)_{\rho}}\right\}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{\rho,k=0}^{\infty} P_\rho(v,\alpha)P_\rho(\zeta,\delta)P_k(\zeta,q^\rho\delta)(-1)^\rho q^{-\binom{\rho}{2}}\frac{\beta^k\, e^{\rho+k}q^{-\rho k}}{(q;q)_k(q;q)_\rho}\right\}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{-\binom{\rho}{2}}P_\rho(v,\alpha)P_\rho(\zeta,\delta)e^\rho}{(q;q)_\rho}\sum_{k=0}^{\infty}P_k(\zeta,q^\rho\delta)\frac{(\beta e q^{-\rho})^k}{(q;q)_k}\right\}$$
$$= L(\lambda\theta_{\delta\zeta})\left\{\sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{-\binom{\rho}{2}}P_\rho(v,\alpha)P_\rho(\zeta,\delta)e^\rho}{(q;q)_\rho}\frac{(\delta\beta e;q)_\infty}{(\zeta\beta e q^{-\rho};q)_\infty}\right\}\quad \text{(by using (1.4))}$$
$$= \sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{-\binom{\rho}{2}}P_\rho(v,\alpha)\, e^\rho}{(q;q)_\rho}L(\lambda\theta_{\delta\zeta})\left\{\frac{P_\rho(\zeta,\delta)(\delta\beta e q^{-\rho};q)_\infty}{(\delta\beta e q^{-\rho};q)_\rho\,(\zeta\beta e q^{-\rho};q)_\infty}\right\}$$
$$= \sum_{\rho=0}^{\infty}\frac{(-1)^\rho q^{-\binom{\rho}{2}}P_\rho(v,\alpha)\, e^\rho}{(q;q)_\rho}\frac{(\lambda\beta e q^{-\rho},\delta\beta e;q)_\infty}{(\zeta\beta e q^{-\rho};q)_\infty}\sum_{k=0}^{\rho}\begin{bmatrix}\rho\\k\end{bmatrix}(-1)^k q^{\binom{k}{2}}\frac{(\zeta\beta e q^{-\rho};q)_k(\delta/\zeta;q)_{\rho-k}}{(\lambda\beta e q^{-\rho};q)_k}\zeta^{\rho-k}\,\lambda^k$$

(by using (4.1))

$$= \sum_{k=0}^{\infty}\sum_{\rho=k}^{\infty} \frac{(-1)^{\rho} q^{-\binom{\rho}{2}} P_{\rho}(v,\alpha)\, e^{\rho}}{(q;q)_{\rho}(q;q)_{\rho-k}} \frac{(\lambda\beta e q^{-\rho+k}, \delta\beta e;q)_{\infty}}{(\zeta\beta e q^{-\rho+k};q)_{\infty}} (-1)^k q^{\binom{k}{2}} (\delta/\zeta;q)_{\rho-k}\, \zeta^{\rho-k}\, \lambda^k$$

$$= \sum_{\rho=0}^{\infty}\sum_{k=0}^{\infty} \frac{(-1)^{\rho} q^{-\binom{\rho}{2}-\rho k} P_{\rho+k}(v,\alpha)\, e^{\rho+k}}{(q;q)_{\rho}(q;q)_{k}} \frac{(\lambda\beta e q^{-\rho}, \delta\beta e;q)_{\infty}}{(\zeta\beta e q^{-\rho};q)_{\infty}} (\delta/\zeta;q)_{\rho}\, \zeta^{\rho}\, \lambda^k$$

$$= \frac{(\lambda\beta e, \delta\beta e;q)_{\infty}}{(\zeta\beta e;q)_{\infty}} \sum_{\rho=0}^{\infty} \frac{(-1)^{\rho} q^{-\binom{\rho}{2}} (\lambda\beta e q^{-\rho}, \delta/\zeta;q)_{\rho}\, P_{\rho}(v,\alpha)\, (e\zeta)^{\rho}}{(q;q)_{\rho}(\zeta\beta e q^{-\rho};q)_{\rho}} \sum_{k=0}^{\infty} P_k(v,\alpha q^{\rho}) \frac{(e\lambda q^{-\rho})^k}{(q;q)_k}$$

$$= \frac{(\lambda\beta e, \delta\beta e, \alpha\lambda e;q)_{\infty}}{(\zeta\beta e, v\lambda e;q)_{\infty}} \sum_{\rho=0}^{\infty} \frac{(-1)^{\rho} q^{-\binom{\rho}{2}} (\lambda\beta e q^{-\rho}, \delta/\zeta;q)_{\rho}\, P_{\rho}(v,\alpha)\, (e\zeta)^{\rho}}{(q;q)_{\rho}(\zeta\beta e q^{-\rho};q)_{\rho}(\lambda v e q^{-\rho};q)_{\rho}}. \quad \text{(by using (1.4))}$$

By using (1.1), we get the required identity. □

Using the operator $E(\delta/\zeta,\zeta;\theta)$, the second Mehler's formula for $S_n(\delta,\zeta,\lambda|q)$ can be determined.

Theorem 4.3 (Second Mehler's formula for $S_n(\delta,\zeta,\lambda;q)$)**.** *We have*

$$\sum_{t=0}^{\infty} S_t(\delta,\zeta,\lambda;q)\, S_t(u,v,w;q) \frac{(-1)^t\, q^{-\binom{t}{2}}\, a^t}{(q;q)_t}$$
$$= \frac{(\delta q/\lambda, \lambda u a, \lambda w a;q)_{\infty}}{(\zeta q/\lambda, \lambda v a;q)_{\infty}}\, {}_3\phi_2\left(\begin{matrix} \delta/\zeta, q/\lambda u a, q/\lambda w a \\ \delta q/\lambda, q/\lambda v a \end{matrix} ; q, \zeta u w a/v\right), \tag{4.3}$$

where $\max\{|\zeta q/\lambda|, |\lambda v a|, |\zeta u w a/v|\} < 1$.

Proof.

$$\sum_{t=0}^{\infty} S_t(\delta,\zeta,\lambda;q)\, S_t(u,v,w;q) \frac{(-1)^t\, q^{-\binom{t}{2}}\, a^t}{(q;q)_t}$$
$$= \sum_{t=0}^{\infty} E(\delta/\zeta,\zeta;\theta)\{(-1)^t\, q^{\binom{t}{2}}\, \lambda^t\}\, S_t(u,v,w;q) \frac{(-1)^t\, q^{-\binom{t}{2}}\, a^t}{(q;q)_t}$$
$$= E(\delta/\zeta,\zeta;\theta) \left\{ \sum_{t=0}^{\infty} S_t(u,v,w;q) \frac{(\lambda a)^t}{(q;q)_t} \right\}$$
$$= E(\delta/\zeta,\zeta;\theta) \left\{ \frac{(\lambda u a, \lambda w a;q)_{\infty}}{(\lambda v a;q)_{\infty}} \right\} \quad \text{(by using (1.16))}$$
$$= \frac{(\delta q/\lambda, \lambda u a, \lambda w a;q)_{\infty}}{(\zeta q/\lambda, \lambda v a;q)_{\infty}}\, {}_3\phi_2\left(\begin{matrix} \delta/\zeta, q/\lambda u a, q/\lambda w a \\ \delta q/\lambda, q/\lambda v a \end{matrix} ; q, \zeta u w a/v\right) \quad \text{(by using (1.12))}.$$

□

By using the following transformation [11], we can demonstrate that the two Mehler's formulas (4.2) and (4.3) are equivalent.

$$
{}_3\phi_2\left(\begin{matrix}\zeta/\delta, q/a\lambda t, q/avt \\ \zeta q/a, q/aut\end{matrix}; q, \delta\lambda vt/u\right) = \frac{(\lambda\zeta t, \delta q/a)_\infty}{(\lambda\delta t, \zeta q/a)_\infty}\,{}_3\phi_2\left(\begin{matrix}\zeta/\delta, v/u, q/a\lambda t \\ q/\delta\lambda t, q/aut\end{matrix}; q, q\right). \tag{4.4}
$$

5. Conclusions

For a single polynomial, several q-shift operators can be used. The generating function, Rogers formula, and Mehler's formula are just a few examples of significant polynomials identities that may be efficiently derived using the q-shift operators. The generating function, two Rogers formulas, the extension of one of Rogers formulas, the linearization formula, and two Mehler's formulas for the polynomials $S_n(\delta, \zeta, \lambda|q)$ have all been systematically supplied using the q-shift operators. We anticipate that this study will stimulate more investigation into q-polynomials and their practical uses.

Authors' Contributions

All authors have read and approved the final version of the manuscript. The authors contributed equally to this work.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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